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# Level sets of univalent functions

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## 1. Introduction

Let $w=f(z)$ be a univalent (one to one analytic) map from $\Delta:|z|<1$ onto a domain $\Omega$ in the closed complex plane. By a level set of $f$ we mean the preimage of $f^{-1}(\Omega\cap L)$ for some straight line or circle $L$. Piranian and Weitsman asked in [7] if every level set of $f$ has finite length. Here we give an affirmative answer. We shall denote by $A_0, A_1, \ldots$ positive absolute constants. If $E$ is any set, $\delta(E)$ is the diameter and $|E|$ is the length or one-dimensional Hausdorff measure of $E$. Then our result is

THEOREM 1. *If $E$ is a subset of a level set of a univalent function, then*

$$|E|\leq A_0\delta(E)\leq 2A_0 \tag{1.1}$$

*where* $A_0<10^{35}$.

A special case of the theorem when $\Omega$ is a Lipschitz domain was proved in [8].

Our argument is rather lengthy and falls naturally into three parts. In the first part (Sections 2 to 5) we assume that $E$ is the full level set $\gamma$. The components, i.e., maximal connected subsets of $\gamma$, will be called level curves and denoted by $\gamma_k$. We shall then prove in Section 5,

$$\sum\delta(\gamma_k)\leq A_1=2.1\times 10^{16}. \tag{1.2}$$

About three weeks after we obtained this result a proof of (1.2) was obtained independently by Gehring and Jones [1]. It is also worth noting that Jones [4] has constructed a bounded analytic function on $\Delta$ for which every level set $|w|=R$ is either empty or has infinite length. Thus the analogue of Theorem 1 for bounded functions is false.

In the second part (Sections 6 to 10), we prove that for any individual level

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curve $\gamma_k$, we have

$$|\gamma_k| \le A_2 < 10^{18}. \tag{1.3}$$

These two parts make up the bulk of the paper.

In the third part (Section 11) we complete the proof of Theorem 1. We show first by an elementary transformations that (1.3) leads very simply to

$$|\gamma_k| \le 2A_2\delta(\gamma_k). \tag{1.4}$$

Using this and (1.2) we deduce at once that

$$|\gamma| = \sum |\gamma_k| \le 2A_2 \sum \delta(\gamma_k) \le 2A_1A_2. \tag{1.5}$$

A similar argument to that leading from (1.3) to (1.4) then shows that if $E$ is any subset of $\gamma$ we have

$$|E| \le 4A_1A_2\delta(E)$$

and this gives Theorem 1. This part is relatively short.

Our arguments in parts I and II are based on harmonic measure. If $D$ is a domain and $E$ is a subset of the closure $\bar{D}$ of $D$, we denote by $\omega(z, E, D)$ a function which is harmonic in $D \setminus E$ and has boundary value 1 on $E$ and zero on $\partial D \setminus E$. Thus if $E$ is a subset of $\partial D$, $\omega(z, E, D)$ is the harmonic measure of $E$ with respect to $D$ at $z$. Clearly $\omega(z, E, D)$ increases with expanding $E$ for fixed $z$ and $D$ and with expanding $D$ for fixed $z$ and $E$. If no confusion is likely because $D$ is fixed we sometimes write simply $\omega(z, E)$.

## 2. Elementary lemmas on harmonic measure

The only property of the $\gamma_k$ which we shall use in order to prove (1.2) is that the $\gamma_k$ are arcs in $\Delta$ with end points on the boundary of $\Delta$, which satisfy the separation condition involving harmonic measure which is described in Lemma 1. However, in order to use Lemma 1 we need various other properties of harmonic measure.

LEMMA 1. *If $\gamma_k$ are the components of the level set $\gamma$ and $\gamma_k' = \gamma \setminus \gamma_k$, then*

$$\omega(z, \gamma_k', \Delta) < \tfrac{1}{2}, \qquad z \in \gamma_k. \tag{2.1}$$

