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Cyclic group actions on odd-dimensional spheres

C. KEARTON* AND S. M. J. WILSON

Abstract. We show that for any simple $(2q - 1)$ -knot k , $q > 1$, and any positive integer n , the knot $\#_1^n k$ is the fixed-point set of a \mathbf{Z}_n -action on S^{2q+1} . Further, we show that for many values of n there are examples of $(2q - 1)$ -knots, $q \geq 2$, which are the fixed-point sets of inequivalent \mathbf{Z}_n -actions.

0. Introduction

An n -knot is a locally-flat PL pair (S^{n+2}, S^n) , where S^n denotes the n -sphere. A $(2q - 1)$ -knot is *simple* if the complement of S^{2q-1} has the homotopy type of a circle up to but not including dimension q . For $q > 1$ such knots have been classified in [L] in terms of the S -equivalence classes of their Seifert matrices, and in [K, T1, T2] in terms of their Blanchfield pairings. Using these classification results, for any simple $(2q - 1)$ -knot k , with $q > 1$, and for any positive integer n , we construct a simple $(2q - 1)$ -knot k_n such that the n -fold cyclic cover of S^{2q+1} branched over k_n is again S^{2q+1} , and such that k_n lifts to $\#_1^n k$, the sum of n copies of k . An immediate corollary is that for any such k and n , there is a \mathbf{Z}_n -action on S^{2q+1} with fixed point set $\#_1^n k$.

The construction in this paper is purely algebraic, and may be contrasted with the geometric construction in [G], where for any m -knot k ($m \geq 2$) Gordon constructs an m -knot which is the fixed-point set of a \mathbf{Z}_n -action and whose fundamental group is isomorphic to that of $\#_1^n k$.

As an application of our construction we are able for many values of n to find examples of $(2q - 1)$ -knots which are the fixed-point sets of inequivalent \mathbf{Z}_n -actions. The technique is to pick simple $(2q - 1)$ -knots k and l such that $\#_1^n k = \#_1^n l$, and such that $k_n \neq l_n$.

1. The main construction

Let k be a simple $(2q - 1)$ -knot, $q > 1$, and $n > 1$ an integer. Let A be a non-singular Seifert matrix of k , and set $\varepsilon = (-1)^q$. Following Trotter [T1], we set $S = (A + \varepsilon A')^{-1}$, $T = -\varepsilon A' A^{-1}$.

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PROPOSITION 1.1. *The pair (S, T) has the following properties.*

- (i) S is integral, unimodular, ε -symmetric.
- (ii) $(I - T)^{-1}$ exists and is integral.
- (iii) $T'ST = S$.
- (iv) $A = (I - T)^{-1}S^{-1}$.

Moreover, any pair of rational matrices (S, T) satisfying (i)–(iii) yields a Seifert matrix A by the formula (iv).

Proof. It is well known (see [L], [T1]) that $A + \varepsilon A'$ is unimodular, and so S is integral and unimodular. Clearly S is ε -symmetric.

Now $I - T = I + \varepsilon A' A^{-1} = (A + \varepsilon A')A^{-1} = S^{-1}A^{-1}$, from which (ii) and (iv) follow at once. Property (iii) is easily checked.

Now suppose that we are given a pair of rational matrices (S, T) satisfying (i)–(iii); then we can define the matrix $A = (I - T)^{-1}S^{-1}$, which by (i) and (ii) is a non-singular matrix over the integers. We have

$$\begin{aligned} A + \varepsilon A' &= (I - T)^{-1}S^{-1} + \varepsilon(S')^{-1}(I - T')^{-1} \\ &= (I - T)^{-1}S^{-1} + S^{-1}(I - ST^{-1}S^{-1})^{-1} \quad \text{by (i), (iii)} \\ &= (I - T)^{-1}S^{-1} + S^{-1}S(I - T^{-1})^{-1}S^{-1} \\ &= [(I - T)^{-1} - (I - T)^{-1}T]S^{-1} = (I - T)^{-1}(I - T)S^{-1} = S^{-1} \end{aligned}$$

which is unimodular. It follows that A is a Seifert matrix. \square

Now we define matrices U, V by

$$U = \begin{pmatrix} 0 & \cdots & 0 & T \\ I & & & 0 \\ & & & \vdots \\ 0 & & I & 0 \end{pmatrix}, \quad V = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix},$$

there being $n \times n$ blocks in each case.

