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On J. H. C. Whitehead’s aspherical question I

JOE BRANDENBURG and MICHEAL DYER

Abstract. A connected, finite two-dimensional CW-complex with fundamental group isomorphic to G is called a $[G, 2]_f$ -complex. Let $L \triangleleft G$ be a normal subgroup of G . L has weight k if and only if k is the smallest integer such that there exists $\{l_1, \dots, l_k\} \subset L$ such that L is the normal closure in G of $\{l_1, \dots, l_k\}$. We prove that a $[G, 2]_f$ -complex X may be embedded as a subcomplex of an aspherical complex $Y = X \cup \{e_1^2, \dots, e_k^2\}$ if and only if G has a normal subgroup L of weight k such that $H = G/L$ is at most two-dimensional and $\text{def } G = \text{def } H + k$. Also, if X is a *non-aspherical* $[G, 2]_f$ -subcomplex of an aspherical 2-complex, then there exists a non-trivial superperfect normal subgroup P such that G/P has cohomological dimension ≤ 2 . In this case, any torsion in G must be in P .

0. Introduction

A $[G, 2]_{(f)}$ -complex X is any (finite) connected two dimensional CW-complex with fundamental group isomorphic to G . Sometimes we will abuse the notation and say $X \in [G, 2]_{(f)}$. Let X be a connected subcomplex of an $[H, 2]$ -complex Y . J. H. C. Whitehead’s question is this: if Y is aspherical, is X also aspherical? [W_1 , p. 428].

The question seems very hard. We say that a group G satisfies the (finite) *Whitehead condition* ($G \in WC_{(f)}$) if any $[G, 2]$ -complex X , which is the subcomplex of an aspherical (finite) 2-complex, is aspherical. Thus $G \in WC_{(f)}$ iff for any $[G, 2]_{(f)}$ -complex X , either X is aspherical or, if not, then no $[H, 2]_{(f)}$ -complex $Y > X$ is aspherical. The philosophy of this paper is to isolate properties of a group G which imply that $G \in WC$ or WC_f .

There are a number of results in this direction. W. Cockcroft [C, Theorem 2] showed that if G is one-relator group, then G has WC. One crucial observation he made was: *Let X be a $[G, 2]$ -complex such that the Hurewicz homomorphism $h_2: \pi_2 X \rightarrow H_2 X$ is non-zero. Then no 2-complex $Y > X$ is aspherical.* It follows that any group G which admits a $[G, 2]$ -complex X which is the subcomplex of an aspherical 2-complex has $H_2 G$ free abelian. (Here $H_2 G$ means the homology of G with coefficients in the trivial module \mathbf{Z} .)

Let $\chi_{\min}(G, 2) = \min \{\chi(X) \mid X \text{ is a } [G, 2]_f\text{-complex}\}$. A complex X whose Euler characteristic $\chi(X)$ is minimal is called a *minimal* $[G, 2]_f$ -complex. *For simplicity, all $[G, 2]$ -complexes will have a single vertex, and this will be the base*

point for all homotopy groups. Any $[G, 2]$ -complex has the (simple) homotopy type of such a complex.

We observe for later use that WC_f can be proved or disproved for a particular group G as follows: choose your favorite minimal $[G, 2]_f$ -complex X and, if it is not aspherical, check that no $[H, 2]_f$ -complex containing X is aspherical (see Lemma 1.4).

It follows from Cockcroft's result above that if $X < Y$, where $Y \in [H, 2]_f$ is aspherical, then X is a *minimal* $[G, 2]_f$ -complex. To see this, we simply observe that if X is any non-minimal $[G, 2]_f$ -complex, then $h_2: \pi_2 X \rightarrow H_2 X$ is *not* zero. For let Y be a minimal $[G, 2]_f$ -complex, with $\chi(Y) < \chi(X)$. Let $\Sigma_2(-) = \text{im}\{h_2: \pi_2(-) \rightarrow H_2(-)\}$ be the image of the Hurewicz homomorphism. Then, by a result of H. Hopf, $H_2 G \cong H_2 X / \Sigma_2 X \cong H_2 Y / \Sigma_2 Y$. Because $H_2 X$ and $H_2 Y$ are finitely generated free abelian groups with $\text{rank}_Z H_2 Y < \text{rank}_Z H_2 X$, we must have $\Sigma_2 X \neq 0$.

J. F. Adams' approach [A, p. 483] was to assume that a non-aspherical $X < Y = X \cup \{e_\alpha^2 \mid \alpha \in \mathcal{A}\}$, with Y aspherical, and to study $L = \ker\{\pi_1 X \rightarrow \pi_1 Y\}$. Adams proved that $H_1 L$ is a free abelian group and L is not transfinite metabelian; i.e., L has a non-trivial (normal) subgroup P which is perfect ($H_1 P = P^{ab} = 0$). This shows that *any* solvable group has WC .

In [Co], J. Cohen points out that Adams' perfect subgroup $P < L$ is actually *superperfect*; i.e., $H_2 P = 0$. He also shows [Co, Theorem 3] that if G a group of cohomological dimension 3 and type FL (that is, Z has a finite resolution by finitely generated free G -modules) such that $H_3 G = 0$, then G has WC_f .

In [GR, Theorem 4], M. Gutierrez and J. Ratcliffe show that if $X < Y (Y \in [H, 2]_f)$ which is aspherical, then X is aspherical if and only if the cohomological dimension of $G \leq 2$ and G has type FL .

In [H], J. Howie shows that any torsion element $x \in G (x^n = 1)$ is contained in a *finitely generated* perfect subgroup of L . We show that, in fact, all the torsion of G is contained in Adams' superperfect subgroup.

Finally, in his thesis [Be] W. Beckmann shows that locally finite groups have WC .

