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On J. H. C. Whitehead's aspherical question I

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Abstract. A connected, finite two-dimensional CW-complex with fundamental group isomorphic to G is called a $[G, 2]_f$-complex. Let $L \lhd G$ be a normal subgroup of G. L has weight k if and only if k is the smallest integer such that there exists $\{l_1, \ldots, l_k\} \subset L$ such that L is the normal closure in G of $\{l_1, \ldots, l_k\}$. We prove that a $[G, 2]_f$-complex X may be embedded as a subcomplex of an aspherical complex $Y = X \cup \{e_1^2, \ldots, e_k^2\}$ if and only if G has a normal subgroup L of weight k such that $H = G/L$ is at most two-dimensional and $\operatorname{def} G = \operatorname{def} H + k$. Also, if X is a *non-aspherical* $[G, 2]_f$-subcomplex of an aspherical 2-complex, then there exists a non-trivial superperfect normal subgroup P such that G/P has cohomological dimension ≤ 2. In this case, any torsion in G must be in P.

0. Introduction

A $[G, 2]_{(f)}$-*complex* X is any (finite) connected two dimensional CW-complex with fundamental group isomorphic to G. Sometimes we will abuse the notation and say $X \in [G, 2]_{(f)}$. Let X be a connected subcomplex of an $[H, 2]$-complex Y. J. H. C. Whitehead's question is this: if Y is aspherical, is X also aspherical? [W_1, p. 428].

The question seems very hard. We say that a group G satisfies the (finite) *Whitehead condition* ($G \in WC_{(f)}$) if any $[G, 2]$-complex X, which is the subcomplex of an aspherical (finite) 2-complex, is aspherical. Thus $G \in WC_{(f)}$ iff for any $[G, 2]_{(f)}$-complex X, either X is aspherical or, if not, then no $[H, 2]_{(f)}$-complex $Y > X$ is aspherical. The philosophy of this paper is to isolate properties of a group G which imply that $G \in WC$ or WC_f.

There are a number of results in this direction. W. Cockcroft [C, Theorem 2] showed that if G is one-relator group, then G has WC. One crucial observation he made was: *Let X be a $[G, 2]$-complex such that the Hurewicz homomorphism $h_2 : \pi_2 X \to H_2 X$ is non-zero. Then no* 2-complex $Y > X$ *is aspherical.* It follows that any group G which admits a $[G, 2]$-complex X which is the subcomplex of an aspherical 2-complex has $H_2 G$ free abelian. (Here $H_2 G$ means the homology of G with coefficients in the trivial module $\mathbf{Z}$.)

Let $\chi_{\min}(G, 2) = \min\{\chi(X) \mid X \text{ is a } [G, 2]_f\text{-complex}\}$. A complex X whose Euler characteristic $\chi(X)$ is minimal is called a *minimal* $[G, 2]_f$-complex. *For simplicity, all* $[G, 2]$-*complexes will have a single vertex, and this will be the base* *point for all homotopy groups.* Any $[G, 2]$-complex has the (simple) homotopy type of such a complex.

We observe for later use that WC_f can be proved or disproved for a particular group G as follows: choose your favorite minimal $[G, 2]_f$-complex X and, if it is not aspherical, check that no $[H, 2]_f$-complex containing X is aspherical (see Lemma 1.4).

It follows from Cockcroft's result above that if $X < Y$, where $Y \in [H, 2]_f$ is aspherical, then X is a *minimal* $[G, 2]_f$-complex. To see this, we simply observe that if X is any non-minimal $[G, 2]_f$-complex, then $h_2: \pi_2 X \to H_2 X$ is *not* zero. For let Y be a minimal $[G, 2]_f$-complex, with $\chi(Y) < \chi(X)$. Let $\Sigma_2(-) = \mathrm{im}\{h_2: \pi_2(-) \to H_2(-)\}$ be the image of the Hurewicz homomorphism. Then, by a result of H. Hopf, $H_2 G \cong H_2 X/\Sigma_2 X \cong H_2 Y/\Sigma_2 Y$. Because $H_2 X$ and $H_2 Y$ are finitely generated free abelian groups with $\mathrm{rank}_Z H_2 Y < \mathrm{rank}_Z H_2 X$, we must have $\Sigma_2 X \neq 0$.

J. F. Adams' approach [A, p. 483] was to assume that a non-aspherical $X < Y = X \cup \{e_\alpha^2 \mid \alpha \in \mathcal{A}\}$, with Y aspherical, and to study $L = \ker\{\pi_1 X \to \pi_1 Y\}$. Adams proved that $H_1 L$ is a free abelian group and L is not transfinite metabelian; i.e., L has a non-trivial (normal) subgroup P which is perfect ($H_1 P = P^{ab} = 0$). This shows that *any* solvable group has WC.

In [Co], J. Cohen points out that Adams' perfect subgroup $P < L$ is actually *superperfect*; i.e., $H_2 P = 0$. He also shows [Co, Theorem 3] that if G a group of cohomological dimension 3 and type *FL* (that is, Z has a finite resolution by finitely generated free G-modules) such that $H_3 G = 0$, then G has WC_f.

In [GR, Theorem 4], M. Gutierrez and J. Ratcliffe show that if $X < Y (Y \in [H, 2]_f)$ which is aspherical, then X is aspherical if and only if the cohomological dimension of $G \leq 2$ and G has type *FL*.

In [H], J. Howie shows that any torsion element $x \in G (x^n = 1)$ is contained in a *finitely generated* perfect subgroup of L. We show that, in fact, all the torsion of G is contained in Adams' superperfect subgroup.

Finally, in his thesis [Be] W. Beckmann shows that locally finite groups have WC.

Specifically, we show the following

THEOREM 1. *Let X be a non-aspherical $[G, 2]$-subcomplex of an aspherical 2-complex. Then there is a nontrivial superperfect normal subgroup P (Adams) such that G/P has cohomological dimension ≤ 2.*

COROLLARY. *Any torsion in G must be in P.*

As a second result we characterize when one may add 2-cells to a $[G, 2]_f$-complex to obtain an aspherical complex. Let F_n denote a free group of rank n.