We recall that $w = f(z)$ maps $\Delta$ onto the domain $\Omega$. Then $\gamma_k$ is mapped onto an arc $l$ of $L$ and $\gamma_k'$ is mapped onto the remainder $l'$ of $L \cap \Omega$. Clearly we may assume that $L$ is the real axis, and (2.1) is equivalent to

$$\omega(w, l', \Omega) < \tfrac{1}{2}, \qquad \omega \in l \tag{2.2}$$

in view of the invariance of harmonic measure under conformal mapping. To prove (2.2) we construct the reflection $\Omega^*$ of $\Omega$ in $L$ and define $U$ to be that component of $\Omega \cap \Omega^*$, which contains $l$. Write

$$\omega(w) = \omega(w, l', \Omega), \qquad \omega^*(w) = \omega(w) + \omega(\bar{w});$$

then $\omega^*(w)$ is harmonic in $U$. Let $\xi$ be a boundary point of $U$. Then, since $\Omega$ is simply-connected, $\xi$ cannot lie in $l'$. Thus $\xi$ is either a boundary point of $\Omega^*$, in which case $\omega(\xi) = 0$, $\omega(\bar{\xi}) < 1$, or a boundary point of $\Omega^*$, in which case $\omega(\bar{\xi}) = 0$, $\omega(\xi) < 1$. Thus $\omega^*(\xi) < 1$ and so $\omega^*(w) < 1$ in $U$. Taking for $w$ a point on $l$, we deduce (2.2) and hence Lemma 1.

If $\Omega$ is the region $-\theta < \arg z < 2\pi - \theta$, where $0 < \theta < \pi$, and $l$ and $l'$ are the negative and positive real axes, we have $\omega(w) = (\pi - \theta)/(2\pi - \theta)$ on $l$, so that (2.1) is sharp.

Our next lemma is a special case of the Milloux–Schmidt inequality [3, p. 109] and [6, p. 107].

LEMMA 2. *Let $\eta$ be an arc in $\Delta_R = \{z \mid |z| < R\}$, which passes through the origin and has one end point on $|z| = R$. Then*

$$\omega(z, \eta, \Delta_R) \geq 1 - \frac{4}{\pi} \arctan\, (|z|/R)^{1/2}.$$

We also need a variant of Hall's Lemma [2]. It is pointed out by David Drasin that a theorem which bears some resemblance to Lemma 3 in disks was proved by Maitland [5].

LEMMA 3. *Suppose that $H = \{z \mid z = x + iy,\ y > 0\}$ is the upper half plane. Let $E$ be a relatively closed set in $\{0 < y < a/100\}$ and let $E^* = \{x \mid x + iy \in E\}$ be the projection of $E$ on the real axis. Then for* $\operatorname{Im} z \geq a$,

$$\omega(z, E, H) \geq \tfrac{2}{3}\omega(z, E^*, H).$$

As in the proof of Hall's Lemma we may assume that $E$ is the union of a finite or countable set of line segments $l_k$, which are parallel to the real axis and whose projections have at most endpoints in common. This is also the only case of

Lemma 3 which we use. Further we assume, as we may, that $a=1$. If

$$l_k=\{t+ib_k,\ a_k\le t\le a_k'\}$$

we define for $z\in H$

$$U(z)=\frac{1}{2\pi}\sum_k\frac{1}{b_k}\int_{l_k}G(z,\zeta)\,|d\zeta|,$$

where

$$G(z,\zeta)=\log\left|\frac{\bar z-\zeta}{z-\zeta}\right|$$

is the Green function in $H$. When $\zeta=t+ib$, $z=x+iy$, we set $|z-\zeta|=r$, $|z-t|=\rho$ and assume that $y\ge 1\ge 100b$. Then

$$\frac{1}{b}\log\left|\frac{\bar z-\zeta}{z-\zeta}\right|=\frac{1}{2b}\log\left(1+\frac{4by}{r^2}\right)\ge\frac{1}{2b}\log\left(1+\frac{4by}{\rho^2}\right)$$
$$\ge\frac{2y}{\rho^2}\Big/\left(1+\frac{4by}{\rho^2}\right)\ge\frac{2y}{\rho^2}\Big/\left(1+\frac{1}{25}\right)=\frac{25}{26}\frac{2y}{\rho^2}.$$

Thus

$$U(z)\ge\frac{25}{26}\frac{1}{\pi}\sum_k\int_{a_k}^{a_k'}\frac{y}{\rho^2}\,dt=\frac{25}{26}\,\omega(z,E^*).\tag{2.3}$$

Next, we prove, following Hall, that

$$U(z)\le\frac{\pi}{4}+\frac{2}{\pi},\qquad z\in H.\tag{2.4}$$

We recall that

$$\frac{1}{b}\log\left|\frac{\bar z-\zeta}{z-\zeta}\right|=\frac{1}{2b}\log\left(1+\frac{4by}{r^2}\right).$$