Specifically, we show the following

THEOREM 1. *Let X be a non-aspherical $[G, 2]$ -subcomplex of an aspherical 2-complex. Then there is a nontrivial superperfect normal subgroup P (Adams) such that G/P has cohomological dimension ≤ 2 .*

COROLLARY. *Any torsion in G must be in P .*

As a second result we characterize when one may add 2-cells to a $[G, 2]_f$ -complex to obtain an aspherical complex. Let F_n denote a free group of rank n .

THEOREM 2. *Let X be a minimal $[G, 2]_f$ -complex. One may add n 1-cells and k 2-cells to X to obtain an aspherical 2-complex if and only if $G * F_n$ has a normal subgroup L which is a free-crossed G -module of weight k (see 3.2) such that (1) there is an aspherical $[G/L, 2]_f$ -complex and (2) $\text{def } G + n = \text{def } (G/L) + k$.*

COROLLARY. *Let X be a minimal $[G, 2]_f$ -complex. One may add two-cells to X to obtain a finite contractible space if and only if $\text{weight } G = \text{def } G$.*

The groups in the corollary are of interest because they are (higher) knot and link groups, according to a theorem of M. Kervaire [K]. These groups are all E -groups in the sense of [St₁], [St₂] and [B]. From this it follows that the derived series of G has many interesting properties; such as, each element G^α in the derived series for G is an E -group, and the derived length of G is severely restricted.

The paper is organized as follows. In section one we study complexes X for which the Hurewicz map $h_2: \pi_2 X \rightarrow H_2 X$ is zero and reprove a crucial lemma of W. Cockcroft and R. Swan about minimal aspherical complexes. In section two we study necessary and sufficient conditions for the inclusion $X < Y$ to induce the zero map on the second homotopy groups. In section three we prove Theorem 2 and in section four, Theorem 1. We defer examples and applications of these results to a later paper.

To fix notation, let G be a group and let $\mathbf{Z}G$ be the integral group ring of G . Let IG denote the augmentation ideal, the kernel of the map $\epsilon: \mathbf{Z}G \rightarrow \mathbf{Z}$.

1. Cockcroft complexes and the Cockcroft–Swan lemma

DEFINITION 1.1. A connected CW-complex X is called *Cockcroft* if and only if the Hurewicz homomorphism $h: \pi_2 X \rightarrow H_2 X$ is trivial. A group G is *Cockcroft* if and only if some $[G, 2]$ -complex is Cockcroft.

Note that any non-minimal $[G, 2]_f$ -complex is not Cockcroft. It follows that any group G having $H_2(G; \mathbf{Z})$ not free abelian is not Cockcroft. It was shown in [C, lemma 1] that for any non-Cockcroft $[G, 2]$ -complex X , any $Y = X \cup \{e_\alpha^2\}$ has $\pi_2 X \rightarrow \pi_2 Y$ not zero. It follows that non-Cockcroft groups have WC.

EXAMPLE 1.2. Any finitely generated one-relator group is Cockcroft. Let G

be the group presented by $\{x_1, \dots, x_n; r\}$ and let X be the model associated with the presentation. Write $r = Q^a$, where Q is not a proper power in the free group $F(x_1, \dots, x_n)$. By a theorem of Lyndon [L], $\pi_2 X \cong ZG(\bar{Q} - 1)$ as a left G -module, where \bar{Q} is the image in G of Q under the natural projection $F \rightarrow G$. As the Hurewicz map $h : \pi_2 X \rightarrow H_2 X$ is given by restricting the augmentation $\epsilon : C_2 \tilde{X} = ZG \rightarrow C_2 X = Z$, we see that $h = 0$. Note that if X is a subcomplex of an aspherical two-complex Y , then X must be a Cockcroft $[G, 2]$ -complex. This follows because if $[X \vee \vee S_\beta^1] \cup \{e_\alpha^2\} = Y$ is aspherical, then the Hurewicz homomorphism $\pi_2(X \vee \vee S_\beta^1) \rightarrow H_2(X \vee \vee S_\beta^1)$ is zero [C, lemma 1]. That X is Cockcroft is clear from the commutative diagram:

$$\begin{array}{ccc} \pi_2 X & \longrightarrow & H_2 X \\ \downarrow & & \downarrow \\ \pi_2(X \vee \vee S_\beta^1) & \longrightarrow & H_2(X \vee \vee S_\beta^1). \end{array}$$

Observe that if X and X' are minimal $[G, 2]_f$ -complexes, X is Cockcroft iff X' is.

The following theorem characterizes in several different ways the property that X is Cockcroft.

PROPOSITION 1.3. *Let X be a $[G, 2]$ -complex. The following are equivalent:*

- (a) $h_2 : \pi_2 X \rightarrow H_2 X$ is zero.
- (b) The Hopf epimorphism $H_2 X \rightarrow H_2 G$ is injective.
- (c) The natural inclusion $H_3 G \rightarrow Z \otimes_G \pi_2 X$ is surjective.

Proof. This follows from the exact sequence of [D], which is just a fancy rewrite of two theorems of H. Hopf:

$$0 \rightarrow H_3 G \rightarrow Z \otimes_G \pi_2 X \xrightarrow{\bar{h}_2} H_2 X \rightarrow H_2 G \rightarrow 0,$$

where \bar{h}_2 is induced by h_2 . \square

We now prove the following key lemma of Cockcroft and Swan [CS, p. 197].

LEMMA 1.4. *Let X and X' be minimal $[G, 2]_f$ -complexes. Then X is aspherical iff X' is.*

Proof. As X and X' have the same Euler characteristic, it follows from Schanuel's lemma that $\pi_2 X \oplus ZG^n \cong \pi_2 X' \oplus ZG^n$ for some integer $n > 0$. Then

$\pi_2 X = 0$ implies that

$$\pi_2 X' \twoheadrightarrow ZG^n \twoheadrightarrow ZG^n$$

is exact, which yields that $\pi_2 X' = 0$ by a theorem of I. Kaplansky. \square

The *proof* of Lemma 1.4 clearly breaks down if X is an infinite, but Cockcroft, $[G, 2]$ -complex. *We conjecture that the lemma is still true for such complexes.*

If G is finitely presented, then one may show from Lemma 1.4 that either *all* minimal $[G, 2]_f$ -complexes are subcomplexes of finite aspherical 2-complexes or *none* are.