THEOREM 1.2. *The pair (V, U) determines a simple $(2q - 1)$ -knot k_n . The n -fold branched cyclic cover of k_n is the knot $\#_1^n k = k + \cdots + k$ (n times).*

Proof. We have to check that the pair (V, U) satisfies conditions (i)–(iii) of

Proposition 1.1. Clearly V satisfies (i), and it is easy to check that

$$(I-U)^{-1} = \begin{pmatrix} (I-T)^{-1} & T(I-T)^{-1} & \dots & T(I-T)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (I-T)^{-1} & \dots & \dots & (I-T)^{-1} \end{pmatrix}.$$

But $T(I-T)^{-1} = -\varepsilon A' A^{-1} (I + \varepsilon A' A^{-1})^{-1} = -\varepsilon A' (A + \varepsilon A')^{-1}$, which is an integer matrix. Hence (ii) is satisfied. To check (iii) is a simple matrix multiplication.

Hence (V, U) determine a unique simple $(2q-1)$ -knot k_n . A routine computation shows that

$$U^n = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix},$$

and hence the pair (V, U^n) satisfies (i)–(iii), and in fact represents the knot $\#_1^n k$.

Let K_n denote the complement of k_n , and \tilde{K}_n the infinite cyclic cover of K_n . If u is a generator of the group of covering translations, then K_n is obtained from \tilde{K}_n by quotienting out by the action of u . Similarly the n -fold cyclic cover of K_n is obtained from \tilde{K}_n by quotienting out by the action of u^n .

Algebraically this can be described as follows, using Trotter’s description of $H_q(\tilde{K}_n)$ in [T1]. Let \mathbf{B} be a basis of \mathbf{Q}^m corresponding to (V, U) where T is an $r \times r$ matrix; then $H_q(\tilde{K}_n)$ is the $\mathbf{Z}[u, u^{-1}]$ -module generated by \mathbf{B} , the action of u being given by U . The fact that $(1-u): H_q(\tilde{K}_n) \rightarrow H_q(\tilde{K}_n)$ is an isomorphism means that when we quotient out by the action of u we get a homology circle. But the form of U^n means that $(1-u^n): H_q(\tilde{K}_n) \rightarrow H_q(\tilde{K}_n)$ is also an isomorphism, and hence the n -fold cyclic cover of K_n is a homology circle. Therefore the n -fold branched cyclic cover of k_n is a homotopy sphere, and hence a sphere. \square

COROLLARY 1.3. *If k is a simple $(2q-1)$ -knot, $q > 1$, then $\#_1^n k$ is the fixed point set of a \mathbf{Z}_n -action on S^{2q+1} .*

PROPOSITION 1.4. *Let B be the Seifert matrix of k_n corresponding to (V, U) . Then*

$$B = \begin{pmatrix} A & -\varepsilon A' & \dots & \varepsilon A' \\ \vdots & \vdots & \ddots & \vdots \\ A & \dots & \dots & A \end{pmatrix}.$$

Proof:

$$B = (I - U)^{-1}V^{-1} = \begin{pmatrix} (I - T)^{-1} & T(I - T)^{-1} & \cdots & T(I - T)^{-1} \\ \vdots & \ddots & \ddots & \vdots \\ (I - T)^{-1} & \cdots & \cdots & (I - T)^{-1} \end{pmatrix} \begin{pmatrix} S^{-1} & & & \\ & 0 & & \\ & & \ddots & \\ 0 & & & S^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (I - T)^{-1}S^{-1} & T(I - T)^{-1}S^{-1} & \cdots & T(I - T)^{-1}S^{-1} \\ \vdots & \ddots & \ddots & \vdots \\ (I - T)^{-1}S^{-1} & \cdots & \cdots & (I - T)^{-1}S^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A & TA & \cdots & TA \\ \vdots & \ddots & \ddots & \vdots \\ A & \cdots & \cdots & A \end{pmatrix}$$

$$= \begin{pmatrix} A & -\varepsilon A' & \cdots & \varepsilon A' \\ \vdots & \ddots & \ddots & \vdots \\ A & \cdots & \cdots & A \end{pmatrix} \square$$

Next we prove a result which relates an Alexander matrix of k to one for k_n . Recall that an Alexander matrix $M(t)$ of k is a matrix over $\mathbf{Z}[t, t^{-1}]$ which presents $H_q(\tilde{K})$ as a $\mathbf{Z}[t, t^{-1}]$ -module; that is, there is an exact sequence of $\mathbf{Z}[t, t^{-1}]$ -modules

$$F \xrightarrow{M(t)} G \longrightarrow H_q(\tilde{K})$$

where F and G are free $\mathbf{Z}[t, t^{-1}]$ -modules.

