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Permutation modules and projective resolutions

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1. Introduction

We adopt the convention of [2] that a *group-pair* $(G, \boldsymbol{S})$ consists of a group G and an indexed family $\boldsymbol{S}=\{S_i\}_I$ of subgroups of G (possibly with repetitions). For the most part no explicit indexing is referred to; we then write $\boldsymbol{S}$ as a shorthand for $\{S_i\}_I$. We let $\mathbf{Z}G/\boldsymbol{S}$ denote the left $\mathbf{Z}G$-module $\oplus_I \mathbf{Z}G/S_i$. Any module of the form $\mathbf{Z}G/\boldsymbol{S}$ is called a *permutation module.*

We study projective resolutions $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and we deal with special splittings of the kernels of the boundary map in some dimension n. We focus attention on such splittings where a $\mathbf{Z}G$-permutation module appears as a direct summand. The motivation for this set-up stems from our investigation [6], where a group is termed of finite quasi-projective dimension in case for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and some integer $n>1$, the kernel of ∂_n splits into a $\mathbf{Z}G$-projective and a $\mathbf{Z}G$-permutation module. Some conclusions in [6] only depend on the existence of a direct permutation module summand. We justify the more general set-up taken up here by constructing examples of groups of infinite quasi-projective dimension to which our structure results still apply.

For finite groups, Gruenberg and Roggenkamp [4], [5] investigated non-projective decompositions of the augmentation ideal or of relation modules. Under a more special hypothesis on the decomposition, we avoid the restriction to finite groups and to low dimensions. Dealing with finite groups later on, similarity with results of [5] turns up.

We now introduce another piece of notation. Let $\mathcal{C}: \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$ be a chain complex. Then we denote the kernel of $\partial_{n-1}: C_{n-1} \to C_{n-2}$ by $K_n(\mathcal{C})$.

In *Section* 2 we assume that for some projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and some n, there is a splitting $K_n(\mathcal{P}) \cong M \oplus \mathbf{Z}G/\boldsymbol{S}$. We show that each group in the family $\boldsymbol{S}$ is finite. The Tate cohomology of any non-trivial group in $\boldsymbol{S}$ has a divisor of n as its period. Our key result is a variation of a theorem attributed to Serre in [7].

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THEOREM 5. *Suppose $(G, \mathbf{S})$ is a group-pair and q, r are positive integers such that for every $\mathbf{Z}G$-module M the group $H^q(G, M)$ has a direct summand isomorphic to $\prod_{i\in I} H^r(S_i, M)$. Suppose also that $i, j \in I$ and $g \in G$ are such that $S_i \cap gS_jg^{-1}$ is not torsion-free. Then $i = j$ and $g \in S_i$.*

This result leads to a chain of corollaries some of which state facts proved earlier for groups of finite quasi-projective dimension in [6]. In particular, the hypotheses of Theorem 5 are satisfied for a pair $(G, \mathbf{S})$ if for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \to \mathbf{Z}$ and some $n > 0$, the module $\mathbf{Z}G/\mathbf{S}$ is a direct summand of the kernel $K_n(\mathcal{P})$. The following statement already shows that our investigation extends beyond the class of groups of finite quasi-projective dimension.

We term a pair of groups (G, S) a *Frobenius pair*, if G is a Frobenius group and S a Frobenius complement of G.

COROLLARY 5.4. *Suppose G is a finite group and $\mathbf{Z}G/S$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \to \mathbf{Z}$ and some $n > 0$. Then either $S = 1$, $S = G$ or (G, S) is a Frobenius-pair.*

The objective of *Section* 3 is a discussion of Frobenius groups from the point of view of homological algebra.

THEOREM 7. *Let (G, S) be a Frobenius pair. Then $\mathbf{Z}G/S$ is a $\mathbf{Z}G$-direct summand of $\mathbf{Z}\oplus\mathbf{Z}G$.*

The relationship between splitting of $\mathbf{Z}\oplus\mathbf{Z}G$ and of the augmentation ideal $IG = \ker(\mathbf{Z}G \to \mathbf{Z})$ is investigated in [4], [5].

THEOREM 9. *Let (G, S) be a Frobenius pair and let n denote the period of the Tate cohomology of S. Then there exists a finitely generated $\mathbf{Z}G$-free resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ such that $\mathbf{Z}G/S$ is a $\mathbf{Z}G$-direct summand of $K_n(\mathcal{P})$.*

In a different direction, we use relative (co)-homology of a Frobenius pair (G, S) to compute the (co)-homology of G. We obtain results similar to [1, 55.1] in a rather elementary way from the long exact (co)-homology sequences of the pair (G, S). We refer to the Appendix for the discussion of some special properties of the restriction-corestriction maps in relative cohomology.

COROLLARY 7.2. *Let (G, S) be a Frobenius pair and denote by N the kernel of the projection $G \twoheadrightarrow S$. Then there exist natural isomorphisms of functors on*

$\mathbf{Z}G$-modules for any $r>1$ as follows:

(i) $H^r(G;-)\cong H^r(S;-)\oplus H^r(G,S;-)$.

(ii) $H^r(G,S;-)\cong H^r(N;-)^S$.

(iii) *For all $r\in\mathbf{Z}$, there are natural isomorphisms*

$$\hat{H}^r(S;(-)^N)\cong\hat{H}^r(S;\mathrm{res}^G_S(-))$$

of functors on $\mathbf{Z}G$-modules.

For any finite group H let π_H denote the set of primes dividing $|H|$. Then the decomposition (i) is a functorial splitting of $H^r(G,-)$ into π_S- and π_N-parts ($r>1$).

In *Section* 4, we present some explicit computations related to examples. We first deal with Frobenius pairs of the type $(G,\mathbf{Z}/k\mathbf{Z})$. In this context, we observe decompositions of the relation modules occurring in the Lyndon resolution derived from a finite presentation of G. This is an opportunity to make explicit some connection with results of Gruenberg and Roggenkamp [5]. The easiest non-trivial example of this case is the symmetric group $S_3=\langle x,y\mid x^2=y^3=(xy)^2=1\rangle$.

In analogy with [6, Theorem 5] we use graph products to construct infinite groups G for which permutation modules occur as a direct summand of some $K_n(\mathcal{P})$. Of special interest is the group $S_3\underset{C}{*}S_3$, where $C\cong\mathbf{Z}/2\mathbf{Z}$, since this group has infinite quasi-projective dimension and it cannot be constructed with the help of graph products along the lines of [6, Theorem 5].