THEOREM 2. *Let X be a minimal $[G, 2]_f$-complex. One may add n 1-cells and k 2-cells to X to obtain an aspherical 2-complex if and only if $G * F_n$ has a normal subgroup L which is a free-crossed G-module of weight k* (see 3.2) *such that* (1) *there is an aspherical $[G/L, 2]_f$-complex and* (2) $\operatorname{def} G + n = \operatorname{def}(G/L) + k$.

COROLLARY. *Let X be a minimal $[G, 2]_f$-complex. One may add two-cells to X to obtain a finite contractible space if and only if weight $G = \operatorname{def} G$.*

The groups in the corollary are of interest because they are (higher) knot and link groups, according to a theorem of M. Kervaire [K]. These groups are all E-*groups* in the sense of [St_1], [St_2] and [B]. From this it follows that the derived series of G has many interesting properties; such as, each element G^α in the derived series for G is an E-group, and the derived length of G is severely restricted.

The paper is organized as follows. In section one we study complexes X for which the Hurewicz map $h_2 : \pi_2 X \to H_2 X$ is zero and reprove a crucial lemma of W. Cockcroft and R. Swan about minimal aspherical complexes. In section two we study necessary and sufficient conditions for the inclusion $X < Y$ to induce the zero map on the second homotopy groups. In section three we prove Theorem 2 and in section four, Theorem 1. We defer examples and applications of these results to a later paper.

To fix notation, let G be a group and let $\mathbf{Z}G$ be the integral group ring of G. Let IG denôte the augmentation ideal, the kernel of the map $\epsilon : \mathbf{Z}G \to \mathbf{Z}$.

1. Cockcroft complexes and the Cockcroft–Swan lemma

DEFINITION 1.1. A connected CW-complex X is called *Cockcroft* if and only if the Hurewicz homomorphism $h : \pi_2 X \to H_2 X$ is trivial. A group G is *Cockcroft* if and only if some $[G, 2]$-complex is Cockcroft.

Note that any non-minimal $[G, 2]_f$-complex is not Cockcroft. It follows that any group G having $H_2(G; Z)$ *not* free abelian is not Cockcroft. It was shown in [C, lemma 1] that for any non-Cockcroft $[G, 2]$-complex X, *any* $Y = X \cup \{e_\alpha^2\}$ has $\pi_2 X \to \pi_2 Y$ not zero. It follows that non-*Cockcroft groups have WC.*

EXAMPLE 1.2. Any finitely generated one-relator group is Cockcroft. Let G

be the group presented by $\{x_1, \ldots, x_n; r\}$ and let X be the model associated with the presentation. Write $r = Q^q$, where Q is not a proper power in the free group $F(x_1, \ldots, x_n)$. By a theorem of Lyndon [L], $\pi_2 X \cong ZG(\bar{Q} - 1)$ as a left G-module, where $\bar{Q}$ is the image in G of Q under the natural projection $F \to G$. As the Hurewicz map $h : \pi_2 X \to H_2 X$ is given by restricting the augmentation $\epsilon : C_2 \tilde{X} = ZG \to C_2 X = Z$, we see that $h = 0$. Note that if X is a subcomplex of an aspherical two-complex Y, then X must be a Cockcroft $[G, 2]$-complex. This follows because if $[X \vee \bigvee S^1_\beta) \cup \{e^2_\alpha\} = Y$ is aspherical, then the Hurewicz homomorphism $\pi_2(X \vee \bigvee S^1_\beta) \to H_2(X \vee \bigvee S^1_\beta)$ is zero [C, lemma 1]. That X is Cockcroft is clear from the commutative diagram:

$$\begin{array}{ccc} \pi_2 X & \longrightarrow & H_2 X \\ \downarrow & & \downarrow \\ \pi_2(X \vee \bigvee S^1_\beta) & \rightarrow & H_2(X \vee \bigvee S^1_\beta). \end{array}$$

Observe that if X and X' are minimal $[G, 2]_f$-complexes, X is Cockcroft iff X is.

The following theorem characterizes in several different ways the property that X is Cockcroft.

PROPOSITION 1.3. *Let X be a $[G, 2]$-complex. The following are equivalent:*

(a) *$h_2 : \pi_2 X \to H_2 X$ is zero.*
(b) *The Hopf epimorphism $H_2 X \to H_2 G$ is injective.*
(c) *The natural inclusion $H_3 G \to Z \otimes_G \pi_2 X$ is surjective.*

Proof. This follows from the exact sequence of [D], which is just a fancy rewrite of two theorems of H. Hopf:

$$0 \to H_3 G \to Z \otimes_G \pi_2 X \xrightarrow{\bar{h}_2} H_2 X \to H_2 G \to 0,$$

where $\overline{h_2}$ is induced by h_2. $\square$

We now prove the following key lemma of Cockcroft and Swan [CS, p. 197].

LEMMA 1.4. *Let X and X' be minimal $[G, 2]_f$-complexes. Then X is aspherical iff X' is.*

Proof. As X and X' have the same Euler characteristic, it follows from Schanuel's lemma that $\pi_2 X \oplus ZG^n \cong \pi_2 X' \oplus ZG^n$ for some integer $n > 0$. Then

$\pi_2 X = 0$ implies that

$$\pi_2 X' \rightarrowtail ZG^n \twoheadrightarrow ZG^n$$

is exact, which yields that $\pi_2 X' = 0$ by a theorem of I. Kaplansky. $\square$

The *proof* of Lemma 1.4 clearly breaks down if X is an infinite, but Cockcroft, $[G, 2]$-complex. *We conjecture that the lemma is still true for such complexes.*

If G is finitely presented, then one may show from Lemma 1.4 that either *all* minimal $[G, 2]_f$-complexes are subcomplexes of finite aspherical 2-complexes or *none* are.