PROPOSITION 1.5. *Let $M(t)$ be a square Alexander matrix for the knot k ; then $M(t^n)$ is an Alexander matrix for k_n .*

Proof. We can describe the $\mathbf{Z}[u, u^{-1}]$ -module structure of $H_q(\tilde{K}_n)$ in the following way. Let L_1, \dots, L_n be n copies of the $\mathbf{Z}[t, t^{-1}]$ -module $H_q(\tilde{K})$. Then for $1 \leq i < n$, $u : L_i \rightarrow L_{i+1}$ is a $\mathbf{Z}[t, t^{-1}]$ -isomorphism, and $u : L_n \rightarrow L_1$ is defined so that $u^n : L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_n \rightarrow L_1$ coincides with $t : L_1 \rightarrow L_1$. Thus a presentation matrix for $H_q(\tilde{K}_n)$ as a $\mathbf{Z}[u, u^{-1}]$ -module is

$$\left(\begin{array}{ccccccc} M(u^n) & 0 & \cdots & \cdots & 0 & & \\ 0 & \diagdown & & & & & \\ 0 & \cdots & 0 & M(u^n) & & & \\ uI & -I & 0 & \cdots & \cdots & 0 & \\ 0 & uI & -I & 0 & \cdots & \cdots & \\ 0 & \diagdown & & & & & \\ 0 & \cdots & 0 & & -I & & \\ -u^n I & 0 & \cdots & \cdots & 0 & uI & \end{array} \right).$$

Two elementary row operations give

$$\left(\begin{array}{ccccccc} M(u^n) & 0 & \cdots & \cdots & 0 & & \\ 0 & \diagdown & & & & & \\ 0 & \cdots & 0 & M(u^n) & & & \\ u^{n-1} M(u^n) & 0 & \cdots & \cdots & 0 & & \\ uI & -I & 0 & \cdots & \cdots & 0 & \\ 0 & uI & -I & 0 & \cdots & \cdots & \\ -u^{n-1} I & 0 & \cdots & \cdots & uI & & 0 \\ -u^n I & 0 & \cdots & \cdots & 0 & uI & \end{array} \right).$$

We eliminate the final row and column to give

$$\left(\begin{array}{ccc} M(u^n) & 0 & 0 \\ 0 & M(u^n) & 0 \\ 0 & 0 & M(u^n) \\ u^{n-1}M(u^n) & 0 & 0 \\ uI & -I & 0 \\ 0 & uI & -I \\ 0 & 0 & -I \\ -u^{n-1}I & 0 & uI \end{array} \right).$$

Now subtract u^{n-1} times the first row from the n^{th} to obtain a row of zeros, which may be eliminated. Continuing in this way we eventually arrive at the matrix $M(u^n)$. \square

THEOREM 1.6. *The knot k_n depends only on k and n , and not upon the choice of Seifert matrix A .*

Proof. Let A be an $r \times r$ matrix, and let $\Lambda = \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$, a subring of the field $\mathbf{Q}(t)$, the field of rational functions in one variable over the rationals. According to Trotter's viewpoint [T1], k gives rise to an ε -symmetric bilinear form $[,]$ on \mathbf{Q}^r represented by the matrix S , and a Λ -module M contained in \mathbf{Q}^r where the action of t is represented by T . A choice of Seifert matrix corresponds to an admissible lattice contained in M (see [T1] for definitions). Although our construction is given in terms of the matrices S and T , it is clear that it could be phrased in terms of M and $[,]$, and hence that it does not depend upon the choice of A .

