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## Cyclic group actions on odd-dimensional spheres

C. Kearton\* and S. M. J. Wilson

Abstract. We show that for any simple (2q-1)-knot k, q > 1, and any positive integer n, the knot  $\#_1^n k$  is the fixed-point set of a  $\mathbb{Z}_n$ -action on  $S^{2q+1}$ . Further, we show that for many values of n there are examples of (2q-1)-knots,  $q \ge 2$ , which are the fixed-point sets of inequivalent  $\mathbb{Z}_n$ -actions.

### 0. Introduction

An *n*-knot is a locally-flat PL pair  $(S^{n+2}, S^n)$ , where  $S^n$  denotes the *n*-sphere. A (2q-1)-knot is *simple* if the complement of  $S^{2q-1}$  has the homotopy type of a circle up to but not including dimension q. For q>1 such knots have been classified in [L] in terms of the S-equivalence classes of their Seifert matrices, and in [K, T1, T2] in terms of their Blanchfield pairings. Using these classification results, for any simple (2q-1)-knot k, with q>1, and for any positive integer n, we construct a simple (2q-1)-knot  $k_n$  such that the n-fold cyclic cover of  $S^{2q+1}$  branched over  $k_n$  is again  $S^{2q+1}$ , and such that  $k_n$  lifts to  $\#_1^n k$ , the sum of n copies of k. An immediate corollary is that for any such k and n, there is a  $\mathbb{Z}_n$ -action on  $S^{2q+1}$  with fixed point set  $\#_1^n k$ .

The construction in this paper is purely algebraic, and may be contrasted with the geometric construction in [G], where for any m-knot k ( $m \ge 2$ ) Gordon constructs an m-knot which is the fixed-point set of a  $\mathbb{Z}_n$ -action and whose fundamental group is isomorphic to that of  $\#_1^n k$ .

As an application of our construction we are able for many values of n to find examples of (2q-1)-knots which are the fixed-point sets of inequivalent  $\mathbb{Z}_n$ -actions. The technique is to pick simple (2q-1)-knots k and l such that  $\#_1^n k = \#_1^n l$ , and such that  $k_n \neq l_n$ .

### 1. The main construction

Let k be a simple (2q-1)-knot, q>1, and n>1 an integer. Let A be a non-singular Seifert matrix of k, and set  $\varepsilon=(-1)^q$ . Following Trotter [T1], we set  $S=(A+\varepsilon A')^{-1}$ ,  $T=-\varepsilon A'A^{-1}$ .

<sup>\*</sup> This paper was written whilst the first author was in receipt of a Research Grant from the Science Research Council of Great Britain.

PROPOSITION 1.1. The pair (S, T) has the following properties.

- (i) S is integral, unimodular,  $\varepsilon$ -symmetric.
- (ii)  $(I-T)^{-1}$  exists and is integral.
- (iii) T'ST = S.
- (iv)  $A = (I T)^{-1}S^{-1}$ .

Moreover, any pair of rational matrices (S, T) satisfying (i)–(iii) yields a Seifert matrix A by the formula (iv).

**Proof.** It is well known (see [L], [T1]) that  $A + \varepsilon A'$  is unimodular, and so S is integral and unimodular. Clearly S is  $\varepsilon$ -symmetric.

Now  $I - T = I + \varepsilon A' A^{-1} = (A + \varepsilon A') A^{-1} = S^{-1} A^{-1}$ , from which (ii) and (iv) follow at once. Property (iii) is easily checked.

Now suppose that we are given a pair of rational matrices (S, T) satisfying (i)-(iii); then we can define the matrix  $A = (I - T)^{-1}S^{-1}$ , which by (i) and (ii) is a non-singular matrix over the integers. We have

$$A + \varepsilon A' = (I - T)^{-1} S^{-1} + \varepsilon (S')^{-1} (I - T')^{-1}$$

$$= (I - T)^{-1} S^{-1} + S^{-1} (I - ST^{-1} S^{-1})^{-1} \quad \text{by (i), (iii)}$$

$$= (I - T)^{-1} S^{-1} + S^{-1} S (I - T^{-1})^{-1} S^{-1}$$

$$= [(I - T)^{-1} - (I - T)^{-1} T] S^{-1} = (I - T)^{-1} (I - T) S^{-1} = S^{-1}$$

which is unimodular. It follows that A is a Seifert matrix.  $\square$ 

Now we define matrices U, V by

$$U = \begin{bmatrix} 0 & T & T \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad V = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix},$$

there being  $n \times n$  blocks in each case.