LEMMA 1.5. *Let G be a finitely presented group. Let X, X' be minimal $[G, 2]_f$ -complexes and $Y = (X \vee \bigvee_{i=1}^m S_i^1) \cup \{e_1^2, \dots, e_n^2\}$ be aspherical. Then one may add m one-cells and n two-cells to X' to obtain an aspherical complex. \square*

2. Killing $\pi_2 X \rightarrow \pi_2 Y$ for X a subcomplex of Y

DEFINITION 2.1. For any subgroup $A < G$, let $K_A = \mathbf{Z}G \cdot IA$ be the left ideal in $\mathbf{Z}G$ generated by $\{a - 1 \mid a \in A\}$. Note that if A is a *normal* subgroup of G , then K_A is a *two-sided* ideal. In any case, $K_A = \ker \{\mathbf{Z}G \rightarrow \mathbf{Z}(G/A)\}$ induced by the coset function $G \rightarrow G/A$.

DEFINITIONS 2.2. Let M be any (left) submodule of a free G -module. The *Fox ideal* of M , $F(M)$, is the two-sided ideal in $\mathbf{Z}G$ generated by the coordinates of each element (of a generating set) of M . We say that a *subgroup* $A < G$ *kills* M if the Fox ideal $F(M)$ is contained in the kernel K_A . Note that $F(M)$ is independent of any chosen basis.

EXAMPLE. For a $[G, 2]$ -complex X , G itself kills $\pi_2 X$ iff $F(\pi_2 X) \subset K_G = IG$. This happens iff the Hurewicz map $h_2: \pi_2 X \rightarrow H_2 X$ is zero.

Now let $X \in [G, 2]$ be a subcomplex of an $[H, 2]$ -complex Y . Let $\iota: X \rightarrow Y$ denote the inclusion map and $L = \ker \pi_1(\iota)$. If $C_* \tilde{X}$ is the cellular chain complex (considered as left G -modules) of the universal cover \tilde{X} of X , let $R_X = \ker \{\partial_1: C_1 \tilde{X} \rightarrow C_0 \tilde{X} = \mathbf{Z}G\}$ be a so-called relation module for G .

THEOREM 2.3. *The following are equivalent:*

- (1) $i_\#: \pi_2 X \rightarrow \pi_2 Y$ is zero,
- (2) the Fox ideal $F(\pi_2 X) \subset K_L$ (L kills $\pi_2 X$),

(3) the Hurewicz homomorphism $h_L : \pi_2 X \rightarrow H_2 X_L$ is zero, where X_L is a covering of X corresponding to the subgroup L .

(4) The natural surjection $\bar{\partial}_2 : C_2 \tilde{X} \rightarrow R_X$ induces an isomorphism $\mathbf{Z} \otimes_L C_2 \tilde{X} \rightarrow \mathbf{Z} \otimes_L R$ of free G/L -modules.

Proof. Let $(\mathbf{Z}H)^{|\mathcal{A}|}$ denote $\bigoplus_{\alpha \in \mathcal{A}} (\mathbf{Z}H)_\alpha$. Consider the universal covering \tilde{X} of X , the cellular chain complex $C_*(\tilde{X})$ of \tilde{X} (viewed as free left G -modules and homomorphisms), and the cellular chain complexes $C_* X_L = \mathbf{Z} \otimes_L C_* \tilde{X} \rightarrow C_* \tilde{Y}$ (viewed as free G/L and H modules, respectively). Let $N = G/L$ denote the image of $\pi_1(i) : G \rightarrow H$ and $\eta : G \rightarrow G/L$ be the natural map. Also let

$$Y = \left(X \vee \bigvee_{\beta \in \mathcal{B}} S^1_\beta \right) \cup \{e_\alpha^2\}_{\alpha \in \mathcal{A}}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 \pi_2 X \hookrightarrow & C_2 \tilde{X} & \xrightarrow{\partial_2^X} & C_1 \tilde{X} & \xrightarrow{\partial_1^X} & \mathbf{Z}G & \longrightarrow \mathbf{Z} \\
 \downarrow & \downarrow & & \downarrow & & \downarrow & \parallel \\
 & \mathbf{Z} \otimes_L C_2 \tilde{X} & \xrightarrow{1 \otimes \partial_2^X} & \mathbf{Z} \otimes_L C_1 \tilde{X} & \xrightarrow{1 \otimes \partial_1^X} & \mathbf{Z} \otimes_L \mathbf{Z}G & \longrightarrow \mathbf{Z} \\
 & \parallel & & \parallel & & \parallel & \parallel \\
 & & & & & \mathbf{Z}N & \\
 & & & & & \parallel & \\
 H_2 X_L \hookrightarrow & C_2 X_L & \xrightarrow{\partial_2^L} & C_1 X_L & \xrightarrow{\partial_1^L} & C_0 X_L & \longrightarrow \mathbf{Z} \\
 \downarrow & \downarrow & & \downarrow & & \downarrow & \parallel \\
 \pi_2 Y \hookrightarrow & C_2 \tilde{Y} & \xrightarrow{\partial_2^Y} & C_1 \tilde{Y} & \xrightarrow{\partial_1^Y} & \mathbf{Z}H & \longrightarrow \mathbf{Z} \\
 & \parallel & & \parallel & & & \\
 & \mathbf{Z}H \otimes_N C_2 X_L \oplus \mathbf{Z}H^{|\mathcal{A}|} & & \mathbf{Z}H \otimes_N C_1 X_L \oplus \mathbf{Z}H^{|\mathcal{B}|} & & &
 \end{array}$$

The chain map $i_* : C_2 \tilde{X} = (\mathbf{Z}G)^m \rightarrow C_2 \tilde{Y}$ factors as

$$\mathbf{Z}G^m = C_2 \tilde{X} \twoheadrightarrow C_2 X_L = \mathbf{Z}N^m \twoheadrightarrow \mathbf{Z}H^m \oplus \mathbf{Z}H^{|\mathcal{A}|} = C_2 \tilde{Y}.$$

with the first map being $\bigoplus \mathbf{Z}\eta : \mathbf{Z}G^m \rightarrow \mathbf{Z}N^m$. Hence, the kernel of $i_\# : \pi_2 X \rightarrow \pi_2 Y$ is $\pi_2 X \cap (K_L)^m$. Also, $H_2 X_L \twoheadrightarrow \pi_2 Y$ (it is a direct summand as \mathbf{Z} -modules).