Alternatively, one can use the formula of Proposition 1.4 to show that if A is S -equivalent to A_1 , then B is S -equivalent to B_1 . \square

2. Knots having distinct Z_n -actions, n odd

In this section we shall show that for many odd integers n , there exist simple $(4q+1)$ -knots ($q \geq 1$), k and l , such that $\#_1^n k = \#_1^n l$ but $k_n \neq l_n$.

Let $\lambda_m(t)$ denote the m^{th} cyclotomic polynomial, where m is not a prime

power. Let ζ be a primitive m^{th} root of unity, $K = \mathbf{Q}(\zeta)$ and $F = \mathbf{Q}(\zeta + \zeta^{-1})$, the fixed field of K under complex conjugation. Let h_K denote the class number of K , h_F that of F , and $h_- = h_K/h_F$. According to the work of Bayer [Ba: Example 6.2], the number of distinct simple $(4q + 1)$ -knots ($q \geq 1$) with Alexander polynomial $\lambda_m(t)$ is $h_- 2^d$ if $m = 2p^\alpha$ and $h_- 2^{d-1}$ otherwise, where $2d = [K : \mathbf{Q}]$. The factor h_- represents the number of isomorphism classes of $\mathbf{Z}[t, t^{-1}]$ -modules supporting a Blanchfield pairing [Ba: Corollary 1.3], and the factor $2^d(2^{d-1})$ represents the number of non-isometric pairings which a given module supports. Note that Bayer's work is couched in terms of pairings on $\mathbf{Z}[\zeta]$ -modules which are hermitian with respect to complex conjugation ($t \rightarrow t^{-1}$ becomes $\zeta \rightarrow \zeta^{-1} = \bar{\zeta}$), and we shall adopt this viewpoint.

Let U be the group of units of (the ring of integers of) K , U_0 the group of units of F , and $N : K \rightarrow F$ the norm. If I is a principal ideal, then let $\langle u \rangle$ denote the hermitian form h on I given by $h(a, b) = ua\bar{b}$. As in [Ba: Prop. 2.1], the set of isometry classes of unimodular hermitian forms on a given ideal (not necessarily principal) is in one-one correspondence with $U_0/N(U)$.

Now suppose that h_- has a factor $n > 1$, where n is odd and $(m, n) = 1$. Let a be an ideal of $\mathbf{Q}(\zeta)$ admitting a non-singular hermitian form h , with a being of order n in $\ker N : C_K \rightarrow C_F$. Then $\perp_1^n(a, h)$ has determinant $\langle u \rangle$ for some $u \in U_0/N(U)$; see [Ba: Definition 1.9] for the definition of determinant. Since the order of $U_0/N(U)$ is 2^d or 2^{d-1} , and n is odd, there exists $v \in U_0/N(U)$ such that $v^n = u$. Then $\perp_1^n\langle v \rangle$ has determinant $\langle v^n \rangle = \langle u \rangle$.

Set $K = (a, h) \perp (a, -h)$, $L = \langle v \rangle \perp \langle -v \rangle$. Then $\perp_1^n K$, $\perp_1^n L$ are indefinite and have the same rank, signatures and determinant. Hence by [Ba: Corollary 4.10] they are isometric. But K is not isometric to L , for the determinant of K is (a^2, α) , and a^2 is non-zero in $\ker N : C_K \rightarrow C_F$ since n is odd.

In fact, if k, l are the simple $(4q + 1)$ -knots corresponding to K, L respectively, we can show that $k_n \neq l_n$. For let $M(t)$ be an Alexander matrix of k , so that by Proposition 1.5 $M(t^n)$ is an Alexander matrix of k_n . The work of Fox and Smythe [F-S] enables us to obtain a row ideal class from the matrix $M(\zeta)$, and the work of Hillman [H: Chap. III, Theorem 12] identifies this with the ideal a^2 in the determinant of K . But the Alexander polynomial of k_n is $\lambda_m(t^n)$, which has $\lambda_m(t)$ as one of its factors since $(m, n) = 1$. Let τ be a primitive m^{th} root of unity such that $\tau^n = \zeta$. Setting $t = \tau$ in the Alexander matrix $M(t^n)$, we obtain $M(\tau^n) = M(\zeta)$, and hence obtain a Fox-Smythe invariant a^2 again. In the case of l_n , these ideal invariants are all trivial, hence $k_n \neq l_n$.