We finally remark that our working hypothesis is based on $\mathbf{Z}G$-projective resolutions. A natural source for such resolutions is free G-actions on acyclic spaces. It would be desirable to know the topological circumstances under which actions produce the phenomena discussed here since they lead from geometric considerations to structure results in group theory.

2. Projective resolutions

2.1. *Preliminaries and general facts*

Suppose $(G,\boldsymbol{S})$ is a group pair, where G is a subgroup of a group H. Then there is a $\mathbf{Z}H$-isomorphism $\mathbf{Z}H\otimes_G\mathbf{Z}G/\boldsymbol{S}\cong\mathbf{Z}H/\boldsymbol{S}$.

Suppose $(G,\boldsymbol{S})$ is a group pair, and U is a subgroup of G. If we restrict the $\mathbf{Z}G$-action on $\mathbf{Z}G/\boldsymbol{S}$ to a $\mathbf{Z}U$-action, we obtain a permutation module $\mathbf{Z}U/\boldsymbol{T}$, where the family $\boldsymbol{T}$ can be constructed as follows. Let I be the index set of the family $\boldsymbol{S}=\{S_i\}_I$. For each $i \in I$, choose a set $\{t_\beta; \beta \in J_i\}$ of representatives of the double cosets UgS_i, and for each β in J_i define $T_{i\beta}=U \cap t_\beta S_i t_\beta^{-1}$. Then $\boldsymbol{T}$ is the family $\{T_{i\beta};\ i \in I, \beta \in J_i\}$. This construction also appears in [2, p. 305]. We call $(U, \boldsymbol{T})$ the pair *induced* from $(G, \boldsymbol{S})$ by U.

We now consider a $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow A$ of some $\mathbf{Z}G$-module A, and suppose that $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a submodule of $K_n(\mathcal{P})$ for some positive integer n. Then, by restricting the $\mathbf{Z}G$-action, we obtain a $\mathbf{Z}U$-projective resolution $\mathcal{P} \twoheadrightarrow A$, and a $\mathbf{Z}U$-submodule of $K_n(\mathcal{P})$ isomorphic to $\mathbf{Z}U/\boldsymbol{T}$, where $(U, \boldsymbol{T})$ is the induced pair.

In particular, if we take $U=S_i$ for some $i \in I$, then one subgroup in the family $\boldsymbol{T}$ is just $S_i \cap S_i = S_i$, so that the $\mathbf{Z}S_i$-projective P_{n-1} contains a $\mathbf{Z}S_i$-submodule isomorphic to $\mathbf{Z}S_i/S_i \cong \mathbf{Z}$. This is possible only if S_i is a finite group. We have thus proved the following.

PROPOSITION 1. *Suppose $\mathcal{P} \twoheadrightarrow A$ is a $\mathbf{Z}G$-projective resolution, and $\mathbf{Z}G/\boldsymbol{S}$ is a submodule of some $K_n(\mathcal{P})$. Then $\boldsymbol{S}$ is a family of finite subgroups.*

Note that any projective resolution $\mathcal{P}$ can be modified so that $K_n(\mathcal{P})$ contains a free direct summand. Hence only the non-trivial subgroups in $\boldsymbol{S}$ are of interest.

Next we observe that, if $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some projective resolution $\mathcal{P} \twoheadrightarrow A$, then, modulo a slight adjustment, the same is true for any other projective resolution of A. In particular, $\mathcal{P}$ may always be assumed to be free.

PROPOSITION 2. *Suppose A is a $\mathbf{Z}G$-module, $\mathcal{P} \twoheadrightarrow A$ and $\mathcal{Q} \twoheadrightarrow A$ are $\mathbf{Z}G$-projective resolutions, and $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_n(\mathcal{P})$. Then one can form a $\mathbf{Z}G$-projective resolution $\mathcal{Q}' \twoheadrightarrow A$ by adding a free $\mathbf{Z}G$-module F to $\mathcal{Q}$ in dimensions n and $n-1$, such that $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_n(\mathcal{Q}')$. Furthermore, if the index set I of the family $\boldsymbol{S}$ is finite, then F may be chosen of finite rank.*

Proof. By Schanuel's lemma, there are $\mathbf{Z}G$-projectives P and Q such that $K_n(\mathcal{P}) \oplus P \cong K_n(\mathcal{Q}) \oplus Q$. The first part of the proposition follows by choosing F large enough to contain Q as a direct summand, and noting that $K_n(\mathcal{Q}') \cong K_n(\mathcal{Q}) \oplus F$.

Now suppose I is finite. then $\mathbf{Z}G/\boldsymbol{S}$ is finitely generated. Hence, for any choice of basis for F, only finitely many basis elements are involved in the images of elements of $\mathbf{Z}G/\boldsymbol{S}$ in $K_n(\mathcal{Q}) \oplus F$. We may therefore replace F in the construction of $\mathcal{Q}'$ by a suitable free direct summand of finite rank.

2.2. *Resolutions of* $\mathbf{Z}$

From now on we restrict our attention to $\mathbf{Z}G$-projective resolutions $\mathscr{P}\twoheadrightarrow\mathbf{Z}$. Our next observation is that an integer n for which $\mathbf{Z}G/\boldsymbol{S}$ is a direct summand of $K_n(\mathscr{P})$ for some $\mathscr{P}$ is by no means unique. In fact, if there is one such integer, there are infinitely many.

PROPOSITION 3. *Suppose $\mathscr{P}\twoheadrightarrow\mathbf{Z}$ is a $\mathbf{Z}G$-projective resolution such that $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_n(\mathscr{P})$. Then there exists, for each positive integer r, a $\mathbf{Z}G$-projective resolution $\mathscr{P}^{(r)}\twoheadrightarrow\mathbf{Z}$, such that $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_{rn}(\mathscr{P}^{(r)})$.*

Proof. We define $\mathscr{P}^{(1)}=\mathscr{P}$, and $\mathscr{P}^{(r)}$ $(r\geq 2)$ inductively as follows. Suppose $\mathscr{P}^{(r-1)}$ has been defined, and $K_{(r-1)n}(\mathscr{P}^{(r-1)})=\mathbf{Z}G/\boldsymbol{S}\oplus A$. Let $\mathscr{Q}\twoheadrightarrow A$ be any $\mathbf{Z}G$-projective resolution, and define $\mathscr{P}^{(r)}\twoheadrightarrow\mathbf{Z}$ to be

$$
\begin{tikzcd}
\left(\bigoplus_{i\in I}\mathbf{Z}G\otimes_{S_i}\mathscr{P}\right)\oplus\mathscr{Q} \arrow[rr] \arrow[dr, two heads] & & P^{(r-1)}_{(r-1)n-1} \arrow[r] & \cdots \arrow[r] & P^{(r-1)}_0 \arrow[r] & \mathbf{Z} \arrow[r] & 0 \\
& \left(\bigoplus_{i\in I}\mathbf{Z}G\otimes_{S_i}\mathbf{Z}\right)\oplus A. \arrow[ur, tail] & & & & &
\end{tikzcd}
$$

Then $K_{nr}(\mathscr{P}^{(r)})\cong(\oplus_{i\in I}\mathbf{Z}G\otimes_{Si}K_n(\mathscr{P}))\oplus K_n(\mathscr{Q})$.