So $i_{\#} : \pi_2 X \rightarrow \pi_2 Y$ is zero if and only if $h_L : \pi_2 X = \pi_2 X_L \rightarrow H_2 X_L$ is zero. This happens if and only if $\pi_2 X \subset K_L^m$, which in turn is true if and only if $F(\pi_2 X) \subset K_L$.

In order to prove (4) \Leftrightarrow (1), consider the following exact sequences (see [D]).

$$\begin{array}{ccccc}
 \mathbf{Z} \otimes_L \pi_2 X & \xrightarrow{\bar{h}_L} & H_2 X_L & \longrightarrow & H_2 L \\
 \parallel & & \downarrow & & \downarrow \\
 \mathbf{Z} \otimes_L \pi_2 X & \longrightarrow & \mathbf{Z} \otimes_L C_2 \tilde{X} & \longrightarrow & \mathbf{Z} \otimes_L R.
 \end{array}$$

It is easily shown that both squares commute. Also

$$\pi_2 X \xrightarrow{h_L} H_2 X_L \text{ factors as } \pi_2 X \longrightarrow \mathbf{Z} \otimes_L \pi_2 X \xrightarrow{\bar{h}_L} H_2 X_L.$$

Hence

$$\begin{aligned}
 \pi_2 X \xrightarrow{0} \pi_2 Y \Leftrightarrow \pi_2 X \xrightarrow{0} H_2 X_L \\
 \Leftrightarrow \mathbf{Z} \otimes_L \pi_2 X \xrightarrow{0} H_2 X_L \\
 \Leftrightarrow \mathbf{Z} \otimes_L \pi_2 X \xrightarrow{0} \mathbf{Z} \otimes_L C_2 \tilde{X}. \quad \square
 \end{aligned}$$

Note 2.4. The proof of Theorem 2.3 shows that conditions (2), (3), and (4) are equivalent for any (not necessarily normal) subgroup $L \leq G$, provided we restate (4) as an isomorphism of (not necessarily free) G -modules. In fact, it is clear that (2)–(4) are *hereditary* in the sense that, if they are true for some subgroup $L \leq G$, then they hold for any subgroup $M \leq G$ containing L .

3. Subcomplexes of aspherical complexes

DEFINITION 3.1. Let G be a (finitely presented) group. G is *at most (finitely) two-dimensional* if and only if there exists an aspherical $[G, 2]_{(f)}$ -complex.

For any group G and element $g \in G$, denote the image of g in G^{ab} by \bar{g} .

DEFINITION 3.2. Let L be a normal subgroup of G ($L \triangleleft G$) with quotient

H. L is said to be a *free crossed G-module of weight k* ($k \leq \infty$) if and only if there exists elements $\{g_1, \dots, g_k\} \subset L$ such that *L* is the normal closure $\langle\langle g_1, \dots, g_k \rangle\rangle_G$ of $\{g_i\}$ in *G*, H_1L is a free *H*-module with basis $\{\bar{g}_1, \dots, \bar{g}_k\}$, and $H_2L = 0$.

It is a very nice theorem of J. Ratcliffe [R, Theorem 2.2] that this is equivalent to the usual definition of a free crossed *G*-module (in this setting). The normal generators $\{g_1, \dots, g_k\}$ of *L* are called a *basis* for the free crossed module *L*.

An interesting special case is when *G* is a free crossed *G*-module of weight *k*. Examples of weight 1 self free crossed modules are knot groups. By a theorem of M. Kervaire [K], any finitely presented self free crossed module of weight *k* is the fundamental group of a *k*-link of 3-spheres embedded in S^5 .

Note that if *L* is a free crossed *G*-module of weight *k*, then *L* is a free crossed *L*-module of weight $k \cdot |G/L|$. Furthermore, if *L* is a free crossed *L*-module of weight *k* and *H* is any group, then, for $G = L * H$, the normal closure *N* of *L* (in *G*) is a free crossed *G*-module of weight *k*. To see this, notice that *N* is the free product $\ast_{h \in H} hLh^{-1}$ in *G* and that $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ is a split extension. $H_1L \cong \mathbf{Z}^k$, so $H_1N \cong \mathbf{Z}H^k$; $H_2L = 0$ implies $H_2N = 0$. If the normal closure of $\{l_1, \dots, l_k\}$ in *L* is equal to *L*, then $\langle\langle\{l_1, \dots, l_k\}\rangle\rangle_G = N$.

As another example, one may show the following proposition.

PROPOSITION 3.3. *Let G be a 1-relator group with presentation $\{x_1, \dots, x_n; Q^a\}$, where Q is not a proper power. Let $e_{x_i}(Q)$ denote the exponent sum of Q with respect to x_i . Then G is a free crossed G-module if and only if $[q = 1$ and $E = \gcd\{e_{x_i}(Q)\} = 1]$ if and only if $[H_1G \cong \mathbf{Z}^{n-1}]$. \square*

Note. Let *G* be a finitely generated 1-relator group. Any two 1-relator presentations of *G* have the same number of generators. This follows because the models associated with both presentations are Cockcroft (Example 1.2) and are therefore minimal.