THEOREM 1.2. The pair (V, U) determines a simple (2q-1)-knot  $k_n$ . The n-fold branched cyclic cover of  $k_n$  is the knot  $\#_1^n k = k + \cdots + k$  (n times).

*Proof.* We have to check that the pair (V, U) satisfies conditions (i)-(iii) of

Proposition 1.1. Clearly V satisfies (i), and it is easy to check that

$$(I-U)^{-1} = \begin{cases} (I-T)^{-1} & T(I-T)^{-1} - T(I-T)^{-1} \\ \vdots & T(I-T)^{-1} \\ (I-T)^{-1} - T(I-T)^{-1} \\ \vdots & T(I-T)^{-1} \end{cases}.$$

But  $T(I-T)^{-1} = -\varepsilon A'A^{-1}(I+\varepsilon A'A^{-1})^{-1} = -\varepsilon A'(A+\varepsilon A')^{-1}$ , which is an integer matrix. Hence (ii) is satisfied. To check (iii) is a simple matrix multiplication.

Hence (V, U) determine a unique simple (2q-1)-knot  $k_n$ . A routine computation shows that

$$U^n = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix},$$

and hence the pair  $(V, U^n)$  satisfies (i)-(iii), and in fact represents the knot  $\#_1^n k$ .

Let  $K_n$  denote the complement of  $k_n$ , and  $\tilde{K}_n$  the infinite cyclic cover of  $K_n$ . If u is a generator of the group of covering translations, then  $K_n$  is obtained from  $\tilde{K}_n$  by quotienting out by the action of u. Similarly the n-fold cyclic cover of  $K_n$  is obtained from  $\tilde{K}_n$  by quotienting out by the action of  $u^n$ .

Algebraically this can be described as follows, using Trotter's description of  $H_q(\tilde{K}_n)$  in [T1]. Let **B** be a basis of  $\mathbb{Q}^m$  corresponding to (V, U) where T is an  $r \times r$  matrix; then  $H_q(\tilde{K}_n)$  is the  $\mathbb{Z}[u, u^{-1}]$ -module generated by **B**, the action of u being given by U. The fact that  $(1-u): H_q(\tilde{K}_n) \to H_q(\tilde{K}_n)$  is an isomorphism means that when we quotient out by the action of u we get a homology circle. But the form of  $U^n$  means that  $(1-u^n): H_q(\tilde{K}_n) \to H_q(\tilde{K}_n)$  is also an isomorphism, and hence the n-fold cyclic cover of  $K_n$  is a homology circle. Therefore the n-fold branched cyclic cover of  $k_n$  is a homotopy sphere, and hence a sphere.  $\square$ 

COROLLARY 1.3. If k is a simple (2q-1)-knot, q > 1, then  $\#_1^n k$  is the fixed point set of a  $\mathbb{Z}_n$ -action on  $S^{2q+1}$ .

PROPOSITION 1.4. Let B be the Seifert matrix of  $k_n$  corresponding to (V, U). Then

$$B = \begin{bmatrix} A & -\varepsilon A' - - \varepsilon A' \\ \vdots & \ddots & \vdots \\ A - - \varepsilon A' \\ A - - - - A \end{bmatrix}.$$