LEMMA 1.5. *Let G be a finitely presented group. Let X, X' be minimal $[G, 2]_f$-complexes and $Y = (X \vee \bigvee_{i=1}^m S_i^1) \cup \{e_1^2, \ldots, e_n^2\}$ be aspherical. Then one may add m one-cells and n two-cells to X' to obtain an aspherical complex.* $\square$

2. Killing $\pi_2 X \to \pi_2 Y$ for X a subcomplex of Y

DEFINITION 2.1. For any subgroup $A < G$, let $K_A = \mathbf{Z}G \cdot IA$ be the left ideal in $\mathbf{Z}G$ generated by $\{a - 1 \mid a \in A\}$. Note that if A is a *normal* subgroup of G, then K_A is a *two-sided* ideal. In any case, $K_A = \ker \{\mathbf{Z}G \to \mathbf{Z}(G/A)\}$ induced by the coset function $G \to G/A$.

DEFINITIONS 2.2. Let M be any (left) submodule of a free G-module. The *Fox ideal of M, $F(M)$*, is the two-sided ideal in $\mathbf{Z}G$ generated by the coordinates of each element (of a generating set) of M. We say that a *subgroup $A < G$ kills M* if the Fox ideal $F(M)$ is contained in the kernel K_A. Note that $F(M)$ is independent of any chosen basis.

EXAMPLE. For a $[G, 2]$-complex X, G itself kills $\pi_2 X$ iff $F(\pi_2 X) \subset K_G = IG$. This happens iff the Hurewicz map $h_2 : \pi_2 X \to H_2 X$ is zero.

Now let $X \in [G, 2]$ be a subcomplex of an $[H, 2]$-complex Y. Let $\iota : X \to Y$ denote the inclusion map and $L = \ker \pi_1(\iota)$. If $C_* \tilde{X}$ is the cellular chain complex (considered as left G-modules) of the universal cover $\tilde{X}$ of X, let $R_X = \ker \{\partial_1 : C_1 \tilde{X} \to C_0 \tilde{X} = \mathbf{Z}G\}$ be a so-called relation module for G.

THEOREM 2.3. *The following are equivalent:*

(1) *$i_\# : \pi_2 X \to \pi_2 Y$ is zero,*
(2) *the Fox ideal $F(\pi_2 X) \subset K_L$ (L kills $\pi_2 X$),*

(3) *the Hurewicz homomorphism* $h_L : \pi_2 X \to H_2 X_L$ *is zero, where* X_L *is a covering of* X *corresponding to the subgroup* L.

(4) *The natural surjection* $\bar{\partial}_2 : C_2\tilde{X} \twoheadrightarrow R_X$ *induces an isomorphism* $\mathbf{Z}\otimes_L C_2\tilde{X} \to Z\otimes_L R$ *of free* G/L*-modules.*

Proof. Let $(\mathbf{Z}H)^{|\mathcal{A}|}$ denote $\oplus_{\alpha\in\mathcal{A}} (\mathbf{Z}H)_\alpha$. Consider the universal covering $\tilde{X}$ of X, the cellular chain complex $C_*(\tilde{X})$ of $\tilde{X}$ (viewed as free left G-modules and homomorphisms), and the cellular chain complexes $C_*X_L = \mathbf{Z}\otimes_L C_*\tilde{X} \to C_*\tilde{Y}$ (viewed as free G/L and H modules, respectively). Let $N = G/L$ denote the image of $\pi_1(i) : G \to H$ and $\eta : G \to G/L$ be the natural map. Also let

$$Y = \Big(X \vee \bigvee_{\beta\in\mathcal{B}} S^1_\beta\Big) \cup \{e^2_\alpha\}_{\alpha\in\mathcal{A}}.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_2 X \arrow[r, tail] \arrow[ddd, "h_L"] & C_2\tilde{X} \arrow[r, "\partial_2^X"] \arrow[d] & C_1\tilde{X} \arrow[r, "\partial_1^X"] \arrow[d] & \mathbf{Z}G \arrow[r] \arrow[d] & \mathbf{Z} \arrow[d, equal] \\
& \mathbf{Z}\otimes_L C_2\tilde{X} \arrow[r, "1\otimes\partial_2^X"] \arrow[dd, equal] & \mathbf{Z}\otimes_L C_1\tilde{X} \arrow[r, "1\otimes\partial_1^X"] \arrow[dd, equal] & \mathbf{Z}\otimes_L \mathbf{Z}G \arrow[r] \arrow[d, equal] & \mathbf{Z} \arrow[dd, equal] \\
& & & \mathbf{Z}N \arrow[d, equal] & \\
H_2X_L \arrow[r, tail] \arrow[d, tail] & C_2X_L \arrow[r, "\partial_2^L"] \arrow[d, tail] & C_1X_L \arrow[r, "\partial_1^L"] \arrow[d, tail] & C_0X_L \arrow[r] \arrow[d, tail] & \mathbf{Z} \arrow[d, equal] \\
\pi_2 Y \arrow[r, tail] & C_2\tilde{Y} \arrow[r, "\partial_2^Y"] \arrow[d, equal] & C_1\tilde{Y} \arrow[r, "\partial_1^Y"] \arrow[d, equal] & \mathbf{Z}H \arrow[r] & \mathbf{Z} \\
& \mathbf{Z}H\otimes_N C_2X_L \oplus \mathbf{Z}H^{|\mathcal{A}|} & \mathbf{Z}H\otimes_N C_1X_L \oplus \mathbf{Z}H^{|\mathcal{B}|} & &
\end{tikzcd}$$

The chain map $i_* : C_2\tilde{X} = (\mathbf{Z}G)^m \to C_2\tilde{Y}$ factors as

$$\mathbf{Z}G^m = C_2\tilde{X} \twoheadrightarrow C_2X_L = \mathbf{Z}N^m \rightarrowtail \mathbf{Z}H^m \oplus \mathbf{Z}H^{|\mathcal{A}|} = C_2\tilde{Y}.$$

with the first map being $\oplus\mathbf{Z}\eta : \mathbf{Z}G^m \to \mathbf{Z}N^m$. Hence, the kernel of $i_\# : \pi_2 X \to \pi_2 Y$ is $\pi_2 X \cap (K_L)^m$. Also, $H_2X_L \rightarrowtail \pi_2 Y$ (it is a direct summand as Z-modules).

So $i_\#: \pi_2 X \to \pi_2 Y$ is zero if and only if $h_L: \pi_2 X = \pi_2 X_L \to H_2 X_L$ is zero. This happens if and only if $\pi_2 X \subset K_L^m$, which in turn is true if and only if $F(\pi_2 X) \subset K_L$.