Let $\text{def } G$ denote the *deficiency* of the finitely presented group *G*. The following theorem characterizes when one may add finitely many one-cells and two-cells to a $[G, 2]_f$ -complex *X* to obtain an aspherical 2-complex. Let F_l denote a free group of rank *l*.

THEOREM 3.4. *Let X be a minimal $[G, 2]_f$ -complex. One may add l 1-cells and k 2-cells to X to obtain an aspherical two-complex Y if and only if (1) there exists a normal subgroup $L < G * F_l$ which is a free crossed $G * F_l$ -module of weight k having $H = (G * F_l)/L$ at most finitely two-dimensional and (2) $\text{def } G + l = \text{def } H + k$.*

Note that this theorem is true whether X is aspherical or not. For another equivalent condition see Theorem 3.6.

Proof. (\Rightarrow) Suppose $X < Y = (X \vee \bigvee_{i=1}^l S_i^1) \cup \{e_1^2, \dots, e_k^2\}$ and Y is aspherical. Consider the homotopy sequence for the pair (Y, \bar{X}) , where $\bar{X} = X \cup Y^{(1)}$:

$$\begin{array}{ccccccccccc} \pi_3(Y) & \rightarrow & \pi_3(Y, \bar{X}) & \rightarrow & \pi_2(\bar{X}) & \rightarrow & \pi_2(Y) & \rightarrow & \pi_2(Y, \bar{X}) & \xrightarrow{\partial} & \pi_1(\bar{X}) & \rightarrow & \pi_1(Y) & \rightarrow & 0. \\ \parallel & & & & & & \parallel & & & & \parallel & & \parallel & & \\ 0 & & & & & & 0 & & & & G * F_l & & H & & \end{array}$$

The group $\pi_2(Y, \bar{X})$ is a free crossed $\pi_1(\bar{X})$ -module on the characteristic maps for the k added two cells. We let $L = \text{im } \partial$. Then $H = G/L$ is at most 2-dimensional. Because $\chi(X) = \chi_{\min}(G, 2)$, $\chi(Y) = \chi_{\min}(H, 2)$ and $\chi(X) + k - l = \chi(Y)$ we have (as $\chi(X) = 1 - \text{def } G$) $k + \text{def } H = \text{def } G + l$.

(\Leftarrow) Let X be a minimal $[G, 2]_f$ -complex and identify $\pi_1 X$ with G . Let $\{g_1, \dots, g_k\}$ be a basis for $L < G * F_l$ as a free crossed $G * F_l$ -module. Attach e_1^2, \dots, e_k^2 to $\bar{X} = X \vee \bigvee_{i=1}^l S_i^1$ using maps $\alpha_i : S_i^1 \rightarrow \bar{X}^{(1)}$ which represent $g_i \in G * F_l$ ($i = 1, \dots, k$). Then $X < Y = \bar{X} \cup \{e_1^2, \dots, e_k^2\}$ has $\pi_1 Y = H$ at most a finitely two-dimensional group. Thus there is an $[H, 2]_f$ -complex W which is aspherical. Because $\text{def } H = \text{def } G - k + l = 1 - \chi(Y) = 1 - \chi(W)$, we have $\chi(Y) = \chi(W) = \chi_{\min}(H, 2)$. Therefore, Y is aspherical by Lemma 1.4. \square

Note. One sees from the proof that really only the following was used:

$G * F_l$ has a normal subgroup L of weight k over $G * F_l$ such that

(a) $G * F_l / L$ is at most finitely 2-dimensional and

(b) $\text{def} [(G * F_l) / L] + k = \text{def } G + l$.

That L is a free crossed $G * F_l$ -module of weight k is a consequence of the above statement.

COROLLARY 3.5. *Let X be a minimal $[G, 2]_f$ -complex. One may add k two-cells to X to obtain a contractible space if and only if $\text{det } G = k = \text{weight } G$. This is true iff G is a free crossed G -module (= higher dimensional link group) of weight k which is Cockcroft.*

Proof. The first statement follows by specializing Theorem 3.1 to $l = 0$ and $H = \{1\}$. To see the second, we observe that a group G with $H_1 G \cong \mathbf{Z}^k$ and which is Cockcroft has $H_2 X \cong H_2 G \cong \mathbf{Z}^{k - \text{def } G}$. So $H_2 G = 0$ implies $\text{def } G = k$. A similar argument yields the converse. \square

We would ask, more generally, what does the fundamental group G of a subcomplex X of a finite contractible space Y look like? By the above corollary, we see that $G * F_l$ is a higher dimensional link group with $\text{def}(G * F_l) = \text{def } G + l = \text{weight}(G * F_l)$. Does this imply that $\text{def } G = \text{weight } G$? It is easy to see that such groups are E -groups (see [B], 123–130, for facts about E -groups).

We also have (using the same geometric techniques)

THEOREM 3.6. *Let X be a Cockcroft $[G, 2]$ -complex. One may add ($k \leq \infty$) 2-cells to X to obtain an aspherical two-complex Y if and only if there exists a free crossed G -module $L \triangleleft G$ (of weight k) which kills $\pi_2 X$.*

Proof. From the exactness of

$$\pi_2 X \rightarrow \pi_2 Y \rightarrow \pi_2(Y, X) \rightarrow \pi_1 X$$

we see that $\pi_2 Y = 0$ if and only if $\pi_2 X \rightarrow \pi_2 Y$ is zero and $\pi_2(Y, X) \rightarrow \pi_1 X$ is monic. But, by the theorem of J. H. C. Whitehead [W₂], $\pi_2(Y, X)$ is a free crossed $\pi_1 X = G$ module of weight k . \square

Thus, a counter example to the Whitehead conjecture would arise if there is a group G with a “large” free crossed module L as a subgroup (in the sense that L kills $\pi_2 X$ for some Cockcroft $[G, 2]$ -complex X).