For each $i\in I$, $K_n(\mathscr{P})$ contains $\mathbf{Z}G/S_i$, and so $\mathbf{Z}$, as a $\mathbf{Z}S_i$-direct summand. Hence $K_{rn}(\mathscr{P})$ contains $\oplus_I\mathbf{Z}G\otimes_{S_i}\mathbf{Z}\cong\mathbf{Z}G/\boldsymbol{S}$ as a $\mathbf{Z}G$-direct summand, as claimed.

The next proposition is a straightforward calculation of cohomology, and we omit the proof.

PROPOSITION 4. *Suppose $\mathscr{P}\twoheadrightarrow\mathbf{Z}$ is a $\mathbf{Z}G$-projective resolution and $K_n(\mathscr{P})=\mathbf{Z}G/\boldsymbol{S}\oplus A$. Then there are, for each $q>0$, natural isomorphisms*

$$H^{n+q}(G;-)\simeq\left(\prod_{i\in I}H^q(S_i;-)\right)\oplus\operatorname{Ext}^q_{\mathbf{Z}G}(A;-)$$

$$H_{n+q}(G;-)\cong\left(\bigoplus_{i\in I}H_q(S_i;-)\right)\oplus\operatorname{Tor}_q^{\mathbf{Z}G}(A;-).$$

COROLLARY 4.1. *Suppose $\mathscr{P}\twoheadrightarrow\mathbf{Z}$ is a $\mathbf{Z}G$-projective resolution such that $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_n(\mathscr{P})$. Then either n is even or $\boldsymbol{S}$ consists entirely of copies of $\{1\}$.*

Proof. Suppose n is odd and, for some $i \in I$, $S_i \neq \{1\}$.

By Proposition 1, S_i is finite. Choose a subgroup C of S_i of prime order. Then, regarded as a $\mathbf{Z}C$-module, $K_n(\mathcal{P})$ has a direct summand isomorphic to $\mathbf{Z}$, say $K_n(\mathcal{P}) \cong \mathbf{Z} \oplus A'$. Apply Proposition 4 to C with $q = 1$. Since n is odd, we have

$$0 = H_{n+1}(C; \mathbf{Z}) \cong H_1(C; \mathbf{Z}) \oplus \mathrm{Tor}_1^{\mathbf{Z}C}(A'; \mathbf{Z}) \neq 0.$$

This is a contradiction, so the proof is complete.

The importance of Proposition 4 is that it may be used together with our next result to deduce certain group-theoretic properties of the pair $(G, \boldsymbol{S})$. This result is a partial generalisation of a theorem of Serre [7].

THEOREM 5. *Suppose $(G, \boldsymbol{S})$ is a group-pair and q, r are positive integers such that, for every $\mathbf{Z}G$-module M, the group $H^q(G; M)$ has a direct summand isomorphic to $\prod_{i \in I} H^r(S_i; M)$. Suppose also that $i, J \in I$, $g \in G$ are such that $S_i \cap gS_jg^{-1}$ is not torsion-free. Then $i = j$ and $g \in S_i$.*

Proof. Choose a cyclic subgroup $C \neq 1$ of finite order in $S_i \cap gS_jg^{-1}$. Then we have, for any $\mathbf{Z}C$-module M, an isomorphism $H^q(C; M) \cong H^q(G; \mathrm{Hom}_{\mathbf{Z}C}(\mathbf{Z}G; M))$. By hypothesis, therefore, $H^q(C; M)$ has a direct summand isomorphic to

$$\begin{aligned}\prod_{i \in I} H^r(S_i; \mathrm{Hom}_{\mathbf{Z}C}(\mathbf{Z}G; M)) &\cong \mathrm{Ext}^r_{\mathbf{Z}G}(\mathbf{Z}G/\boldsymbol{S}; \mathrm{Hom}_{\mathbf{Z}C}(\mathbf{Z}G; M)) \\ &\cong \mathrm{Ext}^r_{\mathbf{Z}C}(\mathbf{Z}G/\boldsymbol{S}; M).\end{aligned}$$

Choose $M = \mathbf{Z}$ or $M = IC$ (the augmentation ideal of $\mathbf{Z}C$) according as r is even or odd. Then $H^q(C; M)$ is either cyclic of order $|C|$ or zero, according as $(q - r)$ is even or odd. In any case $H^q(C; M)$, and so also $\mathrm{Ext}^r_{\mathbf{Z}C}(\mathbf{Z}G/\boldsymbol{S}; M)$, is cyclic. But $\mathrm{Ext}^r_{\mathbf{Z}C}(\mathbf{Z}G/\boldsymbol{S}; M)$ contains a direct summand isomorphic to $\mathbf{Z}/|C|\mathbf{Z}$ for each point of the C-set $\bigcup_{i \in I} G/S_i$ fixed by C. Hence there is at most one such point, so the fixed points $1 \cdot S_i$ and $g \cdot S_j$ coincide. In other words, $i = j$ and $g \in S_i$.

COROLLARY 5.1. *Suppose $\mathbf{Z}G/\boldsymbol{S}$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and some positive integer n. Suppose also that $i, j \in I$, $g \in G$ are such that $S_i \cap gS_jg^{-1} \neq 1$. Then $i = j$ and $g \in S_i$.*

Proof. Follows from Propositions 1 and 4, and Theorem 5.