The right hand side decreases with increasing $b$ when $z$ and $r$ are fixed and so assumes its maximum value for $b=y-r$ if $r<y$ and for $b=0$ when $r\geq y$. Thus

$$\frac{1}{b}\log\left|\frac{\bar{z}-\zeta}{z-\zeta}\right|\leq M(r,z),$$

where

$$M(r,z)=\frac{1}{y-r}\log\left(\frac{2y}{r}-1\right),\qquad r<y$$

$$M(r,z)=\frac{2y}{r^2},\qquad r\geq y.$$

Also $M(r,z)$ decreases with increasing $r$ and so

$$\frac{1}{b}\log\left|\frac{\bar{z}-\zeta}{z-\zeta}\right|\leq M(r,z)\quad\text{if}\quad |z-\zeta|\geq r.$$

In particular this inequality holds if $\zeta=t+ib$, where $t-x=\mp r$. Thus

$$\begin{aligned}U(z)&\leq\frac{1}{\pi}\int_0^\infty M(r,z)\,dr\\&=\frac{1}{\pi}\int_0^y\frac{1}{y-r}\log\left(\frac{2y}{r}-1\right)dr+\frac{1}{\pi}\int_y^\infty\frac{2y}{r^2}\,dr.\end{aligned}$$

The second integral is $2/\pi$. To evaluate the first integral set $y-r=ty$, then

$$\begin{aligned}\frac{1}{\pi}\int_0^y\frac{1}{y-r}\log\left(\frac{2y-r}{r}\right)dr&=\frac{1}{\pi}\int_0^1\log\left(\frac{1+t}{1-t}\right)\frac{dt}{t}\\&=\frac{1}{\pi}\sum_0^\infty\frac{2}{(2n+1)^2}=\frac{\pi}{4}.\end{aligned}$$

This proves (2.4). Now the maximum principle yields

$$U(z)\leq\left(\frac{\pi}{4}+\frac{2}{\pi}\right)\omega(z,E),\qquad z\in H. \tag{2.5}$$

Combining (2.3) and (2.5), we conclude for $y\geq 1$,

$$\omega(z,E)\geq\left(\frac{\pi}{4}+\frac{2}{\pi}\right)^{-1}\tfrac{25}{26}\omega(z,E^*)\geq\tfrac{2}{3}\omega(z,E^*)$$

and this proves Lemma 3.

LEMMA 4. *Suppose that $E$ is a subset of the real axis, that $a>0$ and that $I$ is the interval* $[-10a, 10a]$. *Then if*

$$\omega(ia, E, H)<7/8$$

*we have*

$$|I\setminus E|\geq \tfrac{1}{120}|I|.$$

Write $E'=I\setminus E$, $|E'|=2\eta$. We suppose without loss of generality that $a=1$. Then if $\lambda(t)$ denotes the measure of $E'\cap[-t, t]$ we have

$$\omega(i, E')=\frac{1}{\pi}\int_0^{10}\frac{d\lambda(t)}{1+t^2}.$$

Our hypotheses imply that $\lambda(t)\leq 2\min(t, \eta)$. Thus

$$\omega(i, E')\leq\frac{2}{\pi}\int_0^{\eta}\frac{dt}{1+t^2}=\frac{2}{\pi}\tan^{-1}\eta.$$

Also

$$\tfrac{7}{8}>\omega(i, E)\geq\omega(i, I)-\omega(i, E')\geq\frac{2}{\pi}(\tan^{-1}10-\tan^{-1}\eta).$$

Thus

$$\frac{2}{\pi}\tan^{-1}\eta\geq\frac{2}{\pi}\tan^{-1}10-\tfrac{7}{8}=\tfrac{1}{8}-\frac{2}{\pi}\tan^{-1}\tfrac{1}{10}>\tfrac{1}{8}-\frac{1}{5\pi},$$

$$\eta>\frac{\pi}{16}-\tfrac{1}{10}>0.09>\tfrac{1}{12}.$$

This proves Lemma 4.