**Proof**:

$$B = (I - U)^{-1}V^{-1} = \begin{pmatrix} (I - T)^{-1} & T(I - T)^{-1} & T(I - T)^{-1} \\ (I - T)^{-1} & T(I - T)^{-1} \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (I - T)^{-1}S^{-1} & T(I - T)^{-1}S^{-1} & T(I - T)^{-1}S^{-1} \\ (I - T)^{-1}S^{-1} & T(I - T)^{-1}S^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A & TA - \cdots - TA \\ \vdots & \vdots & \vdots \\ A - \cdots - A \end{pmatrix}$$

$$= \begin{bmatrix} A & -\varepsilon A' - \cdots - \varepsilon A' \\ \vdots & \ddots & \vdots \\ A - \cdots - \varepsilon A' \\ A - \cdots - A \end{bmatrix}$$

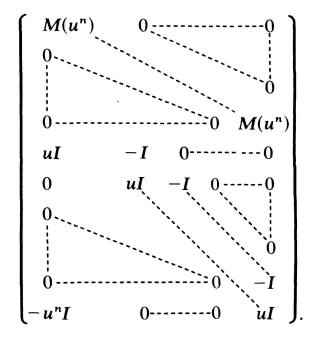
Next we prove a result which relates an Alexander matrix of k to one for  $k_n$ . Recall that an Alexander matrix M(t) of k is a matrix over  $\mathbf{Z}[t, t^{-1}]$  which presents  $H_q(\tilde{K})$  as a  $\mathbf{Z}[t, t^{-1}]$ -module; that is, there is an exact sequence of  $\mathbf{Z}[t, t^{-1}]$ -modules

$$F \xrightarrow{M(t)} G \longrightarrow H_q(\tilde{K})$$

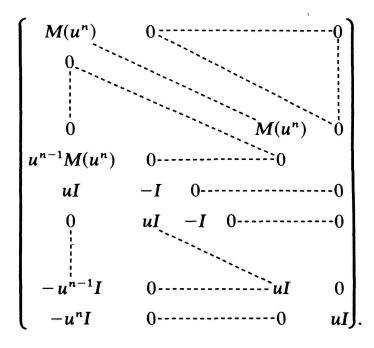
where F and G are free  $\mathbb{Z}[t, t^{-1}]$ -modules.

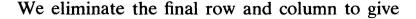
PROPOSITION 1.5. Let M(t) be a square Alexander matrix for the knot k; then  $M(t^n)$  is an Alexander matrix for  $k_n$ .

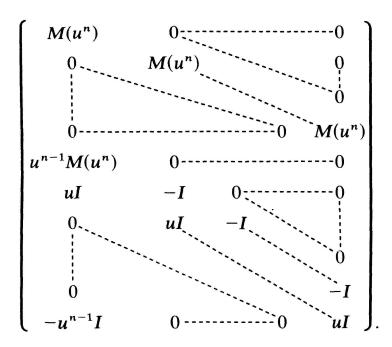
*Proof.* We can describe the  $\mathbb{Z}[u, u^{-1}]$ -module structure of  $H_q(\tilde{K}_n)$  in the following way. Let  $L_1, \ldots, L_n$  be n copies of the  $\mathbb{Z}[t, t^{-1}]$ -module  $H_q(\tilde{K})$ . Then for  $1 \le i < n, u: L_i \to L_{i+1}$  is a  $\mathbb{Z}[t, t^{-1}]$ -isomorphism, and  $u: L_n \to L_1$  is defined so that  $u^n: L_1 \to L_2 \to \cdots \to L_n \to L_1$  coincides with  $t: L_1 \to L_1$ . Thus a presentation matrix for  $H_q(\tilde{K}_n)$  as a  $\mathbb{Z}[u, u^{-1}]$ -module is



Two elementary row operations give







Now subtract  $u^{n-1}$  times the first row from the  $n^{th}$  to obtain a row of zeros, which may be eliminated. Continuing in this way we eventually arrive at the matrix  $M(u^n)$ .  $\square$ 

THEOREM 1.6. The knot  $k_n$  depends only on k and n, and not upon the choice of Seifert matrix A.