COROLLARY 5.2. *Suppose $\mathbf{Z}G/S$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$, and some positive integer n. Then either $S=1$ or S is its own normaliser in G.*

COROLLARY 5.3. *Suppose G is an infinite group with non-trivial centre, and $\mathbf{Z}G/S$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and some positive integer n. Then $S=1$.*

COROLLARY 5.4. *Suppose G is a finite group and $\mathbf{Z}G/S$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and some positive integer n. Then either $S=1$, $S=G$, or (G, S) is a Frobenius pair.*

COROLLARY 5.5. *Suppose $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and $\mathcal{Q} \twoheadrightarrow \mathbf{Z}$ are $\mathbf{Z}G$-projective resolutions and m and n are positive integers, such that $\mathbf{Z}G/S$ is a direct summand of $K_m(\mathcal{P})$, and $\mathbf{Z}G/T$ is a direct summand of $K_n(\mathcal{Q})$. Suppose also that $g \in G$ is such that $U = S \cap gTg^{-1}$ is non-trivial. Then either $U=S$ or (S, U) is a Frobenius pair.*

Proof. Follows from Corollary 5.4 by regarding $\mathcal{Q}$ as a $\mathbf{Z}S$-resolution.

Remarks. Serre's Theorem states that, under a hypothesis somewhat stronger than that of Theorem 5, any finite subgroup is contained in precisely one conjugate of precisely one of the subgroups S_i. Under this stronger hypothesis, the third possibility in Corollary 5.4, that (G, S) is a Frobenius pair, would be ruled out. Under our weaker hypothesis, however, we cannot rule the possibility out. Indeed, the permutation modules $\mathbf{Z}G/S$, where (G, S) is a Frobenius pair, all occur as direct summands of kernels in projective resolutions of $\mathbf{Z}$. We will prove this, and other facts about Frobenius groups, in the next section.

Of course, it is well known that any Frobenius complement has periodic Tate cohomology. Our next result states that, if $\mathbf{Z}G/S$ is isomorphic to a direct summand of some $K_n(\mathcal{P})$, then S has periodic Tate cohomology.

PROPOSITION 6. *Suppose S is a non-trivial subgroup of G, and $\mathbf{Z}G/S$ is isomorphic to a direct summand of $K_n(\mathcal{P})$ for some $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$ and some positive integer n. Then S has periodic Tate cohomology, with period dividing n.*

Proof. By Proposition 1, S is finite. Also, regarded as a $\mathbf{Z}S$-module, $K_n(\mathcal{P})$ has a direct summand isomorphic to $\mathbf{Z}$. Since $n \neq 1$ by Corollary 4.1, we may apply Proposition 2 to find a finitely-generated free $\mathbf{Z}S$-resolution $\mathscr{F} \twoheadrightarrow \mathbf{Z}$ with $F_0 \cong \mathbf{Z}S$ and $K_n(\mathscr{F}) = \mathbf{Z} \oplus A$ for some $\mathbf{Z}S$-module A.

Form a complete $\mathbf{Z}S$-resolution

$$\cdots \to F_1 \xrightarrow{\partial_0} \mathbf{Z}S \xrightarrow{\nu} \mathbf{Z}S \xrightarrow{\partial_0} F_1^* \to \cdots \qquad (*)$$

from $\mathscr{F}$ in the usual way by means of the dual $\mathscr{F}^*$ and the norm-map ν.

From the exact sequences

$$0 \to K_{n-1}(\mathscr{F})^* \to F_{n-1}^* \to K_n(\mathscr{F})^* \to \mathrm{Ext}^n_{\mathbf{Z}S}(\mathbf{Z}; \mathbf{Z}S) = 0$$

(see for example [3, p. 90, Theorem 6.1]), we deduce that $(*)$ is exact and $\mathrm{Coker}\, \partial_{n-1}^* = K_n(\mathscr{F})^* \cong \mathbf{Z} \oplus A^*$. We use this to obtain natural isomorphisms.

$$H^q(S; -) \oplus \mathrm{Ext}^q_{\mathbf{Z}S}(A^*; -) \cong \hat{H}^{q-n}(S; -) \qquad (q \geq 1),$$

where $\hat{H}$ denotes Tate cohomology.

For any $\mathbf{Z}S$-module M and any $q \geq 1$, we have, from the above and Proposition 4, that $H^q(S; M)$ and $H^{q+n}(S; M)$ are direct summands of one another. It follows that $H^q(S; -)$ and $H^{q+n}(S; -)$ agree on finitely generated modules (and hence on all modules, since $H^q(S; -)$ commutes with direct limits).

Thus the Tate cohomology of S is periodic, of period dividing n, as claimed.

3. Frobenius groups

THEOREM 7. *Let (G, S) be a Frobenius pair. Then $\mathbf{Z}G/S$ is a direct summand of $\mathbf{Z} \oplus \mathbf{Z}G$.*

Remark. This result should be compared with [5, Lemma 4.1], which expresses the augmentation ideal of a Frobenius group as a direct sum of factors corresponding to the Frobenius complement and kernel respectively. As is shown in [5], there is a close correspondence between direct sum decompositions of $\mathbf{Z} \oplus \mathbf{Z}G$ and of the augmentation ideal of G.

Proof. By the definition of Frobenius pair, G acts transitively on a set X such that no element of $G \backslash 1$ fixes more than one element of X, and S is the stabiliser of some element x_0 of X. By Frobenius' Theorem, G is a semi-direct product $N \lhd S$, where N acts freely and transitively on the G-orbit X of x_0. Hence X is the N-orbit of x_0, and there is an identification between elements of X and of N such that the action of S on X corresponds to the S-action by conjugation on N, and such that $x_0 \in X$ is identified with $1 \in N$. It follows that S acts freely by conjugation on $N \backslash 1$. Let $\Omega_1, \ldots, \Omega_t$ denote the orbits of the S-action on $N \backslash 1$, and choose orbit representatives $g_r \in \Omega_r$ for $1 \leq r \leq t$.

We define $\mathbf{Z}G$-morphisms $\phi : \mathbf{Z}G/S \to \mathbf{Z} \oplus \mathbf{Z}G$ and $\psi : \mathbf{Z} \oplus \mathbf{Z}G \to \mathbf{Z}G/S$ by

$$\phi : 1 \cdot S \mapsto \left(1, \sum_{r=1}^{t} \sum_{s \in S} s g_r\right)$$

$$\psi|_{\mathbf{Z}} : 1 \mapsto \sum_{g \in N} g \cdot S$$

$$\psi|_{\mathbf{Z}G} : 1 \mapsto -1 \cdot S.$$

Then we have

$$(\psi \circ \phi)(1 \cdot S) = \sum_{g \in N} g \cdot S - \sum_{r=1}^{t} \sum_{s \in S} s g_r \cdot S. \tag{1}$$

Since S acts freely on $N \backslash 1$ we have, for $1 \leq r \leq t$,

$$\sum_{s \in S} s g_r \cdot S = \sum_{s \in S} (s g_r s^{-1}) \cdot S = \sum_{g \in \Omega_r} g \cdot S.$$

Summing over all r and substituting in (1), we deduce that

$$(\psi \circ \phi)(1 \cdot S) = 1 \cdot S.$$

Hence $\psi \circ \phi = 1_{\mathbf{Z}G/S}$ and so $\mathbf{Z}G/S$ is isomorphic to $\mathbf{Z}G$-direct summand of $\mathbf{Z} \oplus \mathbf{Z}G$, as claimed.