In order to prove (4)$\Leftrightarrow$(1), consider the following exact sequences (see [D]).

$$\begin{tikzcd}
\mathbf{Z}\otimes_L \pi_2 X \arrow[r, "\bar{h}_L"] \arrow[d, equal] & H_2 X_L \arrow[r, two heads] \arrow[d, tail] & H_2 L \arrow[d, tail] \\
\mathbf{Z}\otimes_L \pi_2 X \arrow[r] & \mathbf{Z}\otimes_L C_2\tilde{X} \arrow[r, two heads] & \mathbf{Z}\otimes_L R.
\end{tikzcd}$$

It is easily shown that both squares commute. Also

$$\pi_2 X \xrightarrow{h_L} H_2 X_L \quad \text{factors as} \quad \pi_2 X \longrightarrow \mathbf{Z}\otimes_L \pi_2 X \xrightarrow{\bar{h}_L} H_2 X_L.$$

Hence

$$\begin{aligned}
\pi_2 X \xrightarrow{0} \pi_2 Y \Leftrightarrow \pi_2 X \xrightarrow{0} H_2 X_L \\
\Leftrightarrow \mathbf{Z}\otimes_L \pi_2 X \xrightarrow{0} H_2 X_L \\
\Leftrightarrow \mathbf{Z}\otimes_L \pi_2 X \xrightarrow{0} \mathbf{Z}\otimes_L C_2\tilde{X}. \quad \square
\end{aligned}$$

Note 2.4. The proof of Theorem 2.3 shows that conditions (2), (3), and (4) are equivalent for any (not necessarily normal) subgroup $L \leq G$, provided we restate (4) as an isomorphism of (not necessarily free) G-modules. In fact, it is clear that (2)–(4) are *hereditary* in the sense that, if they are true for some subgroup $L \leq G$, then they hold for any subgroup $M \leq G$ containing L.

3. Subcomplexes of aspherical complexes

DEFINITION 3.1. Let G be a (finitely presented) group. G is *at most* (*finitely*) *two-dimensional* if and only if there exists an aspherical $[G,2]_{(f)}$-complex.

For any group G and element $g \in G$, denote the image of g in G^{ab} by $\bar{g}$.

DEFINITION 3.2. Let L be a normal subgroup of G ($L \lhd G$) with quotient

H. L is said to be a *free crossed G-module of weight k* ($k \le \infty$) if and only if there exists elements $\{g_1, \ldots, g_k\} \subset L$ such that L is the normal closure $\langle\langle g_1, \ldots, g_k \rangle\rangle_G$ of $\{g_i\}$ in G, H_1L is a free H-module with basis $\{\bar{g}_1, \ldots, \bar{g}_k\}$, and $H_2L = 0$.

It is a very nice theorem of J. Ratcliffe [R, Theorem 2.2] that this is equivalent to the usual definition of a free crossed G-module (in this setting). The normal generators $\{g_1, \ldots, g_k\}$ of L are called a *basis* for the free crossed module L.

An interesting special case is when G is a free crossed G-*module* of weight k. Examples of weight 1 self free crossed modules are knot groups. By a theorem of M. Kervaire [K], any finitely presented self free crossed module of weight k is the fundamental group of a k-link of 3-spheres embedded in S^5.

Note that if L is a free crossed G-module of weight k, then L is a free crossed L-module of weight $k \cdot |G/L|$. Furthermore, if L is a free crossed L-module of weight k and H is any group, then, for $G = L * H$, the normal closure N of L (in G) is a free crossed G-module of weight k. To see this, notice that N is the free product $\ast_{h \in H} hLh^{-1}$ in G and that $1 \to N \to G \to H \to 1$ is a split extension. $H_1L \cong \mathbf{Z}^k$, so $H_1N \cong \mathbf{Z}H^k$; $H_2L = 0$ implies $H_2N = 0$. If the normal closure of $\{l_1, \ldots, l_k\}$ in L is equal to L, then $\langle\langle \{l_1, \ldots, l_k\} \rangle\rangle_G = N$.

As another example, one may show the following proposition.

PROPOSITION 3.3. *Let G be a 1-relator group with presentation $\{x_1, \ldots, x_n; Q^q\}$, where Q is not a proper power. Let $e_{x_i}(Q)$ denote the exponent sum of Q with respect to x_i. Then G is a free crossed G-module if and only if* $[q = 1$ *and* $E = \gcd\{e_{x_i}(Q)\} = 1]$ *if and only if* $[H_1G \cong Z^{n-1}]$. □

Note. Let G be a finitely generated 1-relator group. Any two 1-relator presentations of G have the same number of generators. This follows because the models associated with both presentations are Cockcroft (Example 1.2) and are therefore minimal.

Let $\operatorname{def} G$ denote the *deficiency* of the finitely presented group G. The following theorem characterizes when one may add finitely many one-cells and two-cells to a $[G, 2]_f$-complex X to obtain an aspherical 2-complex. Let F_l denote a free group of rank l.

THEOREM 3.4. *Let X be a minimal $[G, 2]_f$-complex. One may add l 1-cells and k 2-cells to X to obtain an aspherical two-complex Y if and only if* (1) *there exists a normal subgroup $L < G * F_l$ which is a free crossed $G * F_l$-module of weight k having $H = (G * F_l)/L$ at most finitely two-dimensional and* (2) $\operatorname{def} G + l = \operatorname{def} H + k$.

Note that this theorem is true whether X is aspherical or not. For another equivalent condition see Theorem 3.6.