**Proof.** Let A be an  $r \times r$  matrix, and let  $\Lambda = \mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$ , a subring of the field  $\mathbb{Q}(t)$ , the field of rational functions in one variable over the rationals. According to Trotter's viewpoint [T1], k gives rise to an  $\varepsilon$ -symmetric bilinear form [,] on  $\mathbb{Q}^r$  represented by the matrix S, and a  $\Lambda$ -module M contained in  $\mathbb{Q}^r$  where the action of t is represented by T. A choice of Seifert matrix corresponds to an admissible lattice contained in M (see [T1] for definitions). Although our construction is given in terms of the matrices S and T, it is clear that it could be phrased in terms of M and [,], and hence that it does not depend upon the choice of A.

Alternatively, one can use the formula of Proposition 1.4 to show that if A is S-equivalent to  $A_1$ , then B is S-equivalent to  $B_1$ .  $\square$ 

# 2. Knots having distinct $Z_n$ -actions, n odd

In this section we shall show that for many odd integers n, there exist simple (4q+1)-knots  $(q \ge 1)$ , k and l, such that  $\#_1^n k = \#_1^n l$  but  $k_n \ne l_n$ .

Let  $\lambda_m(t)$  denote the  $m^{th}$  cyclotomic polynomial, where m is not a prime

power. Let  $\zeta$  be a primitive  $m^{\text{th}}$  root of unity,  $K = \mathbf{Q}(\zeta)$  and  $F = \mathbf{Q}(\zeta + \zeta^{-1})$ , the fixed field of K under complex conjugation. Let  $h_K$  denote the class number of K,  $h_F$  that of F, and  $h_- = h_K/h_F$ . According to the work of Bayer [Ba: Example 6.2], the number of distinct simple (4q+1)-knots  $(q \ge 1)$  with Alexander polynomial  $\lambda_m(t)$  is  $h_-2^d$  if  $m = 2p^\alpha$  and  $h_-2^{d-1}$  otherwise, where  $2d = [K:\mathbf{Q}]$ . The factor  $h_-$  represents the number of isomorphism classes of  $\mathbf{Z}[t, t^{-1}]$ -modules supporting a Blanchfield pairing [Ba: Corollary 1.3], and the factor  $2^d(2^{d-1})$  represents the number of non-isometric pairings which a given module supports. Note that Bayer's work is couched in terms of pairings on  $\mathbf{Z}[\zeta]$ -modules which are hermitian with respect to complex conjugation  $(t \to t^{-1})$  becomes  $\zeta \to \zeta^{-1} = \overline{\zeta}$ , and we shall adopt this viewpoint.

Let U be the group of units of (the ring of integers of) K,  $U_0$  the group of units of F, and  $N: K \to F$  the norm. If I is a principal ideal, then let  $\langle u \rangle$  denote the hermitian form h on I given by  $h(a, b) = ua\bar{b}$ . As in [Ba: Prop. 2.1], the set of isometry classes of unimodular hermitian forms on a given ideal (not necessarily principal) is in one-one correspondence with  $U_0/N(U)$ .

Now suppose that  $h_-$  has a factor n > 1, where n is odd and (m, n) = 1. Let a be an ideal of  $\mathbf{Q}(\zeta)$  admitting a non-singular hermitian form h, with a being of order n in ker  $N: C_K \to C_F$ . Then  $\frac{1}{1}^n(a, h)$  has determinant  $\langle u \rangle$  for some  $u \in U_0/N(U)$ ; see [Ba: Definition 1.9] for the definition of determinant. Since the order of  $U_0/N(U)$  is  $2^d$  or  $2^{d-1}$ , and n is odd, there exists  $v \in U_0/N(U)$  such that  $v^n = u$ . Then  $\frac{1}{1}^n \langle v \rangle$  has determinant  $\langle v^n \rangle = \langle u \rangle$ .

Set  $K = (a, h) \perp (a, -h)$ ,  $L = \langle v \rangle \perp \langle -v \rangle$ . Then  $\frac{1}{1}{}^n K$ ,  $\frac{1}{1}{}^n L$  are indefinite and have the same rank, signatures and determinant. Hence by [Ba: Corollary 4.10] they are isometric. But K is not isometric to L, for the determinant of K is  $(a^2, \alpha)$ , and  $a^2$  is non-zero in ker  $N: C_K \to C_F$  since n is odd.