COROLLARY 7.1. *Let (G, S) be a Frobenius pair and $r \geq 2$ an integer. Then there are natural isomorphisms*

$$H^r(G; -) \cong H^r(S; -) \oplus H^r(G, S; -)$$
$$H_r(G; -) \cong H_r(S; -) \oplus H_r(G, S; -).$$

Proof. We denote by A the complement of $\mathbf{Z}G/S$ in $\mathbf{Z} \oplus \mathbf{Z}G$. The following commutative diagram has exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Delta \arrow[r] \arrow[d] & \mathbf{Z}G/S \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Z}G \arrow[r] \arrow[d] & \mathbf{Z}G\oplus\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] & 0 \\
 & A \arrow[r, equal] \arrow[d] & A \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

For $r \geq 1$ we read off the natural isomophism

$$H^r(G; -) \cong \mathrm{Ext}^r_{\mathbf{Z}G}(\mathbf{Z}G\oplus\mathbf{Z}); -) \cong \mathrm{Ext}^r_{\mathbf{Z}G}(\mathbf{Z}G/S; -) \oplus \mathrm{Ext}^r_{\mathbf{Z}G}(A; -).$$

By Shapiro's Lemma,

$$\mathrm{Ext}^r_{\mathbf{Z}G}(\mathbf{Z}G/S; -) \cong H^r(S; -),$$

and from the long exact sequence induced by the first column we get, for $r \geq 2$,

$$\mathrm{Ext}^r_{\mathbf{Z}G}(A; -) \cong \mathrm{Ext}^{r-1}_{\mathbf{Z}G}(\Delta; -) \cong H^r(G, S; -).$$

This establishes the first of the stated natural isomorphisms. The second is proved in an analogous way.

COROLLARY 7.2. *Let (G, S) be a Frobenius pair and denote by N the kernel of the projection $G \twoheadrightarrow S$. For any finite group H let π_H denote the set of primes dividing $|H|$. Then the following hold:*

(i) *The decompositions in Corollary 7.1 are functorial splittings of $H^r(G; -)$, resp. $H_r(G; -)$, into π_S- and π_N-parts $(r \geq 2)$.*

(ii) *For all $r \geq 2$, there are natural isomorphisms of functors on $\mathbf{Z}G$-modules*

$$H^r(G, S; -) \cong H^r(N; -)^S.$$

(iii) *For all $r \in \mathbf{Z}$, there are natural isomorphisms of functors on $\mathbf{Z}G$-modules*

$$\hat{H}^r(S, (-)^N) \cong \hat{H}^r(S, \mathrm{res}^G_S(-)).$$

Proof. Statement (i) follows from the fact that $|S|$ annihilates $H^r(S;-)$ and $|N|=|G:S|$ annihilates $H^r(G,S;-)$ (because (G,S) is Frobenius – see Appendix). Statements (ii) and (iii) result from (i) when one compares the splitting of $H^r(G;A)$ in Corollary 7.1 with that given by the restriction-inflation sequence for the normal Hall subgroup N [1, p. 191, 55.1]. For (iii) one uses the periodicity of S to eliminate the lower bound on the dimension r.

Remark. For the particular coefficient module $\mathbf{Z}G/S$ and for $r=0$, statement (iii) of Corollary 7.2 takes the form

$$\hat{H}^0(S;\mathbf{Z})\cong\hat{H}^0(S;\mathbf{Z}G/S). \tag{2}$$

Conversely, if (G,S) is a group-pair with G finite and $1\neq S\neq G$, such that (2) holds, then (G,S) is a Frobenius pair. For the module $\mathbf{Z}G/S$ considered as a $\mathbf{Z}S$-module has the form

$$\mathbf{Z}G/S\cong\mathbf{Z}\oplus\left(\bigoplus_{i=1}^{n}\mathbf{Z}S/S_i\right)$$

and hence

$$\begin{aligned}\hat{H}^0(S;\mathbf{Z})&\cong\hat{H}^0(S;\mathbf{Z}G/S)\quad\text{(by (2))}\\&\cong\hat{H}^0(S;\mathbf{Z})\oplus\left(\bigoplus_{i=1}^{n}\hat{H}^0(S_i;\mathbf{Z})\right).\end{aligned}$$

Thus all the S_i are trivial and S acts freely on the set $(G/S)\backslash(1\cdot S)$. Essentially the same computation gives the following converse of Theorem 7.

PROPOSITION 8. *Let (G,S) be a group pair, with G finite, such that $\mathbf{Z}G/S$ is a $\mathbf{Z}S$-direct summand of $\mathbf{Z}\oplus M$, where M is a $\mathbf{Z}S$-module satisfying $\hat{H}^0(S;M)=0$. Then (G,S) is a Frobenius pair.*

THEOREM 9. *Let (G,S) be a Frobenius pair, and let n denote the period of the Tate cohomology of S. Then there exists a finitely-generated $\mathbf{Z}G$-free resolution $\mathscr{P}\twoheadrightarrow\mathbf{Z}$ such that $\mathbf{Z}G/S$ is a $\mathbf{Z}G$-direct summand of $K_n(\mathscr{P})$.*

Proof. Since S has cohomological period n, one can construct, using the methods of [8, section 2], a finitely-generated $\mathbf{Z}S$-free resolution $\mathscr{F}\twoheadrightarrow\mathbf{Z}$ such that $K_n(\mathscr{F})\cong\mathbf{Z}\oplus Q$ for some projective $\mathbf{Z}S$-module Q.

