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Autor: Dahlberg, B.E.J. / Trubowitz, E.
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A remark on two dimensional periodic potentials

B. E. J. DAHLBERG¹ and E. TRUBOWITZ^{1,2}

1. Introduction

Let $A = a_1\mathbf{Z} \oplus a_2\mathbf{Z}$ be the lattice generated by the linearly independent vectors $a_1, a_2 \in \mathbf{R}^2$. Suppose q is a bounded, real-valued function periodic with respect to A . It is known that (see [2] and [5]) the spectrum of the self-adjoint operator $-\Delta + q$ acting on $L^2(\mathbf{R}^2)$ is purely absolutely continuous and is the union of the bands $B_n(q)$ ($n \geq 1$). The n th band $B_n(q) \subset \mathbf{R}^1$ ($n \geq 1$) is by definition the image of the function $\lambda_n(\cdot, q)$ defined on \mathbf{R}^2 , where $\lambda_n(k, q)$ ($k \in \mathbf{R}^2$) is the n th eigenvalue (counting with multiplicity) of the boundary value problem

$$\begin{aligned} (-\Delta + q)f &= \lambda f, \\ f(x + a_j) &= e^{ik \cdot a_j} f(x), \quad j = 1, 2, \dots \end{aligned}$$

Let $|\mathfrak{F}|$ be the area of a fundamental cell \mathfrak{F} for the lattice B dual to A .

THEOREM. *There exist positive constants $c(q), C(q)$ such that for all $n \geq N(q)$*

$$\left[\frac{|\mathfrak{F}|}{\pi} n - cn^{1/4}, \frac{|\mathfrak{F}|}{\pi} n + cn^{1/4} \right] \subset B_n(q) \subset \left[\frac{|\mathfrak{F}|}{\pi} n - Cn^{1/3}, \frac{|\mathfrak{F}|}{\pi} n + Cn^{1/3} \right].$$

Suppose $q(x)$ is a bounded, real-valued, periodic function on \mathbf{R}^1 . Then the spectrum of $-(d^2/dx^2) + q(x)$ acting on $L^2(\mathbf{R}^1)$ is purely absolutely continuous and its complement generically consists of an infinite number of intervals tending to $+\infty$ [see [1], p. 161]; i.e., there are infinitely many gaps. In two dimensions, however, we have the following immediate consequence of the Theorem.

COROLLARY. *The spectrum of $-\Delta + q$ contains a ray $[\lambda^*, \infty)$. In particular, there are at most a finite number of gaps.*

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The corollary has been announced in [3] for the special case of rational lattices.

2. Proofs

Let $N_B(r, k)$ be the number of lattice points of B in the open disk of radius r centered at $k \in \mathbf{R}^2$. Denote by c, C generic positive constants.

LEMMA 1. For every $\delta > 0$ there is a constant $c = c(\delta) > 0$ such that

$$\sup_{k \in \mathbf{R}^2} N_B(r, k) \geq \frac{\pi}{|\mathfrak{F}|} r^2 + cr^{1/2},$$

$$\inf_{k \in \mathbf{R}^2} N_B(r, k) \leq \frac{\pi}{|\mathfrak{F}|} r^2 - cr^{1/2}.$$

for all $r \geq \delta$.

Proof. Let h_r be the characteristic function of the open disk $\{k \in \mathbf{R}^2 : |k| < r\}$ and set $E_B(r, k) = N_B(r, k) - (\pi/|\mathfrak{F}|)r^2$. If $a \in A \setminus \{0\}$ we have

$$\begin{aligned} \int_{\mathfrak{F}} E_B(r, k) e^{-ik \cdot a} dk &= \int_{\mathfrak{F}} \left(-\pi \frac{r^2}{|\mathfrak{F}|} + \sum_{b \in B} h_r(k-b) \right) e^{-ik \cdot a} dk \\ &= \int_{\mathbf{R}^2} h_r(k) e^{-ik \cdot a} dk \\ &= \frac{2\pi r}{|a|} J_1(r|a|) \end{aligned}$$

where J_1 is the Bessel function of order 1. A similar calculation yields

$$\int_{\mathfrak{F}} E_B(r, k) dk = 0.$$

It follows that for all $a \in A \setminus \{0\}$

$$\begin{aligned} \int_{\mathfrak{F}} |E_B(r, k)| dk &\geq \max \left(\left| \int_{\mathfrak{F}} E_B(r, k) e^{-ik \cdot a} dk \right|, \left| \int_{\mathfrak{F}} E_B(r, k) e^{-2ik \cdot a} dk \right| \right) \\ &\geq 2\pi r \max \left(\frac{|J_1(r|a|)|}{|a|}, \frac{|J_1(2r|a|)|}{2|a|} \right) \\ &\geq (8\pi r)^{1/2} \max \left(\frac{|\sin(r|a| - \pi/4)|}{|a|^{3/2}}, \frac{|\sin(2r|a| - \pi/4)|}{|2a|^{3/2}} \right) \\ &\quad - Cr^{-1/2} |a|^{-5/2} \\ &\geq cr^{1/2} |a|^{-3/2} - Cr^{-1/2} |a|^{-5/2}. \end{aligned}$$

The asymptotic estimate

$$J_1(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right) + O(x^{-3/2})$$

was used in the third line and

$$\inf_{t \in \mathbb{R}} \max\left(\left|\sin\left(t - \frac{\pi}{4}\right)\right|, \frac{|\sin(2t - \pi/4)|}{\sqrt{8}}\right) > 0$$

in the fourth. Consequently, for any fixed $a^* \in A \setminus \{0\}$ there is an r^* such that

$$\int_{\mathfrak{F}} |E_B(r, k)| dk \geq cr^{1/2}$$

for all $r \geq r^*$. Moreover, for all $0 < \delta < 1$ there is a $c(\delta)$ for which

$$\int_{\mathfrak{F}} |E_B(r, k)| dk \geq c(\delta)r^{1/2}$$

when $\delta < r < r^*$. This is achieved by picking an $a \in A$ with $|a|$ sufficiently large. Therefore,

$$\int_{\mathfrak{F}} E_B^+(r, k) dk = \int_{\mathfrak{F}} E_B^-(r, k) dk > cr^{1/2}.$$

Here, $E_B^+ = \max(E_B, 0)$ and $E_B^- = \min(-E_B, 0)$.