LEMMA 5. *Suppose that $\eta$ is an arc in $\{z\mid |x|\leq 32, 0<y<1\}$ which connects the lines $x=\mp 32$. Then if $1<y_0\leq 2$, we have*

$$\omega(iy_0, \eta, H)>\tfrac{18}{19}.$$

Let $l=32$. Let $\eta_1$, $\eta_2$ denote the segments $\{-l+iy, 0\le y\le 1\}$, $\{l+iy, 0\le y\le 1\}$ respectively. Let $\eta_3$, $\eta_4$ denote the segments $\{-l-1\le x\le -l+1, y=0\}$ and $\{l-1\le x\le l+1, y=0\}$. Then

$$\omega(z,\eta_3)\ge\tfrac{1}{2} \text{ on } \eta_1 \quad\text{and}\quad \omega(z,\eta_4)\ge\tfrac{1}{2} \text{ on } \eta_2.$$

Thus for $z\in H$

$$\omega(z,\eta_1)+\omega(z,\eta_2)\le 2(\omega(z,\eta_3)+\omega(z,\eta_4)).$$

Finally if $\eta_5$ denote the segment $\{-l\le x\le l, y=0\}$ then the maximum principle shows that for $z=x_0+iy_0$, where $y_0>1$,

$$\omega(z,\eta_5)\le\omega(z,\eta)+\omega(z,\eta_1)+\omega(z,\eta_2),$$

so that

$$\omega(z,\eta)\ge\omega(z,\eta_5)-2(\omega(z,\eta_3)+\omega(z,\eta_4)).$$

Setting $z=iy_0$ where $1\le y_0\le 2$, we obtain

$$\begin{aligned}\omega(z,\eta)&\ge\frac{2}{\pi}\left(\tan^{-1}\frac{l}{2}-2\tan^{-1}\frac{l+1}{2}+2\tan^{-1}\frac{l-1}{2}\right)\\ &=1-\frac{2}{\pi}\left(\tan^{-1}\frac{2}{l}+2\tan^{-1}\frac{4}{l^2+3}\right)\\ &>1-\frac{4}{\pi}\left(\frac{1}{l}+\frac{4}{l^2+3}\right)>\tfrac{18}{19}.\end{aligned}$$

This proves Lemma 5.

## 3. Construction of segments

The inequality (2.1) is the only property of the level curves which we use in order to complete our results. It is convenient to work in $H$ rather than $\Delta$ since the noneuclidean metric is easier to visualize in $H$. We consider in the first instance only those level curves which lie entirely in the square

$$R_0: 0\le x\le 1, \qquad 0\le y\le 1. \tag{3.1}$$

More precisely we assume that $w = F(z)$ is a univalent map from $H$ into the complex plane, whose image does not contain the whole real axis $L$ in the $\omega$ plane, and denote by $l_j$ those components of $F^{-1}(L)$ which lie entirely in $R_0$. (The simpler case $L \subseteq F(H)$ will be considered at the end of Section 11.) If

$$l'_k = \bigcup_{j \neq k} l_j,$$

it follows from Lemma 1, the invariance of harmonic measure and the fact that $\omega(z, E, H)$ increases with expanding $E$ that

$$\omega(z, l'_k, H) \leq \tfrac{1}{2} \quad \text{on} \quad l_k. \tag{3.2}$$

Apart from (3.2) we only use the fact that the $l_k$ are arcs which lie in $R_0$ and have both end points on the segment $0 \leq x \leq 1$, $y = 0$.

It proves convenient to replace each $l_k$ by certain horizontal line segments $q_k$ of diameter comparable to the diameter of $l_k$. We can do this at the cost of replacing $\frac{1}{2}$ by $\frac{7}{12}$ in (3.2). Using Lemma 3 we shall deduce that the projection of a suitable subset of $\bigcup_{j \neq k} q_j$ has harmonic measure less than $\frac{7}{8}$ on $q_k$ and so by Lemma 4 must leave uncovered $\frac{1}{120}$ of a neighborhood of the projection of $q_k$. From this we can deduce that the sum of the diameters of the $q_k$ and so that of the $l_k$ is bounded by a (very large) absolute constant and (1.2) will follow.

Let $m, n$ be integers such that

$$n \geq 10, \qquad 0 < m \leq 2^n.$$

We shall call a dyadic *segment* the set

$$q(m, n) : (m-1)2^{-n} < x < m2^{-n}, \qquad y = 600 \cdot 2^{-n}. \tag{3.3}$$

The projection

$$J(m, n) : (m-1)2^{-n} < x < m2^{-n}, \qquad y = 0 \tag{3.4}$$

of $q(m, n)$ on the real axis will be called a dyadic *interval*. We note that two dyadic segments are disjoint, unless they are identical. Two dyadic intervals are disjoint unless one is contained in the other. Also all dyadic segments lie in $R_0$.