By Theorem 7, there exists a $\mathbf{Z}G$-module A such that $\mathbf{Z}\oplus\mathbf{Z}G\cong A\oplus\mathbf{Z}G/S$ as

$\mathbf{Z}G$-modules. Let $\mathcal{F}' \twoheadrightarrow A$ be any finitely-generated $\mathbf{Z}G$-free resolution. Then $\mathcal{P}' = \mathcal{F}' \oplus (\mathbf{Z}G \otimes_S \mathcal{F})$ is a finitely-generated $\mathbf{Z}G$-free resolution of $A \oplus \mathbf{Z}G/S \cong \mathbf{Z} \oplus \mathbf{Z}G$. Furthermore, $\mathbf{Z}G/S$ is a $\mathbf{Z}G$-direct summand of $K_n(\mathcal{P}')$.

We use the epimorphism $\varepsilon : P_0' \twoheadrightarrow \mathbf{Z} \oplus \mathbf{Z}G$ to produce a splitting $P_0' \cong Q' \oplus \mathbf{Z}G$, where Q' is the full preimage of $\mathbf{Z}$ under ε. Since Q' contains the kernel of ε, we may define a finitely-generated $\mathbf{Z}G$-free resolution

$$\begin{array}{ccccccccccc}
\mathcal{P}: \cdots & \longrightarrow & P_2' & \longrightarrow & P_1' & \longrightarrow & Q' & \longrightarrow & \mathbf{Z}' & \longrightarrow & 0 \\
 & & & & \oplus & & \oplus & & & & \\
 & & & & \mathbf{Z}G & \longrightarrow & \mathbf{Z}G & & & &
\end{array}$$

such that $K_n(\mathcal{P}) \cong (K_n(\mathcal{P}')$ has a $\mathbf{Z}G$-direct summand isomorphic to $\mathbf{Z}G/S$.

4. Examples

4.1. *Frobenius pairs*

Suppose (G, S) is a Frobenius pair, and S has cohomological period n. Then Theorem 9 tells us that $\mathbf{Z}G/S$ is a direct summand of $K_n(\mathcal{F})$ for some $\mathbf{Z}G$-free resolution $\mathcal{F} \twoheadrightarrow \mathbf{Z}$. The proof of Theorem 9 relies on the decomposition $\mathbf{Z} \oplus \mathbf{Z}G \cong \mathbf{Z}G/S \oplus A$ to construct a suitable resolution $\mathcal{F}$. It does not help us to decide whether, for any given resolution $\mathcal{F} \twoheadrightarrow \mathbf{Z}$, the module $\mathbf{Z}G/S$ appears as a direct summand of $K_n(\mathcal{F})$.

We examine this situation more closely in the special case where $n = 2$ (that is, S is cyclic) and $\mathcal{F} \twoheadrightarrow \mathbf{Z}$ is the Lyndon resolution arising from a finite presentation of G. In this case, $K_2(\mathcal{F})$ is the relation module of the presentation.

Gruenberg and Roggenkamp [5, Propositions 5(i), 6] have shown that, in this situation, any relation module decomposes as the direct sum of two non-projective factors. In fact, following the argument in [5] reveals that one of the factors is isomorphic to $\mathbf{Z}G/S$.

Explicitly, there is an isomorphism [5, Lemma 4.1]

$$K_1(F) = IG \cong \mathbf{Z}G \otimes_S IS \oplus IN,$$

where N is the kernel of the projection $G \twoheadrightarrow S$; the modules IG, IS, IN are the augmentation ideals in $\mathbf{Z}G$, $\mathbf{Z}S$, $\mathbf{Z}N$ respectively, and the N-action on IN is extended to a G-action by letting S act by conjugation on elements of N.

Given a free presentation $\phi : \Phi \to G$ of finite rank, one chooses a basis $\{x_1, \ldots, x_n\}$ of Φ such that $g_1 = \phi(x_1) \in S$ and $g_i = \phi(x_i) \in N$ for $2 \leq i \leq n$. This allows one to decompose the exact sequence

$$0 \longrightarrow K_2(\mathcal{F}) \longrightarrow (\mathbf{Z}G)^n \xrightarrow[\begin{pmatrix}1-g_1\\ \vdots \\ 1-g_n\end{pmatrix}]{} K_1(\mathcal{F}) = IG \longrightarrow 0$$

as the direct sum of two terms:

$$0 \longrightarrow \mathbf{Z}G/S \longrightarrow \mathbf{Z}G \xrightarrow[(1-g_1)]{} \mathbf{Z}G \otimes_S IS \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow (\mathbf{Z}G)^{n-1} \xrightarrow[\begin{pmatrix}1-g_2\\ \vdots \\ 1-g_n\end{pmatrix}]{} IN \longrightarrow 0$$

EXAMPLE 1. Let G be the symmetric group of degree 3, $\mathcal{F} \twoheadrightarrow \mathbf{Z}$ the Lyndon resolution associated to the presentation

$$\langle x, y \mid x^2 = (xy)^2 = y^3 = 1\rangle,$$

and S the subgroup of order 2 generated by x.

Then, $\partial_1 : (\mathbf{Z}G)^2 \to \mathbf{Z}G$ is given by the matrix

$$\begin{pmatrix}1-x\\ 1-y\end{pmatrix}$$

and $K_2(\mathcal{F}) = \operatorname{Ker} \partial_1$ has a direct summand isomorphic to $\mathbf{Z}G/S$. This is embedded in $F_1 = (\mathbf{Z}G)^2$ as the cyclic submodule generated by the element $(1+x, 0)$.

4.2. *Infinite groups*

Using Theorem 5, Corollary 5.4 and Theorem 9, we can completely classify those group pairs (G, S), with G finite, for which $\mathbf{Z}G/S$ occurs as a direct summand of some $K_n(\mathcal{P})$. They are pairs $(G, \{S, 1, 1, \ldots\})$ such that either

(a) (G, S) is a Frobenius pair;

(b) $S = G$, a group with periodic cohomology; or

(c) $S = 1$, in which case $\mathbf{Z}G/S$ is free, and no further information about G can be deduced.

For group pairs $(G, \mathbf{S})$ with G infinite, the position is less clear, and a complete classification would appear to be impossible to find. A restricted insight into the complexity of the class of groups considered may be obtained from looking at constructions under which the class is closed. In this light, the following proposition and examples give some idea of how complicated the situation is.