Proof. ($\Rightarrow$) Suppose $X < Y = (X \vee \bigvee_{i=1}^{l} S_i^1) \cup \{e_1^2, \ldots, e_k^2\}$ and Y is aspherical. Consider the homotopy sequence for the pair $(Y, \bar{X})$, where $\bar{X} = X \cup Y^{(1)}$:

$$\begin{array}{ccccccccccccccc}
\pi_3(Y) & \to & \pi_3(Y,\bar{X}) & \to & \pi_2(\bar{X}) & \to & \pi_2(Y) & \to & \pi_2(Y,\bar{X}) & \xrightarrow{\partial} & \pi_1(\bar{X}) & \to & \pi_1(Y) & \to & 0. \\
\| & & & & & & \| & & & & \| & & \| & & \\
0 & & & & & & 0 & & & & G * F_l & & H & &
\end{array}$$

The group $\pi_2(Y, \bar{X})$ is a free crossed $\pi_1(\bar{X})$-module on the characteristic maps for the k added two cells. We let $L = \operatorname{im} \partial$. Then $H = G/L$ is at most 2-dimensional. Because $\chi(X) = \chi_{\min}(G, 2)$, $\chi(Y) = \chi_{\min}(H, 2)$ and $\chi(X) + k - l = \chi(Y)$ we have (as $\chi(X) = 1 - \operatorname{def} G$) $k + \operatorname{def} H = \operatorname{def} G + l$.

($\Leftarrow$) Let X be a minimal $[G, 2]_f$-complex and identify $\pi_1 X$ with G. Let $\{g_1, \ldots, g_k\}$ be a basis for $L < G * F_l$ as a free crossed $G * F_l$-module. Attach $e_1^2, \ldots, e_k^2$ to $\bar{X} = X \vee \bigvee_{i=1}^{l} S_i^1$ using maps $\alpha_i : S_i^1 \to \bar{X}^{(1)}$ which represent $g_i \in G * F_l$ $(i = 1, \ldots, k)$. Then $X < Y = \bar{X} \cup \{e_1^2, \ldots, e_k^2\}$ has $\pi_1 Y = H$ at most a finitely two-dimensional group. Thus there is an $[H, 2]_f$-complex W which is aspherical. Because $\operatorname{def} H = \operatorname{def} G - k + l = 1 - \chi(Y) = 1 - \chi(W)$, we have $\chi(Y) = \chi(W) = \chi_{\min}(H, 2)$. Therefore, Y is aspherical by Lemma 1.4. □

Note. One sees from the proof that really only the following was used:
$G * F_l$ has a normal subgroup L of weight k over $G * F_l$ such that
(a) $G * F_l / L$ is at most finitely 2-dimensional and
(b) $\operatorname{def}[(G * F_l)/L] + k = \operatorname{def} G + l$.
That L is a free crossed $G * F_l$-module of weight k is a *consequence* of the above statement.

COROLLARY 3.5. *Let X be a minimal $[G, 2]_f$-complex. One may add k two-cells to X to obtain a contractible space if and only if* $\det G = k =$ *weight G. This is true iff G is a free crossed G-module (= higher dimensional link group) of weight k which is Cockcroft.*

Proof. The first statement follows by specializing Theorem 3.1 to $l = 0$ and $H = \{1\}$. To see the second, we observe that a group G with $H_1 G \cong \mathbf{Z}^k$ and which is Cockcroft has $H_2 X \cong H_2 G \cong \mathbf{Z}^{k - \operatorname{def} G}$. So $H_2 G = 0$ implies $\operatorname{def} G = k$. A similar argument yields the converse. □

We would ask, more generally, *what does the fundamental group G of a subcomplex X of a finite contractible space Y look like*? By the above corollary, we see that $G*F_l$ is a higher dimensional link group with $\operatorname{def}(G*F_l)=\operatorname{def} G+l=$ weight $(G*F_l)$. Does this imply that def $G=$ weight G? It is easy to see that such groups are E-groups (see [B], 123–130, for facts about E-groups).

We also have (using the same geometric techniques)

THEOREM 3.6. *Let X be a Cockcroft $[G,2]$-complex. One may add $(k\leq\infty)$ 2-cells to X to obtain an aspherical two-complex Y if and only if there exists a free crossed G-module $L\lhd G$ (of weight k) which kills π_2X.*

Proof. From the exactness of

$$\pi_2X\to\pi_2Y\to\pi_2(Y,X)\to\pi_1X$$

we see that $\pi_2Y=0$ if and only if $\pi_2X\to\pi_2Y$ is zero and $\pi_2(Y,X)\to\pi_1X$ is monic. But, by the theorem of J. H. C. Whitehead [W_2], $\pi_2(Y,X)$ is a free crossed $\pi_1X=G$ module of weight k. □

Thus, a counter example to the Whitehead conjecture would arise if there is a group G with a "large" free crossed module L as a subgroup (in the sense that L kills π_2X for some Cockcroft $[G,2]$-complex X).

4. Extending Adams' theorem

In this section we prove Theorem 1 of the introduction.

LEMMA 4.1. *Let X be a $[G,2]$-complex and $\partial_2: C_2\tilde{X}\to C_1\tilde{X}$ be the second boundary operator considered as a left G-module homomorphism of free G-modules $C_i\tilde{X}$. Let N be a subgroup of G. Then $1\otimes\partial_2:\mathbf{Z}\otimes_N C_2\tilde{X}\to\mathbf{Z}\otimes_N C_1\tilde{X}$ is a monomorphism if and only if $F(\pi_2X)\subset K_N$ and $H_2N=0$. This happens iff $H_2(X_N)=0$.*

Proof. Recall that R is the image of $\partial_2^X: C_2\tilde{X}\to C_1\tilde{X}$ and $IG=\operatorname{im}\partial_1^X$. From the exact sequences

$$\pi_2X\rightarrowtail C_2\tilde{X}\twoheadrightarrow R,\qquad R\rightarrowtail C_1\tilde{X}\twoheadrightarrow IG,\quad\text{and}\quad IG\rightarrowtail\mathbf{Z}G\twoheadrightarrow\mathbf{Z},$$

we obtain the following sequences:

$$\begin{tikzcd}
 & & & \operatorname{Tor}_1^N(\mathbf{Z}, IG) = H_2N \arrow[d, tail] & & \\
\operatorname{Tor}_1^N(\mathbf{Z}, R) \arrow[r, tail] \arrow[d, equal] & \mathbf{Z}\otimes_N \pi_2 X \arrow[r, "\bar{h}_N"] & \mathbf{Z}\otimes_N C_2\tilde{X} \arrow[r, two heads, "\alpha"] \arrow[dr, "1\otimes\partial_2"'] & \mathbf{Z}\otimes_N R \arrow[d, "\beta"] & & \\
H_3N & & & \mathbf{Z}\otimes_N C_1\tilde{X} \arrow[d] \arrow[dr, "1\otimes\partial_1"] & & \\
 & & H_1N = \operatorname{Tor}_1^N(\mathbf{Z}, \mathbf{Z}) \arrow[r, tail] & \mathbf{Z}\otimes_N IG \arrow[r] & \mathbf{Z}\otimes_N \mathbf{Z}G \arrow[r, two heads] & \mathbf{Z}
\end{tikzcd} \tag{4.1}$$