Taking the n -fold branched cyclic covers of k_n, l_n we obtain respectively the knots $\#_1^n k, \#_1^n l$. Since $\perp_1^n K$ is isometric to $\perp_1^n L$, we have $\#_1^n k = \#_1^n l$.

Many examples may be obtained from the tables in [Sch].

For the case of $(4q - 1)$ -knots, $q \geq 1$, and $m \neq 2p^r, p^r$, where p is a prime, then as in [Ba: §5], $\zeta - \zeta^{-1}$ is a unit and so we can multiply all the pairings above by $\zeta - \zeta^{-1}$ to obtain skew-hermitian pairings. The argument then goes through as before. We are grateful to Dr. Bayer for pointing out this extension to the case of $(4q - 1)$ -knots.

3. Number theory

This section deals with some results from algebraic number theory, which will be used in the next section to deal with the case $n = 2$.

Let K be an algebraic number field, $R = \text{int}(K)$ its ring of integers, $\mathbf{Z}_{(p)}$ the p -adic integers, $R_p = R \otimes \mathbf{Z}_{(p)}$, $K_p = K \otimes \mathbf{Z}_{(p)}$, $U(R) = \prod_p R_p^\times$ and $J(K) = U(R) \cdot \prod_p K_p^\times$, where \amalg denotes the direct sum. K^\times is considered as a subgroup of $J(K)$ under the ‘‘diagonal’’ map. If $C(K)$ denotes the ideal class group of K , then we have $C(K) \cong J(K)/U(R) \cdot K^\times$ an isomorphism which is natural with respect to ring extensions.

Now suppose that L is an algebraic number field, Γ a group of automorphisms of L , $S = \text{int}(L)$, $K = L^\Gamma$ the subfield of L fixed under Γ , and $R = \text{int}(K) = S^\Gamma$.

LEMMA 3.1. $\ker [C(R) \rightarrow C(S)] \cong \ker [H^1(\Gamma, S^\times) \rightarrow H^1(\Gamma, U(S))]$, where the first map is induced by ring extension, the second by the ‘‘diagonal’’ map $S^\times \rightarrow U(S)$.

Proof. Consider the exact sequence

$$0 \rightarrow U(S) \cdot L^\times \rightarrow J(L) \rightarrow C(S) \rightarrow 0,$$

Since $J(L)^\Gamma = J(K)$, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & U(R) \cdot K^\times & \rightarrow & J(K) & \rightarrow & C(R) & \rightarrow 0 \\ & \downarrow & & \parallel & & \downarrow & \\ 0 \rightarrow & (U(S) \cdot L^\times)^\Gamma & \rightarrow & J(L)^\Gamma & \rightarrow & C(S)^\Gamma & \end{array}$$

Applying the Snake Lemma [Bass: p. 26] we find that

$$\begin{aligned} \ker [C(R) \rightarrow C(S)] &= \ker [C(R) \rightarrow C(S)^\Gamma] \\ &\cong \text{coker} [U(R) \cdot K^\times \rightarrow (U(S) \cdot L^\times)^\Gamma]. \end{aligned}$$

Now consider the exact sequence

$$0 \rightarrow S^\times \rightarrow U(S) \oplus L^\times \rightarrow U(S) \cdot L^\times \rightarrow 0$$

where the first map is $s \mapsto (s, s^{-1})$. From cohomology theory we obtain the exact sequence

$$0 \rightarrow R^\times \rightarrow U(R) \oplus K^\times \rightarrow (U(S) \cdot L^\times)^\Gamma \rightarrow H^1(\Gamma, S^\times) \rightarrow H^1(\Gamma, U(S)) \oplus H^1(\Gamma, L^\times).$$

Since by Hilbert 90, $H^1(\Gamma, L^\times) = 0$, we have

$$\begin{aligned} \text{coker} [U(R) \oplus K^\times \rightarrow (U(S) \cdot L^\times)^\Gamma] &= \text{coker} [U(R) \cdot K^\times \rightarrow (U(S) \cdot L^\times)^\Gamma] \\ &\cong \ker [H^1(\Gamma, S^\times) \rightarrow H^1(\Gamma, U(S))], \end{aligned}$$

and the result follows. \square

Now let

$$L = \mathbf{Q}(\sqrt{-123}, \sqrt{-31}) \quad S = \text{int}(L)$$

$$K = \mathbf{Q}(\sqrt{-123}) \quad \Gamma = \text{Gal}(L/K), \quad R = \text{int}(K)$$

$$K' = \mathbf{Q}(\sqrt{3813}) \quad \Gamma' = \text{Gal}(L/K') \quad R' = \text{int}(K').$$

The action of the non-trivial elements of Γ, Γ' will be denoted respectively by $\bar{}, \bar{}$. Our immediate purpose is to show that $C(R) \rightarrow C(S)$ is injective.