4. Extending Adams’ theorem

In this section we prove Theorem 1 of the introduction.

LEMMA 4.1. *Let X be a $[G, 2]$ -complex and $\partial_2: C_2 \tilde{X} \rightarrow C_1 \tilde{X}$ be the second boundary operator considered as a left G -module homomorphism of free G -modules $C_i \tilde{X}$. Let N be a subgroup of G . Then $1 \otimes \partial_2: \mathbf{Z} \otimes_N C_2 \tilde{X} \rightarrow \mathbf{Z} \otimes_N C_1 \tilde{X}$ is a monomorphism if and only if $F(\pi_2 X) \subset K_N$ and $H_2 N = 0$. This happens iff $H_2(X_N) = 0$.*

Proof. Recall that R is the image of $\partial_2^X: C_2 \tilde{X} \rightarrow C_1 \tilde{X}$ and $IG = \text{im } \partial_1^X$. From the exact sequences

$$\pi_2 X \twoheadrightarrow C_2 \tilde{X} \twoheadrightarrow R, \quad R \twoheadrightarrow C_1 \tilde{X} \twoheadrightarrow IG, \quad \text{and} \quad IG \twoheadrightarrow \mathbf{Z}G \twoheadrightarrow \mathbf{Z},$$

we obtain the following sequences:

$$\begin{array}{ccccccc}
 & & & & \text{Tor}_1^N(\mathbf{Z}, IG) = H_2N & & \\
 & & & & \downarrow & & \\
 \text{Tor}_1^N(\mathbf{Z}, R) & \twoheadrightarrow & \mathbf{Z} \otimes_N \pi_2 X & \xrightarrow{\bar{h}_N} & \mathbf{Z} \otimes_N C_2 \tilde{X} & \xrightarrow{\alpha} & \mathbf{Z} \otimes_N R \\
 \parallel & & & & \searrow^{1 \otimes \partial_2} & \downarrow \beta & \\
 H_3N & & & & & & \mathbf{Z} \otimes_N C_1 \tilde{X} \\
 & & & & & & \downarrow \\
 & & & & & & H_1N = \text{Tor}_1^N(\mathbf{Z}, \mathbf{Z}) \twoheadrightarrow \mathbf{Z} \otimes_N IG \rightarrow \mathbf{Z} \otimes_N \mathbf{Z}G \twoheadrightarrow \mathbf{Z}
 \end{array} \tag{4.1}$$

Note that the triangles commute and that the vertical and horizontal sequences are exact. Clearly $\ker(1 \otimes \partial_2) = \alpha^{-1}H_2N$. So $H_2N = 0$ yields $\ker(1 \otimes \partial_2) = \ker \alpha = \text{im } \bar{h}_N = 0$, if $F(\pi_2 X) \subset K_N$. Similarly $\ker(1 \otimes \partial_2)$ is zero yields $\alpha^{-1}H_2N = 0$ which implies $\alpha^{-1}(0) = \text{im } \bar{h}_N = 0$ and $H_2N = 0$. \square

DEFINITION 4.2 [St₁]. A group G is called an E -group if H_1G is torsion free and the trivial G -module \mathbf{Z} has a projective G -resolution

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_2 \xrightarrow{\partial_2} P_1 \rightarrow P_0 \twoheadrightarrow \mathbf{Z}$$

such that the homomorphism $1 \otimes_G \partial_2: \mathbf{Z} \otimes_G P_2 \rightarrow \mathbf{Z} \otimes_G P_1$ is *injective*.

It follows that if G is an E -group, then $H_2(G) = 0$.

For a given group G , we define P_1G to be the *maximal perfect subgroup of the group G* . It is *uniquely defined* as the group generated by the family of all perfect subgroups of G . This subgroup is perfect because the group generated by any family of perfect subgroups is perfect. Because the normal closure of a perfect group is perfect, P_1G is a *normal subgroup* of G . P_1 is clearly a functor from the category of groups and homomorphisms to the category of perfect groups and homomorphisms.

There is another way to define P_1G . Let $\{G^\alpha \mid \alpha \text{ ordinal}\}$ denote the *derived series*: $G^\alpha = (G^{\alpha-1})'$ for α not a limit ordinal, $G^\alpha = \bigcap_{\beta < \alpha} G^\beta$ for α a limit ordinal. This sequence terminates [Dr, p. 20] at a perfect group, and since G^α contains any perfect subgroup of G , it terminates at P_1G .

The following theorem may have been known to J. F. Adams. It certainly follows from his techniques, when applied to the derived series of L . See [A] and [St₁].

THEOREM 4.3. *Suppose X is a non-aspherical $[G, 2]$ -complex such that $Y = X \cup \{e_\alpha^2 \mid \alpha \in \mathcal{A}\}$ is aspherical. Let $L = \ker \{\pi_1 X \rightarrow \pi_1 Y = H\}$. Then the maximal perfect subgroup P_1L of L is superperfect, kills $\pi_2 X$, and is non-trivial.*

Proof. We first observe that L is an E -group because

$$\begin{array}{ccc} 1 \otimes_L \partial_2^X : \mathbf{Z} \otimes_L C_2 \tilde{X} & \rightarrow & \mathbf{Z} \otimes_L C_1 \tilde{X} \\ \parallel & & \parallel \\ C_2 X_L & & C_1 X_L \end{array}$$

is monic (Lemma 4.1) and $H_1 L \cong \mathbf{Z}H^{|\mathcal{A}|}$. By Theorem A(i) of [St₁] each term of the derived series L^α of L is an E -group; hence P_1L is an E -group, therefore superperfect. In fact, the argument of the theorem cited above shows that

$$1 \otimes_{P_1L} \partial_2^X : \mathbf{Z} \otimes_{P_1L} C_2 \tilde{X} \rightarrow \mathbf{Z} \otimes_{P_1L} C_1 \tilde{X}$$

is monic, hence P_1L kills $\pi_2 X$ by Lemma 4.1. P_1L is non-trivial because, if it were trivial then P_1L kills $\pi_2 X$ implies $F(\pi_2 X) \subset K_{P_1L} = 0$. But $F(\pi_2 X) = 0$ iff $\pi_2 X = 0$, which contradicts the assumption that X was non-aspherical. \square

A group G has cohomological dimension $\leq n$ ($\text{cd } G \leq n$) if $H^i(G; M) = 0$ for all $i > n$ and all $\mathbf{Z}G$ -modules M . Equivalently $\text{cd } G \leq n$ if and only if the trivial G -module \mathbf{Z} has an $\mathbf{Z}G$ -projective resolution of length n :

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbf{Z} \rightarrow 0. \tag{*}$$

A group G has type FP (FL) if and only if G has a projective resolution (*) of finite length with each P_i (free) of finite rank.