LEMMA 6. *If $l_k$ is a level curve in $R_0$ there exists a dyadic segment $q_k$ such that*

$$\delta(q_k) > 10^{-5} \delta(l_k) \tag{3.5}$$

*and*

$$\omega(z, l_k, H) > \tfrac{18}{19} \quad \textit{for all } z \textit{ on } q_k. \tag{3.6}$$

Let $x_1, x_2$ be the lower and upper bounds of $x$ on $l_k$ and let $h$ be the upper bound of $y$ on $l_k$. We write $x_2 - x_1 = 2d$ and distinguish two cases.

(i) Suppose first that $d > 40h$. In this case we define $n$ to be the largest integer such that

$$2^{-n} > \frac{d}{24{,}000},$$

and then define $m$ to be the largest integer such that

$$m2^{-n} \geq \tfrac{1}{2}(x_1 + x_2).$$

Let $q(m, n)$ be the corresponding segment given by (3.3), and suppose that $\zeta = \xi + i\eta \in q(m, n)$. Then

$$\frac{d}{40} < \eta \leq \frac{d}{20}.$$

Also if $x_0 = \frac{1}{2}(x_1 + x_2)$, we have for $\zeta = \xi + i\eta$ on $q(m, n)$

$$|\xi - x_0| \leq 2^{-n} < (12{,}000)^{-1} d < \frac{d}{5}.$$

Thus $l_k$ has a subarc $l'$, lying in the rectangle

$$|x - \xi| \leq \tfrac{32}{40} d, \qquad 0 < y < \frac{d}{40},$$

and joining the sides $x = \xi \mp 32d/40)$ of this rectangle, since by hypothesis $h \leq d/(40)$. Now it follows from Lemma 5, $d/40 < \eta \leq d/20$ and conformal invariance that

$$\omega(\zeta, l_k) \geq \omega(\zeta, l') > \tfrac{18}{19},$$

which gives (3.6). Further, using our choice of $n$ we see

$$\delta(l_k) \leq (h^2 + (2d)^2)^{1/2} \leq 2d(1 + (\tfrac{1}{80})^2)^{1/2} < 50{,}000 \cdot 2^{-n},$$

which yields (3.5). Thus Lemma 6 is proved in this case.

(ii) Suppose next that $d \le 40h$. We choose for $n$ the least integer such that

$$600 \cdot 2^{-n} \le h,$$

and set $\eta = 600 \cdot 2^{-n}$, so that

$$h/2 < \eta \le h.$$

Since $l_k$ has endpoints on the positive axis we see that $l_k$ contains a point $\zeta_0 = \xi_0 + i\eta$. If $\xi_0 = 0$ we choose $m = 1$ and otherwise we choose for $m$ the smallest integer such that $m \cdot 2^{-n} \ge \xi_0$. Then if $\zeta$ is any point on $q(m, n)$, defined by (3.3), we have

$$|\zeta - \zeta_0| \le 2^{-n} = \eta/600 < \eta \tan^2\left(\frac{\pi}{76}\right).$$

Now $l_k$ contains a subarc $l'$ lying in the disk $D_0 : |z - \zeta_0| < \eta$ and joining the point $\zeta_0$ to the circumference of $D_0$. Hence Lemma 2 shows that for $\zeta \in q(m, n)$,

$$\begin{aligned}\omega(\zeta, l_k, H) &\ge \omega(\zeta, l', H) \ge \omega(\zeta, l', D_0)\\ &\ge 1 - \frac{4}{\pi} \arctan \sqrt{\frac{|\zeta - \zeta_0|}{\eta}} \ge 1 - \frac{4}{\pi} \cdot \frac{\pi}{76} = \frac{18}{19}.\end{aligned}$$

Also $\delta(l_k) \le ((2d)^2 + h^2)^{1/2} \le h(1 + (80)^2)^{1/2}$

$$< 81h < 81 \cdot 1200 \cdot 2^{-n}.$$

Thus (3.5) and (3.6) are satisfied and Lemma 6 is proved.

We now prove the required separation property for the segments $q_k$, which is the analogue of Lemma 1.