In fact, if k, l are the simple (4q+1)-knots corresponding to K, L respectively, we can show that  $k_n \neq l_n$ . For let M(t) be an Alexander matrix of k, so that by Proposition 1.5  $M(t^n)$  is an Alexander matrix of  $k_n$ . The work of Fox and Smythe [F-S] enables us to obtain a row ideal class from the matrix  $M(\zeta)$ , and the work of Hillman [H: Chap. III, Theorem 12] identifies this with the ideal  $a^2$  in the determinant of K. But the Alexander polynomial of  $k_n$  is  $\lambda_m(t^n)$ , which has  $\lambda_m(t)$  as one of its factors since (m, n) = 1. Let  $\tau$  be a primitive  $m^{th}$  root of unity such that  $\tau^n = \zeta$ . Setting  $t = \tau$  in the Alexander matrix  $M(t^n)$ , we obtain  $M(\tau^n) = M(\zeta)$ , and hence obtain a Fox-Smythe invariant  $a^2$  again. In the case of  $l_n$ , these ideal invariants are all trivial, hence  $k_n \neq l_n$ .

Taking the *n*-fold branched cyclic covers of  $k_n$ ,  $l_n$  we obtain respectively the knots  $\#_1^n k$ ,  $\#_1^n l$ . Since  $\lim_{l \to \infty} K$  is isometric to  $\lim_{l \to \infty} L$ , we have  $\#_1^n k = \#_1^n l$ .

Many examples may be obtained from the tables in [Sch].

For the case of (4q-1)-knots,  $q \ge 1$ , and  $m \ne 2p^r$ ,  $p^r$ , where p is a prime, then as in [Ba: §5],  $\zeta - \zeta^{-1}$  is a unit and so we can multiply all the pairings above by  $\zeta - \zeta^{-1}$  to obtain skew-hermitian pairings. The argument then goes through as before. We are grateful to Dr. Bayer for pointing out this extension to the case of (4q-1)-knots.

## 3. Number theory

This section deals with some results from algebraic number theory, which will be used in the next section to deal with the case n = 2.

Let K be an algebraic number field,  $R = \operatorname{int}(K)$  its ring of integers,  $\mathbf{Z}_{(p)}$  the p-adic integers,  $R_p = R \otimes \mathbf{Z}_{(p)}$ ,  $K_p = K \otimes \mathbf{Z}_{(p)}$ ,  $U(R) = \prod_P R_p^{\times}$  and  $J(K) = U(R) \cdot \coprod_p K_p^{\times}$ , where  $\coprod$  denotes the direct sum.  $K^{\times}$  is considered as a subgroup of J(K) under the "diagonal" map. If C(K) denotes the ideal class group of K, then we have  $C(K) \cong J(K)/U(R) \cdot K^{\times}$  an isomorphism which is natural with respect to ring extensions.

Now suppose that L is an algebraic number field,  $\Gamma$  a group of automorphisms of L, S = int(L),  $K = L^{\Gamma}$  the subfield of L fixed under  $\Gamma$ , and  $R = \text{int}(K) = S^{\Gamma}$ .

LEMMA 3.1.  $\ker[C(R) \to C(S)] \cong \ker[H^1(\Gamma, S^{\times}) \to H^1(\Gamma, U(S))]$ , where the first map is induced by ring extension, the second by the "diagonal" map  $S^{\times} \to U(S)$ .