If G has type FL , we define the (naive) Euler characteristic $\chi(G) = \sum_{i=0}^n (-1)^i \text{rank}_{\mathbf{Z}G} P_i$. In this case, let $b_i G = \text{rank}_{\mathbf{Z}} H_i G$. Standard arguments show that $\chi(G) = \sum_{i=0}^n (-1)^i b_i G$ as well.

THEOREM 4.4. *Let X be a $[G, 2]$ -complex and P be any superperfect normal subgroup of G such that P kills $\pi_2 X$. Then G/P has cohomological dimension ≤ 2 over \mathbf{Z} . Furthermore, if X is a minimal $[G, 2]_f$ -complex, then G/P has type FL and $\chi_{\min}(G, 2) = \chi(G/P)$.*

Proof. Consider the chain complex $C_*\tilde{X}$. In diagram 4.1 with $P=N$, we see that $H_1P = H_2P = 0$ together with $\bar{h}_p = 0$ (if and only if $K_P \supset F(\pi_2X)$) shows that the sequence

$$0 \rightarrow \mathbf{Z} \otimes_P C_2\tilde{X} \rightarrow \mathbf{Z} \otimes_P C_1\tilde{X} \rightarrow \mathbf{Z} \otimes_P \mathbf{Z}G \rightarrow \mathbf{Z} \rightarrow 0$$

is an exact sequence of $\mathbf{Z}G$ -modules. Because P is normal, $\mathbf{Z} \otimes_P \mathbf{Z}G \cong \mathbf{Z}(G/P)$ as (G/P) -modules. If X is a $[G, 2]_f$ -complex, then clearly G/P has type FL and $\chi(G/P) = \chi(X) = \chi_{\min}(G, 2)$. \square

After proving Theorem 4.4, we noticed that R. Strebel had proved a similar result for E -groups [St₁]. However, groups arising as the fundamental group of a subcomplex of an aspherical complex are *not* necessarily E -groups, as the second homology group is not necessarily zero (it is free abelian). The results do not imply one another, even though the basic trick is the same.

THEOREM 4.5. *Let X be any non-aspherical $[G, 2]$ -complex and $X < Y$, an aspherical $[H, 2]$ -complex. Then there exists a family of distinct non-trivial normal superperfect subgroups $P_i \triangleleft G$, $i \in I$, such that $\text{cd } G/P_i \leq 2$ for $i \in I$ and such that the smallest (normal) subgroup $P = \langle P_i \mid i \in I \rangle$ containing all P_i kills π_2X . Hence, P_1G kills π_2X .*

Proof. $X < X \cup Y^{(1)} = X \vee \bigvee S_\alpha^1 = \bar{X} < Y = \bar{X} \cup \{e_\beta^2\}$. By theorem 4.3 there is a superperfect normal subgroup $\bar{P} \neq 1$ in $G * F$, where F is a free group isomorphic to $\pi_1(\bigvee S_\alpha^1)$. Also \bar{P} kills $\pi_2\bar{X}$. Hence, by 4.4, $\text{cd } G * F / \bar{P} \leq 2$.

By Kuros' theorem, we have $\bar{P} = *_{u}(uGu^{-1} \cap \bar{P})$ for certain $u \in G * F$. The group \bar{P} is superperfect implies that each $uGu^{-1} \cap \bar{P}$ is superperfect. Let $P_u = u^{-1}(uGu^{-1} \cap \bar{P})u$. Each P_u is a superperfect normal subgroup of G . The group $\bar{P} \neq 1$ implies that *some* of the $P_u \neq 1$. Choose the family $\{P_i\}$ to be those $P_u \neq 1$.

Consider the following diagram:

$$\begin{array}{ccc} G * F & \longrightarrow & G * F / \bar{P} \\ \uparrow & & \uparrow \\ uGu^{-1} & \longrightarrow & uGu^{-1} / \bar{P} \cap uGu^{-1} \end{array}$$

Thus $uGu^{-1} / \bar{P} \cap uGu^{-1} \approx G/P_u$ has cohomological dimension ≤ 2 for each u . Note that if *any* $\bar{P} \cap uGu^{-1} = 1$, then G itself has cohomological dimension ≤ 2 .

Let $F_G(M)$ denote the 2-sided ideal in $\mathbf{Z}G$ generated by the coordinates of elements of the G -module $M \subset (\mathbf{Z}G)^\alpha$. We know that $F_{G * F}(\pi_2\bar{X}) \subset K_{\bar{P}} = \mathbf{Z}(G * F) \cdot I_{\bar{P}}$, where $\pi_2\bar{X} \cong \mathbf{Z}(G * F) \otimes_G \pi_2X$, by Theorem 4.3. It follows that