PROPOSITION 10. *Suppose Γ is a graph of groups whose edge groups G_e satisfy* cd $G_e \leq n_0$ *for some fixed integer n_0. Suppose we are given, for each vertex v, a (possibly empty) family $\mathbf{S}_v$ of subgroups of the vertex group G_v, and a $\mathbf{Z}G$-projective resolution $\mathcal{P}_v \twoheadrightarrow \mathbf{Z}$ such that for some $n(v) \leq n_0$, $\mathbf{Z}G_v/\mathbf{S}_v$ is a direct summand of $K_{n(v)}(\mathcal{P}_v)$. Let $\mathbf{S}$ denote the union $\bigcup_v \mathbf{S}_v$, regarded as a family of subgroups of the fundamental group $G = \pi(\Gamma)$. Then there exists a $\mathbf{Z}G$-projective resolution $\mathcal{P} \twoheadrightarrow \mathbf{Z}$, and an integer r, such that $\mathbf{Z}G/\mathbf{S}$ is a direct summand of $K_r(\mathcal{P})$.*

Proposition 10 is a direct generalisation of [6, Theorem 5], and is proved in an analogous way, using mapping cones. We omit the details.

Proposition 10 allows us to construct a large class of examples, beginning with those we already know, such as Frobenius groups, and groups of finite quasi-projective dimension. Our next example does not arise in this way. That is, the group concerned is infinite, has infinite quasi-projective dimension, and cannot be properly expressed as the fundamental group of a graph of groups whose edge groups are torsion-free.

EXAMPLE 2. Define $G = A \underset{S}{*} B$, where A and B are isomorphic copies of the symmetric group of degree 3, and $S = \langle x \rangle$ is a common subgroup of order 2.

If we apply the exact functor $\mathbf{Z}G \otimes_A -$ to the $\mathbf{Z}A$-resolution in Example 1, we obtain an exact sequence

$$0 \longrightarrow \mathbf{Z}G/S \oplus M_A \longrightarrow (\mathbf{Z}G)^2 \longrightarrow \mathbf{Z}G \longrightarrow \mathbf{Z}G/A \longrightarrow 0$$

in which $\mathbf{Z}G/S$ is identified with the cyclic submodule of $(\mathbf{Z}G)^2$ generated by the element $(1+x, 0)$. Hence there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}G/S \arrow[r] \arrow[d, "\lambda_A"] & \mathbf{Z}G \arrow[r, "(1-x)"] \arrow[d, "{(1,0)}"] & \mathbf{Z}G \arrow[r] \arrow[d, equal] & \mathbf{Z}G/S \arrow[r] \arrow[d, "\varepsilon_A"] & 0 \\
0 \arrow[r] & \mathbf{Z}G/S \oplus M_A \arrow[r] & (\mathbf{Z}G)^2 \arrow[r] & \mathbf{Z}G \arrow[r] & \mathbf{Z}G/A \arrow[r] & 0
\end{tikzcd}$$

in which ε_A is the augmentation map, and λ_A is the canonical inclusion corresponding to the direct sum decomposition.

Replacing A by B in the above gives rise to another, similar diagram. From the two diagrams one can obtain a new commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}G/S \arrow[r] \arrow[d, "(\lambda_A, -\lambda_B)"'] & \mathbf{Z}G \arrow[r] \arrow[d, "(1,0,-1,0)"'] & \mathbf{Z}G \arrow[r] \arrow[d, "(1,-1)"] & \mathbf{Z}G/S \arrow[r] \arrow[d, "(\varepsilon_A, -\varepsilon_B)"] & 0 \\
0 \arrow[r] & (\mathbf{Z}G/S \oplus M_A) \oplus (\mathbf{Z}G/S \oplus M_B) \arrow[r] & (\mathbf{Z}G)^4 \arrow[r] & (\mathbf{Z}G)^2 \arrow[r] & \mathbf{Z}G/A \oplus \mathbf{Z}G/B \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact and the vertical maps are all injective. It follows that the induced sequence of cokernels is also exact. This has the form

$$0 \longrightarrow \mathbf{Z}G/S \oplus (M_A \oplus M_B) \longrightarrow (\mathbf{Z}G)^3 \longrightarrow \mathbf{Z}G \longrightarrow \mathbf{Z} \longrightarrow 0.$$

Hence $\mathbf{Z}G/S$ is a direct summand of $K_2(\mathcal{F})$ for some $\mathbf{Z}G$-free resolution $\mathcal{F} \twoheadrightarrow \mathbf{Z}$.

Remark. The group G of Example 2 is expressed as an amalgamated free product with finite amalgamated subgroup, a construction which does not satisfy the hypotheses of Proposition 10. This suggests that it may be possible to weaken the restriction on edge groups in Proposition 10. However, it does not seem easy to find a general condition under which the conclusion of the proposition continues to hold. That some restriction on edge groups is necessary is shown by the following example.

EXAMPLE 3. $G = SL_2(\mathbf{Z}) \cong (\mathbf{Z}/4\mathbf{Z}) \underset{(\mathbf{Z}/2\mathbf{Z})}{*} (\mathbf{Z}/6\mathbf{Z})$ is an infinite group with non-trivial torsion and non-trivial centre. By Corollary 5.3, there is no non-trivial subgroup S such that $\mathbf{Z}G/S$ occurs as a direct summand in any $K_n(\mathcal{P})$.

Appendix on relative cohomology

Suppose G is a finite group of order r. Then the ordinary (co)-homology functors of G with arbitrary $\mathbf{Z}G$-module coefficients are annihilated by r.

Now let G be a cyclic group of order $2n$ and S the subgroup of order n. Then $G/S = C \cong \mathbf{Z}/2\mathbf{Z}$. We shall compute the group $H^2(G, S; \mathbf{Z}C)$. In the sequel, we follow the notation of [2].

Consider the exact sequence $\Delta \rightarrowtail \mathbf{Z}G/S \twoheadrightarrow \mathbf{Z}$, where $\mathbf{Z}G/S = \mathbf{Z}C$. Then $\Delta \cong \tilde{\mathbf{Z}}$, the $\mathbf{Z}G$-module $\mathbf{Z}$ with nontrivial G-action. In terms of a generator g for G we can write down a free $\mathbf{Z}G$-resolution

$$\mathcal{F}: \cdots \longrightarrow \mathbf{Z}G \xrightarrow{1+g} \mathbf{Z}G \xrightarrow{\sum_{k=0}^{2n-1}(-1)^k g^k} \mathbf{Z}G \xrightarrow{1+g} \mathbf{Z}G \longrightarrow \tilde{\mathbf{Z}} \longrightarrow 0.$$

Now the complex $\mathrm{Hom}_{\mathbf{Z}G}(\mathcal{F}, \mathbf{Z}C)$ is of the form

$$\mathcal{F}' \cdots \longrightarrow \mathbf{Z}C \xrightarrow{1+c} \mathbf{Z}C \xrightarrow{n(1-c)} \mathbf{Z}C \xrightarrow{1+c} \mathbf{Z}C \longrightarrow \mathrm{Hom}_{\mathbf{Z}G}(\tilde{\mathbf{Z}}, \mathbf{Z}C) \longrightarrow 0$$

where $c = gS$ is a generator for C. In view of the definitions in [2], we may use $\mathcal{F}'$ to compute

$$H^2(G, S; \mathbf{Z}C) \cong \mathrm{Ext}^1_{\mathbf{Z}G}(\tilde{\mathbf{Z}}; \mathbf{Z}C) \cong \tilde{\mathbf{Z}}/n\tilde{\mathbf{Z}}.$$

Thus, in general, $H^2(G, S; -)$ is *not* annihilated by the index of S in G. However, our investigation will lead to the following result:

COROLLARY 2. *Suppose (G, S) is a Frobenius pair and S has index k in G. Then k annihilates the relative cohomology functors $H^q(G, S; -)$ for all $q \geq 1$.*

The proof of this result involves a slight digression.