LEMMA 7. *Let $q_k$ be the segments defined in Lemma 6 and write*

$$q_k' = \bigcup_{j \ne k} q_j.$$

*Then we have*

$$\omega(z, q_k', H) < \tfrac{7}{12}, \qquad z \in q_k.$$

We note that

$$\omega(z, l_k)+\omega(z, l'_k)\le\tfrac{3}{2}, \qquad z\in H. \tag{3.7}$$

For if $z$ lies on $l_k$, the inequality holds in view of (3.2) and if $z$ lies on $l_j$, where $j\neq k$, we have

$$\omega(z, l_k)\le\omega(z, l'_j)\le\tfrac{1}{2}$$

so that (3.7) still holds. Also the left-hand side of (3.3) is zero on the boundary of $H$ and is harmonic in $H$ except on the $l_j$. Thus (3.7) follows from the maximum principle.

Using (3.7) and (3.6), we deduce that

$$\omega(z, l'_k)\le\tfrac{3}{2}-\tfrac{18}{19}=\tfrac{21}{38} \quad \text{on} \quad q_k. \tag{3.8}$$

Further we have on $q_j$, where $j\neq k$, from (3.6)

$$1=\omega(z, q'_k)<\tfrac{19}{18}\omega(z, l_j)\le\tfrac{19}{18}\omega(z, l'_k).$$

Thus

$$\omega(z, q'_k)<\tfrac{19}{18}\omega(z, l'_k) \quad \text{for} \quad z\in q'_k \tag{3.9}$$

and (3.9) trivially holds on $l'_k$ and on the boundary of $H$. Thus (3.9) holds in $H$. Combining (3.8) and (3.9), we deduce on $q_k$

$$\omega(z, q'_k)<\tfrac{19}{18}\cdot\tfrac{21}{38}=\tfrac{7}{12}$$

and this proves Lemma 7.

Let $S$ be a collection of distinct dyadic intervals $I$ on the real axis. Suppose that $a_1, a_2, a_3$ are positive numbers such that $a_1\ge 1$, $0<a_2\le 1$, $0<a_3\le 1$. If $I$ is an interval $|x-x_0|<\frac{1}{2}|I|$, we write $aI$ for the interval $|x-x_0|<\frac{1}{2}a\,|I|$. We shall say that $S$ satisfies the hypothesis $P(a_1, a_2, a_3)$ if for every $I\in S$ there exists a set $e(I)$ such that

$$e(I)\subseteq a_1 I \tag{3.10}$$

$$|e(I)|\ge a_1 a_2\,|I| \tag{3.11}$$

and if $I'$ is any interval of $S$ such that $|I'| \le a_3 |I|$ we have

$$e(I) \cap I' = \phi. \tag{3.12}$$

We shall prove in Lemma 9 that such a collection $S$ necessarily has finite total length. Thus in order to prove that $\sum \delta(l_k)$ is finite, it is sufficient, in view of Lemma 6, to prove that the projections $J_k$ of the $q_k$ onto the real axis satisfy $P(a_1, a_2, a_3)$ for suitable constants $a_1, a_2, a_3$. We proceed to deduce this result from Lemma 7.

LEMMA 8. *The projections $J_k$ of the dyadic intervals $q_k$ are all distinct and satisfy* $P(12{,}000, \frac{1}{120}, \frac{1}{100})$.

It follows at once from Lemma 7 that $q_k \cap q'_k = \phi$ so that the different $q_k$ are disjoint. Thus their projections are not identical. Suppose that $q = q_k$, and let $z_0 = x_0 + 600 \cdot 2^{-n} i$ be the midpoint of $q$. Let $J$ be the projection of $q$ on the real axis and let $E$ be the union of all those $q_j$ whose length is less than $10^{-2} \cdot 2^{-n}$. Then it follows from Lemma that

$$\omega(z_0, E) \le \omega(z_0, q'_k) < \tfrac{7}{12}.$$

Also by our construction $E$ lies in $0 > y > 6 \cdot 2^{-n}$. Thus if $E^*$ is the projection of $E$ on the real axis, we deduce from Lemma 3 and conformal invariance that

$$\omega(z_0, E^*) \le \tfrac{3}{2} \omega(z_0, E) < \tfrac{7}{8}.$$

It now follows from Lemma 4 that if $I$ is the interval $12{,}000\, J$, i.e. $|x - x_0| < 6000 \cdot 2^{-n}$, then

$$|I - E^*| \ge \tfrac{1}{120} |I| = 12{,}000 \cdot \tfrac{1}{120} |J|.$$

Letting $e(J) = I \setminus E^*$, we conclude Lemma 8.