$F_{G * F}(\pi_2 \bar{X}) = F_{G * F}(\pi_2 X)$, with $\pi_2 X$ considered as a $G * F$ -module via the projection $\eta : G * F \rightarrow G$. Notice that $P = \eta(\bar{P})$. The surjection $\mathbf{Z}\eta : \mathbf{Z}(G * F) \rightarrow \mathbf{Z}(G)$ clearly carries $K_{\bar{P}} = \mathbf{Z}(G * F) \cdot \bar{I}\bar{P}$ onto $K_P = \mathbf{Z}(G) \cdot IP$. Also $F_{G * F}(\pi_2 \bar{X}) = F_{G * F}(\pi_2 X)$ is carried onto $F_G(\pi_2 X)$. Thus $F_{G * F}(\pi_2 X) \subset K_{\bar{P}}$ implies $F_G(\pi_2 X) \subset K_P$ and we are done. \square

For the next corollary let X be a $[G, 2]$ -complex which is not aspherical, but which is a subcomplex of an aspherical $[H, 2]$ -complex. Thus, there must exist a non-trivial superperfect subgroup $P \triangleleft G$ such that $\text{cd } G/P \leq 2$. Because groups of finite cohomological dimension are torsion free, we have

COROLLARY 4.6. *Any element $g \in G$ such that $g^n \in P$ ($n \geq 1$) must be in P . In particular, the torsion of G is contained in P . \square*

In [B, p. 122], R. Bieri shows that the center of a non-abelian group of cohomology dimension ≤ 2 is cyclic. The exact sequence $P \twoheadrightarrow G \twoheadrightarrow \bar{G} = G/P$ induces a monomorphism

$$\mathfrak{Z}G / (P \cap \mathfrak{Z}G) \twoheadrightarrow \mathfrak{Z}(G/P)$$

($\mathfrak{Z}G$ is the center of G). If G/P is non-abelian, then $\mathfrak{Z}(G/P)$ is 0 or \mathbf{Z} ; if G is finitely generated, $\mathfrak{Z}(G/P) = 0, \mathbf{Z}$, or $\mathbf{Z} \oplus \mathbf{Z}$ (this last occurs only if $G/P = \mathbf{Z} \oplus \mathbf{Z}$ is abelian).

COROLLARY 4.7. *Let G be a finitely presented group, X be a minimal $[G, 2]_f$ -complex, and P be a superperfect normal subgroup of G with the cohomological dimension of $G/P \leq 2$. Then $\mathfrak{Z}G / (P \cap \mathfrak{Z}G) = 0, \mathbf{Z}$, or $\mathbf{Z} \oplus \mathbf{Z}$, with this last group occurring only if $\bar{G} = G/P$ is abelian. If $\text{def } G \geq 1$ and P doesn't kill $\pi_2 X$ or if P kills $\pi_2 X$ and $\text{def } G \neq 1$, then $\mathfrak{Z}G \subset P$.*

Proof. First, we assume that P kills $\pi_2 X$ and that $\text{def } G \neq 1$. Then, by Theorem 4.4, \bar{G} has type FL and $\chi(\bar{G}) = \chi(X) = 1 - \text{def } G$. The deficiency of $G \neq 1$ implies that $\chi(\bar{G}) \neq 0$. Then, by corollary 3.6 of [S], we see that $\mathfrak{Z}(\bar{G})$ is trivial. Hence P contains $\mathfrak{Z}(\bar{G})$.

We assume that $\text{def } G \geq 1$ and that P does not kill $\pi_2 X$. Let $R_i = \ker \{ \mathbf{Z} \otimes_P C_i \tilde{X} \rightarrow \mathbf{Z} \otimes_P C_{i-1} \tilde{X} \}$ ($i = 1, 2$). The cohomological dimension of $\bar{G} \leq 2$ implies that R_1 is a projective \bar{G} -module. Because P is superperfect, we have an exact sequence

$$R_2 \twoheadrightarrow \mathbf{Z} \otimes_P C_2 \tilde{X} \twoheadrightarrow R_1.$$

This shows that R_1 and R_2 are both finitely generated projective \bar{G} -modules. Now at this point in the proof, we must use the Euler characteristic of a group defined by J. Stallings in [S]. The rank of a finitely generated projective \bar{G} -module Q is a certain element rQ in the free abelian group T on the set of conjugacy classes of \bar{G} . Then ρQ is defined to be the coefficient of [1] in rQ . Accordingly, $\chi(\bar{G}) = \bigoplus_{i=0}^2 (-1)^i \cdot \rho(\mathbf{Z} \otimes_P C_i \tilde{X}) - \rho R_2 = \chi_{\min}(G, 2) - \rho R_2 = 1 - \text{def } G - \rho R_2$. It follows from proposition 1 of [DV] that $\rho R_2 \geq 0$ and $\rho R_2 = 0$ iff $R_2 = 0$. Now P does not kill $\pi_2 X$ implies that $R_2 \neq 0$. Thus $\rho R_2 > 0$. Hence the deficiency of $G \geq 1$ implies that $\chi(\bar{G}) < 0$ and the result again follows from corollary 3.6 of [S]. \square

We would like to thank the referee for simplifying the hypotheses of 4.7.

One may show that all the higher centers $\mathfrak{Z}^n G \subset P$ as well. To see that $\mathfrak{Z}^2 G \subset P$, notice that the hypotheses of 4.7 imply that $\mathfrak{Z} \bar{G} = 1$. Then the following diagram commutes:

$$\begin{array}{ccccc}
 \mathfrak{Z}G & \twoheadrightarrow & P & \twoheadrightarrow & P/\mathfrak{Z}G \\
 \parallel & & \downarrow & & \downarrow \\
 \mathfrak{Z}G & \twoheadrightarrow & G & \xrightarrow{\eta} & G/\mathfrak{Z}G \\
 & & \downarrow & & \downarrow \\
 & & \bar{G} & \xlongequal{\quad} & \bar{G}
 \end{array}$$

Now $\mathfrak{Z} \bar{G} = 1$ implies that $\mathfrak{Z}(G/\mathfrak{Z}G) \subset P/\mathfrak{Z}G$ and hence that $\mathfrak{Z}^2 G = \eta^{-1} \mathfrak{Z}(G/\mathfrak{Z}G) \subset P$.

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