Proof. Consider the exact sequence

$$0 \rightarrow U(S) \cdot L^{\times} \rightarrow J(L) \rightarrow C(S) \rightarrow 0$$
,

Since  $J(L)^{\Gamma} = J(K)$ , we obtain a commutative diagram

$$0 \to U(R) \cdot K^{\times} \to J(K) \to C(R) \to 0$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \to (U(S) \cdot L^{\times})^{\Gamma} \to J(L)^{\Gamma} \to C(S)^{\Gamma}$$

Applying the Snake Lemma [Bass: p. 26] we find that

$$\ker [C(R) \to C(S)] = \ker [C(R) \to C(S)^{\Gamma}]$$

$$\cong \operatorname{coker} [U(R) \cdot K^{\times} \to (U(S) \cdot L^{\times})^{\Gamma}].$$

Now consider the exact sequence

$$0 \rightarrow S^{\times} \rightarrow U(S) \oplus L^{\times} \rightarrow U(S) \cdot L^{\times} \rightarrow 0$$

where the first map is  $s \mapsto (s, s^{-1})$ . From cohomology theory we obtain the exact sequence

$$0 \to R^{\times} \to U(R) \oplus K^{\times} \to (U(S) \cdot L^{\times})^{\Gamma} \to H^{1}(\Gamma, S^{\times}) \to H^{1}(\Gamma, U(S)) \oplus H^{1}(\Gamma, L^{\times}).$$

Since by Hilbert 90,  $H^1(\Gamma, L^{\times}) = 0$ , we have

$$\operatorname{coker} \left[ U(R) \oplus K^{\times} \to (U(S) \cdot L^{\times})^{\Gamma} \right] = \operatorname{coker} \left[ U(R) \cdot K^{\times} \to (U(S) \cdot L^{\times})^{\Gamma} \right]$$

$$\cong \ker \left[ H^{1}(\Gamma, S^{\times}) \to H^{1}(\Gamma, U(S)) \right],$$

and the result follows.  $\square$ 

Now let

$$L = \mathbf{Q}(\sqrt{-123}, \sqrt{-31}) \qquad S = \text{int } (L)$$

$$K = \mathbf{Q}(\sqrt{-123}) \qquad \Gamma = \text{Gal } (L/K), \quad R = \text{int } (K)$$

$$K' = \mathbf{Q}(\sqrt{3813}) \qquad \Gamma' = \text{Gal } (L/K') \quad R' = \text{int } (K').$$

The action of the non-trivial elements of  $\Gamma$ ,  $\Gamma'$  will be denoted respectively by  $\tilde{}$ ,  $\bar{}$ . Our immediate purpose is to show that  $C(R) \to C(S)$  is injective.

LEMMA 3.2. The fundamental unit of R' is  $v = 247 + 4\sqrt{3813}$ .

*Proof.* Certainly  $247^2 - 16.3813 = 1$ , so v is a unit of R'. If v is not the fundamental unit, then there exist positive integers a, b, c, d such that  $(a + b\sqrt{N})$   $(c + d\sqrt{N}) = 4v$ , where N = 3813. Thus

$$ac + bdN + (ad + bc)\sqrt{N} = 4(247 + 4\sqrt{N}).$$

But  $ac + bdN \ge N > 4.247$ , so this is impossible.  $\square$ 

By the Dirichlet Unit Theorem, rank  $(S^{\times}) = 1$  and  $S^{\times} = \langle \pm 1 \rangle \times \langle u \rangle$  for some u (a fundamental unit).

LEMMA 3.3. S has  $u = \sqrt{-123} + 2\sqrt{-31}$  as a fundamental unit.

**Proof.** Note that  $u\bar{u} = v$ , so  $u \in S^{\times}$ . If  $u = \pm w^n$  for some  $w \in S$  ( $\pm 1$  are the only units of finite order) then  $w\bar{w} \in R'^{\times}$  and so  $w\bar{w} = \pm v^m$  for some  $m \in \mathbb{Z}$ . But then  $(v^m)^n = \pm v$  whence  $m = n = \pm 1$ . Hence the result.  $\square$ 

LEMMA 3.4.  $H^1(\Gamma, S^{\times}) \to H^1(\Gamma, S^{\times}_{31})$  is injective, and hence so is  $H^1(\Gamma, S^{\times}) \to H^1(\Gamma, U(S))$ .