The statement of Lemma 1 follows at once. \square

Let $[x_n, y_n]$, $n \geq 1$, be the n th band corresponding to $q = 0$.

LEMMA 2. For¹ $n \geq 1$

$$\sup_{k \in \mathbb{R}^2} N_B(\sqrt{x_n}, k) \leq n \leq \inf_{k \in \mathbb{R}^2} N_B\left(\sqrt{y_n + \frac{1}{\sqrt{y_n}}}, k\right)$$

Proof. Suppose $\sup_k N_B(\sqrt{x_n}, k) \geq n + 1$. Then there is a k such that $\lambda_{n+1}(k, 0) < x_n$. This contradicts the definition of $\lambda_{n+1}(k, 0)$. Similarly, suppose $\inf_k N_B(\sqrt{y_n + 1/\sqrt{y_n}}, k) \leq n - 1$. Then there is a k such that $\lambda_n(k, 0) > y_n$. But this contradicts the definition of $\lambda_n(k, 0)$. \square

¹Observe that $0 \leq x_n < y_n$, $n \geq 1$.

We can now prove the theorem.

Combining Lemmas 1, 2 with the obvious estimate $(\sqrt{y_n} + 1/\sqrt{y_n})^{1/2} > y_n^{1/4}$, we have

$$\begin{aligned} \frac{\pi}{|\mathcal{F}|} x_n + cx_n^{1/4} \leq n &\leq \frac{\pi}{|\mathcal{F}|} \left(y_n + 2 + \frac{1}{y_n} \right) - c \left(\sqrt{y_n} + \frac{1}{\sqrt{y_n}} \right)^{1/2} \\ &\leq \frac{\pi}{|\mathcal{F}|} y_n - c' y_n^{1/4}. \end{aligned}$$

By similar arguments it is easy to show that

$$n \leq \sup_k N_B \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}, k \right) \leq cx_n$$

and

$$n \leq \sup_k N_B(\sqrt{y_n}, k) \geq cy_n.$$

Consequently,

$$x_n \leq \frac{|\mathcal{F}|}{\pi} n - cn^{1/4}, \quad \frac{|\mathcal{F}|}{\pi} n + cn^{1/4} \leq y_n.$$

This proves the left hand inclusion of the theorem for $B_n(0)$. The statement for general q follows from the min-max characterization of $\lambda_n(k, q)$:

$$\lambda_n(k, q) = \sup_{\Phi} \inf_{\substack{\psi \in \Phi^\perp \\ \|\psi\|=1}} \langle \psi, (-\Delta + q)\psi \rangle$$

where $\Phi = \{\Phi_1, \dots, \Phi_{n-1}\} \subset L^2(\mathbf{R}^2)$, Φ^\perp is the orthogonal complement of the span of Φ and $\psi(x + a_j) = e^{ik \cdot a_j} \psi(x)$, $j = 1, 2$. It gives

$$|\lambda_n(k, q) - \lambda_n(k, 0)| \leq \|q\|_\infty$$

which yields the results.

The right hand inclusion is verified in almost the same way. The only significant difference is that Lemma 1 is replaced by the well known estimate

$$\left| N_B(r, k) - \frac{\pi}{|\mathcal{F}|} r^2 \right| \leq cr^{2/3}$$

one derives from the Poisson summation formula (c is independent of k). The proof of the theorem is finished.

Remark. The overlap of $B_n(0)$ and $B_{n+1}(0)$ is uniformly bounded away from 0; i.e., $|B_n(0) \cap B_{n+1}(0)| \geq c > 0$ for all $n \geq 1$. For

$$\begin{aligned} n &\geq \sup_k N_B\left(\sqrt{x_{n+1}} - \frac{\varepsilon}{\sqrt{x_{n+1}}}, k\right) \\ &\geq \frac{\pi}{|\mathfrak{F}'|} \left(x_{n+1} - 2\varepsilon + \frac{\varepsilon^2}{x_{n+1}}\right) + c \left(\sqrt{x_{n+1}} - \frac{\varepsilon}{\sqrt{x_{n+1}}}\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} n &\leq \inf_k N\left(\sqrt{y_n} + \frac{\varepsilon}{\sqrt{y_n}}, k\right) \\ &\leq \frac{\pi}{|\mathfrak{F}'|} \left(y_n + 2\varepsilon + \frac{\varepsilon^2}{y_n}\right) - c \left(\sqrt{y_n} + \frac{\varepsilon}{\sqrt{y_n}}\right)^{1/2}. \end{aligned}$$

Combining the inequalities and letting ε tend to 0 we find

$$x_{n+1} \leq y_n - c(y_n^{1/4} + x_{n+1}^{1/4}).$$

It is an immediate consequence that the spectrum of $-\Delta + q$ has no gaps at all when $|q|_\infty$ is sufficiently small. this reproves the result of [4].

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Department of Mathematics
Uppsala University
Thunbergsv. 3
S-75238 Uppsala
Sweden

Courant Institute
New York University
251 Mercer St.
New York, NY 10012
U.S.A.

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