Note that the triangles commute and that the vertical and horizontal sequences are exact. Clearly $\ker(1\otimes\partial_2) = \alpha^{-1}H_2N$. So $H_2N = 0$ yields $\ker(1\otimes\partial_2) = \ker\alpha = \operatorname{im}\bar{h}_N = 0$, if $F(\pi_2 X) \subset K_N$. Similarly $\ker(1\otimes\partial_2)$ is zero yields $\alpha^{-1}H_2N = 0$ which implies $\alpha^{-1}(0) = \operatorname{im}\bar{h}_N = 0$ and $H_2N = 0$. □

DEFINITION 4.2 [St_1]. A group G is called an *E-group* if H_1G is torsion free and the trivial G-module $\mathbf{Z}$ has a projective G-resolution

$$\cdots \to P_n \to \cdots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow \mathbf{Z}$$

such that the homomorphism $1\otimes_G \partial_2 : \mathbf{Z}\otimes_G P_2 \to \mathbf{Z}\otimes_G P_1$ is *injective.*

It follows that if G is an E-group, then $H_2(G) = 0$.

For a given group G, we define P_1G to be the *maximal perfect subgroup of the group G*. It is *uniquely defined* as the group generated by the family of all perfect subgroups of G. This subgroup is perfect because the group generated by any family of perfect subgroups is perfect. Because the normal closure of a perfect group is perfect, P_1G is a *normal subgroup* of G. P_1 is clearly a functor from the category of groups and homomorphisms to the category of perfect groups and homomorphisms.

There is another way to define P_1G. Let $\{G^\alpha \mid \alpha \text{ ordinal}\}$ denote the *derived series*: $G^\alpha = (G^{\alpha-1})'$ for α not a limit ordinal, $G^\alpha = \bigcap_{\beta<\alpha} G^\beta$ for α a limit ordinal. This sequence terminates [Dr, p. 20] at a perfect group, and since G^α contains any perfect subgroup of G, it terminates at P_1G.

The following theorem may have been known to J. F. Adams. It certainly follows from his techniques, when applied to the derived series of L. See [A] and [St_1].

THEOREM 4.3. *Suppose X is a non-aspherical $[G, 2]$-complex such that $Y = X \cup \{e^2_\alpha \mid \alpha \in \mathscr{A}\}$ is aspherical. Let $L = \ker \{\pi_1 X \to \pi_1 Y = H\}$. Then the maximal perfect subgroup $P_1 L$ of L is superperfect, kills $\pi_2 X$, and is non-trivial.*

Proof. We first observe that L is an E-group because

$$\begin{array}{ccc} 1\otimes_L \partial_2^X : \mathbf{Z}\otimes_L C_2\tilde{X} & \to & \mathbf{Z}\otimes_L C_1\tilde{X} \\ \| & & \| \\ C_2 X_L & & C_1 X_L \end{array}$$

is monic (Lemma 4.1) and $H_1 L \cong \mathbf{Z}H^{|\mathscr{A}|}$. By Theorem A(i) of [$\mathrm{St}_1$] each term of the derived series L^α of L is an E-group; hence $P_1 L$ is an E-group, therefore superperfect. In fact, the argument of the theorem cited above shows that

$$1\otimes_{P_1 L} \partial_2^X : \mathbf{Z}\otimes_{P_1 L} C_2\tilde{X} \to \mathbf{Z}\otimes_{P_1 L} C_1\tilde{X}$$

is monic, hence $P_1 L$ kills $\pi_2 X$ by Lemma 4.1. $P_1 L$ is non-trivial because, if it were trivial then $P_1 L$ kills $\pi_2 X$ implies $F(\pi_2 X) \subset K_{P_1 L} = 0$. But $F(\pi_2 X) = 0$ iff $\pi_2 X = 0$, which contradicts the assumption that X was non-aspherical. □

A group G has cohomological dimension $\leq n$ ($\mathrm{cd}\, G \leq n$) if $H^i\ (G; M) = 0$ for all $i > n$ and all $\mathbf{Z}G$-modules M. Equivalently $\mathrm{cd}\, G \leq n$ if and only if the trivial G-module $\mathbf{Z}$ has an $\mathbf{Z}G$-protective resolution of length n:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0. \qquad (*)$$

A group G has type *FP* (*FL*) if and only if G has a projective resolution $(*)$ of finite length with each P_i (free) of finite rank.

If G has type *FL*, we define the (naive) Euler characteristic $\chi(G) = \sum_{i=0}^n (-1)^i \operatorname{rank}_{ZG} P_i$. In this case, let $b_i G = \operatorname{rank}_Z H_i G$. Standard arguments show that $\chi(G) = \sum_{i=0}^n (-1)^i b_i G$ as well.

THEOREM 4.4. *Let X be a $[G, 2]$-complex and P be any superperfect normal subgroup of G such that P kills $\pi_2 X$. Then G/P has cohomological dimension ≤ 2 over Z. Furthermore, if X is a minimal $[G, 2]_f$-complex, then G/P has type FL and $\chi_{\min}(G, 2) = \chi(G/P)$.*

Proof. Consider the chain complex $C_*\tilde{X}$. In diagram 4.1 with $P=N$, we see that $H_1P=H_2P=0$ together with $\bar{h}_P=0$ (if and only if $K_P\supset F(\pi_2X)$) shows that the sequence

$$0\to \mathbf{Z}\otimes_P C_2\tilde{X}\to \mathbf{Z}\otimes_P C_1\tilde{X}\to \mathbf{Z}\otimes_P \mathbf{Z}G\to \mathbf{Z}\to 0$$

is an exact sequence of $\mathbf{Z}G$-modules. Because P is normal, $\mathbf{Z}\otimes_P\mathbf{Z}G\cong\mathbf{Z}(G/P)$ as (G/P)-modules. If X is a $[G,2]_f$-complex, then clearly G/P has type FL and $\chi(G/P)=\chi(X)=\chi_{\min}(G,2)$. □

After proving Theorem 4.4, we noticed that R. Strebel had proved a similar result for E-groups [St$_1$]. However, groups arising as the fundamental group of a subcomplex of an aspherical complex are *not* necessarily E-groups, as the second homology group is not necessarily zero (it is free abelian). The results do not imply one another, even though the basic trick is the same.