*Proof.* For an abelian  $\Gamma$ -group A we use the representation

$$H^1(\Gamma, A) \cong \frac{\{a \in A : a\tilde{a} = 1\}}{\{a/\tilde{a} : a \in A\}}.$$

This representation is natural with respect to extension of A. Now

$$u\tilde{u} = (\sqrt{-123} + 2\sqrt{-31})(\sqrt{-123} - 2\sqrt{-31}) = -123 + 124 = 1,$$

and  $u/\tilde{u} = u^2/u\tilde{u} = u^2$  so

$$H^1(\Gamma, S^{\times}) = S^{\times}/\langle u^2 \rangle = \{(1), (-1), (u), (-u)\}.$$

We must show that none of -1, u, -u is of the form  $s/\tilde{s}$  for some  $s \in S_{31}^{\times}$ .

If for some  $s \in S_{31}^{\times}$ ,  $s/\tilde{s} = -1$ , then  $s = -\tilde{s}$  and so  $s = r\sqrt{-31}$  for some  $r \in R_{31}$ . Hence s is not a unit.

If for some  $s \in S_{31}^{\times}$ ,  $s/\tilde{s} = u$ , then, as  $S_{31} = R_{31}[u]$ , s = a + bu with  $a, b \in R_{31}$ , and so  $a + bu = (a + b\tilde{u})u = au + b$ . Hence a = b and s = a(1 + u). As s is a unit,  $N_{L/\mathbb{Q}}(s)$  is also a unit. But  $N_{L/\mathbb{Q}}(s) = (a\bar{a})^2(1 + u)(1 + \tilde{u})(1 + \bar{u})(1 + \tilde{u}) = (a\bar{a})^2 \cdot 16.31$  and this is not a unit in  $\mathbb{Z}_{31}$ .

A similar argument disposes of the possibility  $s/\tilde{s} = -u$  and the result is proved.  $\square$ 

Remark 3.5. We can now see that the primes of S above 31 are principal. By our calculation above,  $N_{L/\mathbb{Q}}((1+u)/2) = 31$  and so, since (1+u)/2 is integral  $(N_{L/K}((1+u)/2) = (1+\sqrt{-123})/2 \in R)$ ,  $((1+u)/2)_S$  is a prime of S above 31. But  $L/\mathbb{Q}$  is galois so all the primes of S above 31 are conjugate and hence principal. In fact,  $(31)_S = ((1+u)/2)_S^2((1-u)/2)_S^2$ .

PROPOSITION 3.6.  $(3, \sqrt{-123})_{S[1/31]}$  is not principal.

*Proof.*  $(3, \sqrt{-123})_R$  is clearly not principal, for the equation  $a^2 + 123b^2 = 12$  has no solutions over **Z**. Since  $C(R) \to C(S)$  is injective by Lemmas 3.1 and 3.4,  $(3, \sqrt{-123})_S$  is not principal.

In passing from C(S) to C(S[1/31]) we kill off the ideal classes represented by ideals dividing  $(31)_S$ ; since by Remark 3.5 these ideals are all principal, the map  $C(S) \rightarrow C(S[1/31])$  is injective. The result follows.  $\square$ 

### 4. The case n=2

In this section we construct two simple (4q+1)-knots k and l such that k+k=l+l but  $k_2 \neq l_2$ .

LEMMA 4.1. Let  $\Delta(t) = 31t^2 - 61t + 31$ . Then  $\mathbb{Q}(\sqrt{-123}, \sqrt{-31})$  is a splitting field for  $\Delta(t^2)$ .

*Proof.* Let τ be a root of  $\Delta(t^2)$ ; then we can take  $\tau^2 = (61 + \sqrt{-123})/62$ , so that  $\Delta(t)$  splits in  $\mathbf{Q}(\sqrt{-123})$ . Now  $31\tau^2 = (61 + \sqrt{-123})/2 = -[(1 - \sqrt{-123})/2]^2$ , and so  $\tau = (1 - \sqrt{-123})/2\sqrt{-31}$ . Hence  $\tau \in \mathbf{Q}(\sqrt{-123}, \sqrt{-31})$ . But the conjugates of τ are  $\tau$ ,  $-\tau$ ,  $\bar{\tau} = 1/\tau$  and  $-1/\tau$ , so  $\Delta(t^2)$  splits in  $\mathbf{Q}(\sqrt{-123}, \sqrt{-31})$ . □