LEMMA 3.2. *The fundamental unit of R' is $v = 247 + 4\sqrt{3813}$.*

Proof. Certainly $247^2 - 16 \cdot 3813 = 1$, so v is a unit of R' . If v is not the fundamental unit, then there exist positive integers a, b, c, d such that $(a + b\sqrt{N})(c + d\sqrt{N}) = 4v$, where $N = 3813$. Thus

$$ac + bdN + (ad + bc)\sqrt{N} = 4(247 + 4\sqrt{N}).$$

But $ac + bdN \geq N > 4 \cdot 247$, so this is impossible. \square

By the Dirichlet Unit Theorem, $\text{rank}(S^\times) = 1$ and $S^\times = \langle \pm 1 \rangle \times \langle u \rangle$ for some u (a fundamental unit).

LEMMA 3.3. S has $u = \sqrt{-123} + 2\sqrt{-31}$ as a fundamental unit.

Proof. Note that $u\bar{u} = v$, so $u \in S^\times$. If $u = \pm w^n$ for some $w \in S$ (± 1 are the only units of finite order) then $w\bar{w} \in R'^\times$ and so $w\bar{w} = \pm v^m$ for some $m \in \mathbf{Z}$. But then $(v^m)^n = \pm v$ whence $m = n = \pm 1$. Hence the result. \square

LEMMA 3.4. $H^1(\Gamma, S^\times) \rightarrow H^1(\Gamma, S_{31}^\times)$ is injective, and hence so is $H^1(\Gamma, S^\times) \rightarrow H^1(\Gamma, U(S))$.

Proof. For an abelian Γ -group A we use the representation

$$H^1(\Gamma, A) \cong \frac{\{a \in A : a\bar{a} = 1\}}{\{a/\bar{a} : a \in A\}}.$$

This representation is natural with respect to extension of A . Now

$$u\bar{u} = (\sqrt{-123} + 2\sqrt{-31})(\sqrt{-123} - 2\sqrt{-31}) = -123 + 124 = 1,$$

and $u/\bar{u} = u^2/u\bar{u} = u^2$ so

$$H^1(\Gamma, S^\times) = S^\times / \langle u^2 \rangle = \{(1), (-1), (u), (-u)\}.$$

We must show that none of $-1, u, -u$ is of the form s/\bar{s} for some $s \in S_{31}^\times$.

If for some $s \in S_{31}^\times$, $s/\bar{s} = -1$, then $s = -\bar{s}$ and so $s = r\sqrt{-31}$ for some $r \in R_{31}$. Hence s is not a unit.

If for some $s \in S_{31}^\times$, $s/\bar{s} = u$, then, as $S_{31} = R_{31}[u]$, $s = a + bu$ with $a, b \in R_{31}$, and so $a + bu = (a + b\bar{u})u = au + b$. Hence $a = b$ and $s = a(1 + u)$. As s is a unit, $N_{L/\mathbf{Q}}(s)$ is also a unit. But $N_{L/\mathbf{Q}}(s) = (a\bar{a})^2(1 + u)(1 + \bar{u})(1 + \bar{u})(1 + \bar{\bar{u}}) = (a\bar{a})^2 \cdot 16.31$ and this is not a unit in \mathbf{Z}_{31} .

A similar argument disposes of the possibility $s/\bar{s} = -u$ and the result is proved. \square

Remark 3.5. We can now see that the primes of S above 31 are principal. By our calculation above, $N_{L/\mathbf{Q}}((1 + u)/2) = 31$ and so, since $(1 + u)/2$ is integral ($N_{L/\mathbf{K}}((1 + u)/2) = (1 + \sqrt{-123})/2 \in \mathbf{R}$), $((1 + u)/2)_S$ is a prime of S above 31. But L/\mathbf{Q} is galois so all the primes of S above 31 are conjugate and hence principal. In fact, $(31)_S = ((1 + u)/2)_S^2((1 - u)/2)_S^2$.