THEOREM 4.5. *Let X be any non-aspherical $[G,2]$-complex and $X<Y$, an aspherical $[H,2]$-complex. Then there exists a family of distinct non-trivial normal superperfect subgroups $P_i\lhd G$, $i\in I$, such that* cd $G/P_i\le 2$ *for $i\in I$ and such that the smallest (normal) subgroup $P=\langle P_i\mid i\in I\rangle$ containing all P_i kills π_2X. Hence, P_1G kills π_2X.*

Proof. $X<X\cup Y^{(1)}=X\vee\bigvee S^1_\alpha=\bar{X}<Y=\bar{X}\cup\{e^2_\beta\}$. By theorem 4.3 there is a superperfect normal subgroup $\bar{P}\ne 1$ in $G*F$, where F is a free group isomorphic to $\pi_1(\bigvee S^1_\alpha)$. Also $\bar{P}$ kills $\pi_2\bar{X}$. Hence, by 4.4, cd $G*F/\bar{P}\le 2$.

By Kuros' theorem, we have $\bar{P}=\mathop{\ast}_u(uGu^{-1}\cap\bar{P})$ for certain $u\in G*F$. The group $\bar{P}$ is superperfect implies that each $uGu^{-1}\cap\bar{P}$ is superperfect. Let $P_u=u^{-1}(uGu^{-1}\cap\bar{P})u$. Each P_u is a superperfect normal subgroup of G. The group $\bar{P}\ne 1$ implies that *some* of the $P_u\ne 1$. Choose the family $\{P_i\}$ to be those $P_u\ne 1$.

Consider the following diagram:

$$\begin{tikzcd} G*F \arrow[r, two heads] & G*F/\bar{P} \\ uGu^{-1} \arrow[u, tail] \arrow[r, two heads] & uGu^{-1}/\bar{P}\cap uGu^{-1} \arrow[u, tail] \end{tikzcd}$$

Thus $uGu^{-1}/\bar{P}\cap uGu^{-1}\approx G/P_u$ has cohomological dimension ≤ 2 for each u. Note that if *any* $\bar{P}\cap uGu^{-1}=1$, then G itself has cohomological dimension ≤ 2.

Let $F_G(M)$ denote the 2-sided ideal *in* $\mathbf{Z}G$ generated by the coordinates of elements of the G-module $M\subset(\mathbf{Z}G)^\alpha$. We know that $F_{G*F}(\pi_2\bar{X})\subset K_{\bar{P}}=\mathbf{Z}(G*F)\cdot I\bar{P}$, where $\pi_2\bar{X}\cong\mathbf{Z}(G*F)\otimes_G\pi_2X$, by Theorem 4.3. It follows that

$F_{G*F}(\pi_2\bar{X})=F_{G*F}(\pi_2X)$, with π_2X considered as a $G*F$-module via the projection $\eta: G*F\to G$. Notice that $P=\eta(\bar{P})$. The surjection $\mathbf{Z}\eta:\mathbf{Z}(G*F)\to\mathbf{Z}(G)$ clearly carries $K_{\bar{P}}=\mathbf{Z}(G*F)\cdot I\bar{P}$ onto $K_P=\mathbf{Z}(G)\cdot IP$. Also $F_{G*F}(\pi_2\bar{X})=F_{G*F}(\pi_2X)$ is carried onto $F_G(\pi_2X)$. Thus $F_{G*F}(\pi_2X)\subset K_{\bar{P}}$ implies $F_G(\pi_2X)\subset K_P$ and we are done. □

For the next corollary let X be a $[G,2]$-complex which is not aspherical, but which is a subcomplex of an aspherical $[H,2]$-complex. Thus, there must exist a non-trivial superperfect subgroup $P\lhd G$ such that cd $G/P\le 2$. Because groups of finite cohomological dimension are torsion free, we have

COROLLARY 4.6. *Any element $g\in G$ such that $g^n\in P$ $(n\ge 1)$ must be in P. In particular, the torsion of G is contained in P.* □

In [B, p. 122], R. Bieri shows that the center of a non-abelian group of cohomology dimension ≤ 2 is cyclic. The exact sequence $P\rightarrowtail G\twoheadrightarrow\bar{G}=G/P$ induces a monomorphism

$$\mathfrak{Z}G/(P\cap\mathfrak{Z}G)\rightarrowtail\mathfrak{Z}(G/P)$$

($\mathfrak{Z}G$ is the center of G). If G/P is non-abelian, then $\mathfrak{Z}(G/P)$ is 0 or $\mathbf{Z}$; if G is finitely generated, $\mathfrak{Z}(G/P)=0$, $\mathbf{Z}$, or $\mathbf{Z}\oplus\mathbf{Z}$ (this last occurs only if $G/P=\mathbf{Z}\oplus\mathbf{Z}$ is abelian).

COROLLARY 4.7. *Let G be a finitely presented group, X be a minimal $[G,2]_f$-complex, and P be a superperfect normal subgroup of G with the cohomological dimension of $G/P\le 2$. Then $\mathfrak{Z}G/(P\cap\mathfrak{Z}G)=0$, $\mathbf{Z}$, or $\mathbf{Z}\oplus\mathbf{Z}$, with this last group occurring only if $\bar{G}=G/P$ is abelian. If* def $G\ge 1$ *and P doesn't kill π_2X or if P kills π_2X and* def $G\ne 1$, *then $\mathfrak{Z}G\subset P$.*

Proof. First, we assume that P kills π_2X and that def $G\ne 1$. Then, by Theorem 4.4, $\bar{G}$ has type FL and $\chi(\bar{G})=\chi(X)=1-\operatorname{def}G$. The deficiency of $G\ne 1$ implies that $\chi(\bar{G})\ne 0$. Then, by corollary 3.6 of [S], we see that $\mathfrak{Z}(\bar{G})$ is trivial. Hence P contains $\mathfrak{Z}(\bar{G})$.