Let J denote the ideal  $(3, \sqrt{-123})$  over the ring  $\mathbf{Z}[\tau^2, \tau^{-2}] = R[1/31]$  in the notation of Section 3. Note that  $J = \overline{J}$  and  $J\overline{J} = (3)$ , where here denotes complex conjugation. Hence we can define a non-singular hermitian form  $b: J \times J \to R[1/31]$  by  $b(\alpha, \beta) = \alpha \overline{\beta}/3$ . Let  $(J \oplus J, B)$  denote the orthogonal direct sum  $(J, b) \perp (J, b)$ , and set

$$e = ((6 + \sqrt{-123})/31, (51 + \sqrt{-123})/31)$$
  
$$f = ((51 - \sqrt{-123})/31, (-6 + \sqrt{-123})/31).$$

It is easily checked that B(e, e) = B(f, f) = 1 and that B(e, f) = 0. Hence  $(J, b) \perp (J, b) \cong \langle 1 \rangle \perp \langle 1 \rangle$ .

Let k be the simple (4q+1)-knot  $(q \ge 1)$  represented by (J, b) and l the corresponding knot represented by  $\langle 1 \rangle$ . Then k+k=l+l, but since J is a non-principal ideal by Proposition 3.6,  $k \ne l$ . Let M(t) be a square Alexander matrix for k; then by Proposition 1.5,  $M(t^2)$  is an Alexander matrix for  $k_2$ . The Fox-Smythe row ideal class of  $k_2$  is obtained from the matrix  $M(\tau^2)$  over the ring  $\mathbb{Z}[\tau, \tau^{-1}] = S[1/31]$ , and by [H: Chap. III, Theorem 12] this is the ideal  $J_{S[1/31]}$ . By Proposition 3.6, this ideal is non-principal. Since the corresponding invariant for  $l_2$  is trivial, we have  $k_2 \ne l_2$ .

#### **REFERENCES**

[Bass] H. Bass. Algebraic K-theory. W. A. Benjamin, Inc. (1968).

[Ba] E. BAYER. Unimodular hermitian and skewhermitian forms. Jour. Algebra (to appear).

- [B] R. C. BLANCHFIELD. Intersection theory of manifolds with operators with applications to knot theory. Annals of Math. 65 (1957), 340–356.
- [F-S] R. H. FOX and N. SMYTHE. An ideal class invariant of knots. Proc. Amer. Math. Soc. 15 (1964), 707-709.
- [G] C. McA. GORDON. On the higher-dimensional Smith conjecture. Proc. London Math. Soc. (3) 29 (1974), 98-110.
- [H] J. A. HILLMAN. Alexander ideals. Springer Lecture Notes 895.
- [K] C. KEARTON. Classification of simple knots by Blanchfield duality. Bull. Amer. Math. Soc. 79 (1973), 952-955.
- [L] J. LEVINE. An algebraic classification of some knots of codimension two. Comment. Math. Helv. 45 (1970), 185-198.
- [Sch] G. Schrutka v. Rechtenstamm. Tabelle der (relativ-) Klassenzahlen von Kreiskörpern. Abh. Deutsch Akad. Wiss. Berlin 1964, Math. Nat. Kl. Nr 2.
- [T1] H. F. TROTTER. On S-equivalence classes of Seifert matrices. Invent. math 20 (1973), 173-207.
- [T2] —. Knot modules and Seifert matrices. Knot Theory, ed. J.-C. Hausmann, Lecture Notes in Mathematics 685 (1978), Springer-Verlag.

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Note added in proof: The second author has recently shown that for any integer n there is an integer m, prime to n and not a prime power, such that, if  $\zeta$  is an mth root of 1, there is an ideal class in  $C_{Q[\zeta]}$  of order n with norm 1 in  $C_{Q[\zeta+\zeta^{-1}]}$ . Thus the results of section 2 are valid for any odd n.