PROPOSITION 3.6. $(3, \sqrt{-123})_{S[1/31]}$ is not principal.

Proof. $(3, \sqrt{-123})_R$ is clearly not principal, for the equation $a^2 + 123b^2 = 12$ has no solutions over \mathbf{Z} . Since $C(R) \rightarrow C(S)$ is injective by Lemmas 3.1 and 3.4, $(3, \sqrt{-123})_S$ is not principal.

In passing from $C(S)$ to $C(S[1/31])$ we kill off the ideal classes represented by ideals dividing $(31)_S$; since by Remark 3.5 these ideals are all principal, the map $C(S) \rightarrow C(S[1/31])$ is injective. The result follows. \square

4. The case $n = 2$

In this section we construct two simple $(4q+1)$ -knots k and l such that $k+k=l+l$ but $k_2 \neq l_2$.

LEMMA 4.1. *Let $\Delta(t) = 31t^2 - 61t + 31$. Then $\mathbf{Q}(\sqrt{-123}, \sqrt{-31})$ is a splitting field for $\Delta(t^2)$.*

Proof. Let τ be a root of $\Delta(t^2)$; then we can take $\tau^2 = (61 + \sqrt{-123})/62$, so that $\Delta(t)$ splits in $\mathbf{Q}(\sqrt{-123})$. Now $31\tau^2 = (61 + \sqrt{-123})/2 = -[(1 - \sqrt{-123})/2]^2$, and so $\tau = (1 - \sqrt{-123})/2\sqrt{-31}$. Hence $\tau \in \mathbf{Q}(\sqrt{-123}, \sqrt{-31})$. But the conjugates of τ are $\tau, -\tau, \bar{\tau} = 1/\tau$ and $-1/\tau$, so $\Delta(t^2)$ splits in $\mathbf{Q}(\sqrt{-123}, \sqrt{-31})$. \square

Let J denote the ideal $(3, \sqrt{-123})$ over the ring $\mathbf{Z}[\tau^2, \tau^{-2}] = R[1/31]$ in the notation of Section 3. Note that $J = \bar{J}$ and $J\bar{J} = (3)$, where $\bar{}$ here denotes complex conjugation. Hence we can define a non-singular hermitian form $b: J \times J \rightarrow R[1/31]$ by $b(\alpha, \beta) = \alpha\bar{\beta}/3$. Let $(J \oplus J, B)$ denote the orthogonal direct sum $(J, b) \perp (J, b)$, and set

$$e = ((6 + \sqrt{-123})/31, (51 + \sqrt{-123})/31)$$

$$f = ((51 - \sqrt{-123})/31, (-6 + \sqrt{-123})/31).$$

It is easily checked that $B(e, e) = B(f, f) = 1$ and that $B(e, f) = 0$. Hence $(J, b) \perp (J, b) \cong \langle 1 \rangle \perp \langle 1 \rangle$.

Let k be the simple $(4q+1)$ -knot ($q \geq 1$) represented by (J, b) and l the corresponding knot represented by $\langle 1 \rangle$. Then $k+k=l+l$, but since J is a non-principal ideal by Proposition 3.6, $k \neq l$. Let $M(t)$ be a square Alexander matrix for k ; then by Proposition 1.5, $M(t^2)$ is an Alexander matrix for k_2 . The Fox-Smythe row ideal class of k_2 is obtained from the matrix $M(\tau^2)$ over the ring $\mathbf{Z}[\tau, \tau^{-1}] = S[1/31]$, and by [H: Chap. III, Theorem 12] this is the ideal $J_{S[1/31]}$. By Proposition 3.6, this ideal is non-principal. Since the corresponding invariant for l_2 is trivial, we have $k_2 \neq l_2$.

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Note added in proof: The second author has recently shown that for any integer n there is an integer m , prime to n and not a prime power, such that, if ζ is an m th root of 1, there is an ideal class in $C_{\mathbb{Q}[\zeta]}$ of order n with norm 1 in $C_{\mathbb{Q}[\zeta+\zeta^{-1}]}$. Thus the results of section 2 are valid for any odd n .