We assume that def $G\ge 1$ and that P does not kill π_2X. Let $R_i=\ker\{\mathbf{Z}\otimes_P C_i\tilde{X}\to\mathbf{Z}\otimes_P C_{i-1}\tilde{X}\}$ $(i=1,2)$. The cohomological dimension of $\bar{G}\le 2$ implies that R_1 is a projective $\bar{G}$-module. Because P is superperfect, we have an exact sequence

$$R_2\rightarrowtail\mathbf{Z}\otimes_P C_2\tilde{X}\twoheadrightarrow R_1.$$

This shows that R_1 and R_2 are both finitely generated projective $\bar{G}$-modules. Now at this point in the proof, we must use the Euler characteristic of a group defined by J. Stallings in [S]. The rank of a finitely generated projective $\bar{G}$-module Q is a certain element rQ in the free abelian group T on the set of conjugacy classes of $\bar{G}$. Then ρQ is defined to be the coefficient of [1] in rQ. Accordingly, $\chi(\bar{G}) = \oplus_{i=0}^{2} (-1)^i \cdot \rho(\mathbf{Z} \otimes_P C_i \tilde{X}) - \rho R_2 = \chi_{\min}(G, 2) - \rho R_2 = 1 - \operatorname{def} G - \rho R_2$. It follows from proposition 1 of [DV] that $\rho R_2 \geq 0$ and $\rho R_2 = 0$ iff $R_2 = 0$. Now P does *not* kill $\pi_2 X$ implies that $R_2 \neq 0$. Thus $\rho R_2 > 0$. Hence the deficiency of $G \geq 1$ implies that $\chi(\bar{G}) < 0$ and the result again follows from corollary 3.6 of [S]. $\square$

We would like to thank the referee for simplifying the hypotheses of 4.7.

One may show that all the higher centers $\mathfrak{Z}^n G \subset P$ as well. To see that $\mathfrak{Z}^2 G \subset P$, notice that the hypotheses of 4.7 imply that $\mathfrak{Z}\bar{G} = 1$. Then the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{Z}G \arrow[r, tail] \arrow[d, equal] & P \arrow[r, two heads] \arrow[d, tail] & P/\mathfrak{Z}G \arrow[d, tail] \\
\mathfrak{Z}G \arrow[r, tail] & G \arrow[r, two heads, "\eta"] \arrow[d, two heads] & G/\mathfrak{Z}G . \arrow[d, two heads] \\
 & \bar{G} \arrow[r, equal] & \bar{G}
\end{tikzcd}$$

Now $\mathfrak{Z}\bar{G} = 1$ implies that $\mathfrak{Z}(G/\mathfrak{Z}G) \subset P/\mathfrak{Z}G$ and hence that $\mathfrak{Z}^2 G = \eta^{-1}\mathfrak{Z}(G/\mathfrak{Z}G) \subset P$.

BIBLIOGRAPHY

[A] J. F. Adams, *A new proof of a theorem of W. H. Cockcroft*, J. London Math. Soc., *30* (1955), 482–488.

[B] R. Bieri, Homological dimension of discrete groups, Queen Mary College Notes, 1976, University of London, London E.1.

[Be] W. Beckmann, Completely Aspherical 2-Complexes, Thesis, Cornell University, 1980.

[C] W. H. Cockcroft, *On two-dimensional aspherical complexes*, Proc. London Math. Soc., *4* (1954), 375–384.

[CS] —— and R. Swan, *On the homotopy type of certain two-dimensional complexes*, Proc. London Math. Soc., *11* (1961), 194–202.

[Co] J. Cohen, *Aspherical 2-complexes*, J. of Pure and Applied Algebra *12* (1978), 101–110.

[DV] E. Dyer and A. Vasquez, *An invariant for finitely generated projectives over* $\mathbf{Z}[G]$, J. Pure and Applied Algebra, 7 (1976) 241–248.

[D] M. DYER, *On the second homotopy module of two-dimensional* CW*-complexes*, Proc. AMS, *55* (1976), 400–404.

[Dr] E. DROR, *A generalization of the Whitehead theorem*, Symposium on algebraic topology, vol. 249, Lecture Notes in Math., Springer-Verlag, 13–22.

[GR] M. A. GUTIERREZ and J. G. RATCLIFFE, *On the second homotopy group*, Quart, J. Math., *32* (1981), 45–55.

[H] J. HOWIE, *Aspherical and acyclic 2-complexes*, J. London Math. Soc., *20* (1979), 549–558.

[K] M. KERVAIRE, On higher dimensional knots, *Differential and Combinatorial Topology*, pp. 105–119, Princeton Univ. Press, Princeton, N.J., 1965.

[L] R. LYNDON, *Cohomology theory of groups with a single defining relation*, Ann. Math., *52* (1950), 650–665.

[MKS] W. MAGNUS, A. KARRASS, and D. SOLITAR, *Combinatorial Group Theory*, Interscience, New York, 1966.

[R] J. RATCLIFFE, *Free and projective crossed modules*, J. London Math. Soc., *22* (1980), 66–74.

[Se] J.-P. SERRE, *Cohomologie Des Groupes Discrets, Prospects in Mathematics*, Annals of Math. Studies, *70*, 1971.

[S] J. STALLINGS, *Centerless groups – an algebraic formulation of Gottlieb's theorem*, Topology, *4* (1965), 129–134.

[St_1] R. STREBEL, *Homological methods applied to the derived series of groups*, Comment. Math. Helv., *49* (1974), 302–332.

[St_2] ——, *Die Reihe der Derivierten von E-Gruppen*, Dissertation an der Eidgenössischen Technischen Hochschule Zürich (1973).

[W_1] J. H. C. WHITEHEAD, *On adding relations to homotopy groups*, Annals of Math. *42* (1941), 409–428.

[W_2] ——, *Combinatorial homotopy* II, Bull. American Math. Soc., *55* (1949), 453–496.

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