If U is a subgroup of G, and M is a $\mathbf{Z}G$-module, let $\varepsilon : \mathbf{Z}G \otimes_U M \to M$ denote the "evaluation" map, $\varepsilon(g \otimes m) = g \cdot m$. Let σ denote the natural isomorphism

$$\mathrm{Ext}^*_{\mathbf{Z}G}(\mathbf{Z}G \otimes_U M; -) \xrightarrow{\cong} \mathrm{Ext}^*_{\mathbf{Z}U}(M; -).$$

Then we will refer to the natural transformation

$$\mathrm{res} = \sigma \circ \varepsilon^* : \mathrm{Ext}^*_{\mathbf{Z}G}(M; -) \longrightarrow \mathrm{Ext}_{\mathbf{Z}U}(M; -)$$

as *restriction*, since in the special case $M = \mathbf{Z}$, it is the usual restriction $H^*(G; -) \to H^*(U; -)$.

Similarly, if U has finite index k in G and $\{g_1, \ldots, g_k\}$ is a left transversal, we can define a map

$$\eta : M \longrightarrow \mathbf{Z}G \otimes_U M \quad \text{by} \quad \eta(m) = \sum_{r=1}^{k} g_r \otimes g_r^{-1} m.$$

Transfer or *corestriction* is then defined to be the natural transformation

$$\mathrm{cor} = \eta^* \circ \sigma^{-1} : \mathrm{Ext}^*_{\mathbf{Z}U}(M; -) \longrightarrow \mathrm{Ext}^*_{\mathbf{Z}G}(M; -).$$

Just as in the usual case $M = \mathbf{Z}$, we now have the following result.

PROPOSITION. *Let G be a group and U a subgroup of finite index k in G. Then for any pair of $\mathbf{Z}G$-modules M, A and any integer $q \geq 0$, the composite*

$$\mathrm{cor} \circ \mathrm{res} : \mathrm{Ext}^q_{\mathbf{Z}G}(M; A) \longrightarrow \mathrm{Ext}^q_{\mathbf{Z}U}(M, A) \longrightarrow \mathrm{Ext}^q_{\mathbf{Z}G}(M, A)$$

is just multiplication by the integer k.

Proof. One checks that $\varepsilon \circ \eta$ is multiplication by k on M.

Now let $(G, \boldsymbol{S})$ be a group pair and U a subgroup of G. As in Section 2, we can define a family $\boldsymbol{T}$ of subgroups of U such that there is a $\mathbf{Z}U$-module isomorphism $\mathbf{Z}G/\boldsymbol{S} \cong \mathbf{Z}U/\boldsymbol{T}$. (See also [2, pp. 305–306]). As above, we denote by Δ the kernel of the augmentation $\mathbf{Z}G/\boldsymbol{S} \to \mathbf{Z}$ and define $H^k(G, S; -) \cong \mathrm{Ext}^{k-1}(\Delta, -)$ following [2]. Specializing the above to $M = \Delta$, we obtain natural transformations

$$\mathrm{res} : H^*(G; \boldsymbol{S}; -) \longrightarrow H^*(U, \boldsymbol{T}; -)$$

and, provided U has finite index in G,

$$\mathrm{cor} : H^*(U, \boldsymbol{T}; -) \longrightarrow H^*(G, \boldsymbol{S}; -).$$

The Proposition now yields the following consequence:

COROLLARY 1. *Let $(G, \boldsymbol{S})$ and $(U, \boldsymbol{T})$ be pairs of groups as above and suppose U has finite index k in G. Then for any integer $q \geq 1$, the natural transformation*

$$\mathrm{cor} \circ \mathrm{res} : H^q(G, \boldsymbol{S}; -) \longrightarrow H^q(U, \boldsymbol{T}; -) \longrightarrow H^q(G, \boldsymbol{S}; -)$$

is just multiplication by k.

If (G, S) is a Frobenius pair, we set $U = S$ in Corollary 1. Then the family $\boldsymbol{T}$ consists of one copy of S together with $(k-1)$ copies of the trivial group. Since $H^q(S, \{S, 1, \dots, 1\}; -) \cong H^q(S, S; -) = 0$ for $q \geq 1$, Corollary 2 follows because multiplication by k factors through 0.

REFERENCES

[1] BABAKHANIAN, A., Cohomological methods in group theory. Marcel Dekker Inc., New York 1972.
[2] BIERI, R. and ECKMANN, B., *Relative homology and Poincaré duality for group pairs.* J. Pure and Appl. Alg. *13* (1978) 277–319.
[3] CARTAN, H. and EILENBERG, S., Homological algebra. Princeton University Press 1956.
[4] GRUENBERG, K. W., Free abelianised extensions of finite groups. Homological Group Theory (C. T. C. Wall, ed.), 71–104; London Math. Soc. Lecture Note Series. No. 36, Cambridge Univ. Press 1979.
[5] GRUENBERG, K. W. and ROGGENKAMP, K. W., *Decomposition of the augmentation ideal and of the relation modules of a finite group.* Proc. London. Math. Soc. (3) *31* (1975) 149–166.
[6] HOWIE, J. and SCHNEEBELI, H. R., *Groups of finite quasi-projective dimension.* Comment. Math. Helvetici *54* (1979) 615–628.
[7] HUEBSCHMANN, J., *Cohomology theory of aspherical groups and of small cancellation groups.* J. Pure and Appl. Alg. *14* (1979) 137–143.
[8] SWAN, R. G., *Periodic resolutions for finite groups.* Annals of Math. 72 (1960) 267–291.

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