We can now obtain a bound for the sum of the diameters of the $l_k$ in $R_0$ by proving

LEMMA 9. *If the collection S of dyadic intervals I satisfies* $P(a_1, a_2, a_3)$ *then*

$$\sum_S |I| \le K(a_1, a_2, a_3) \le 288 \frac{a_1}{a_2} \left(5 + \log \frac{a_1}{a_2 a_3}\right). \tag{3.13}$$

*Thus if $l_k$ are the level curves in $R_0$, we have*

$$\sum \delta(l_k) \leq 10^5 K(12{,}000, \tfrac{1}{120}, \tfrac{1}{100}) < 10^{15}. \tag{3.14}$$

If $I$ is a dyadic interval, then $I = J(m, n)$ for some $m$ and $n$ as in (3.4); we call $n = n(I)$ the index of $I$. Thus $|I|$ decreases with increasing $n(I)$.

## 4. Proof of Lemma 9

We first assume that if $J(m, n)$ and $J(m', n')$ are intervals of $S$ then

$$m = m' (\text{mod } k_1) \quad \text{and} \quad n = n' (\text{mod } k_2), \tag{4.1}$$

where $k_1, k_2$ are the least integers satisfying respectively

$$k_1 \geq 3a_1 \tag{4.2}$$

and

$$2^{k_2} \geq \frac{1}{a_3} + \frac{48 a_1}{a_2}. \tag{4.3}$$

If $S$ does not satisfy (4.1) we shall divide $S$ into at most $k_1 k_2$ subclasses each of which satisfies (4.1) and sum over each subclass separately.

Let $F_\nu$ be the subset of all points of $(0, 1)$ which are covered by at least $\nu$ distinct intervals of $S$. Evidently $F_\nu$ is the union of a finite or countable set of intervals in $S$. Also $F_{\nu+1} \subseteq F_\nu$. Let $I_0$ be a component of $F_\nu$. We proceed to show that

$$|I_0 \cap F_{\nu+2}| \leq \left(1 - \frac{a_2}{24}\right) |I_0|. \tag{4.4}$$

To see this, suppose that $n(I_0) = n_0$ and let $n_1$ be the least index greater than $n_0$ of intervals of $S$ which meet $\frac{1}{2} I_0$. We call $V_1$ the class of all such intervals of index $n_1$. We note that by the first relation in (4.1) the intervals $3a_1 I_1$, where $I_1 \in V_1$, are disjoint. We now define

$$H_1 = \tfrac{1}{2} I_0 \setminus \bigcup_{V_1} 3a_1 I_1.$$

More generally if the class $V_k$ of intervals $I_k$ with index $n_k$ has been constructed we write

$$H_k = \tfrac{1}{2}I_0 \Big\backslash \Big(\bigcup_{V_1} 3a_1I_1 \cup \bigcup_{V_2} 3a_1I_2 \cup \cdots \cup \bigcup_{V_k} 3a_1I_k\Big) \tag{4.5}$$

and define $V_{k+1}$ to be the union of all intervals of $S$, meeting $H_k$ and having least index $n_{k+1} > n_0$. By our construction $n_{k+1} > n_k$.

This process continues indefinitely or stops, either because $H_k$ is empty or because $H_k$ meets no intervals of $S$ with index greater than $n_0$. We then define

$$H = \bigcap H_k, \qquad V = \bigcup V_k,$$

where intersection and union are taken over all $k$ for which $H_k$ is defined. We now distinguish two cases.

(i) Suppose that $|H| \geq \frac{1}{4}|I_0|$.

We note that no point of $H$ meets any interval $I$ of $S$ having index greater than $n_0$. For suppose contrary to this that $I$ is such an interval of least index $n$. Let $k$ be the largest integer for which $n_k < n$ and $H_k$ is defined. Then $I$ meets $H_k$ and so belongs to the class $V_{k+1}$, and thus $I$ is disjoint from $H_{k+1}$, contrary to hypothesis. Thus in case (i) $H$ does not meet $F_{\nu+1}$ and

$$|I_0 \cap F_{\nu+1}| \leq |I_0 \backslash H| \leq \tfrac{3}{4}|I_0|.$$

Since $F_{\nu+2} \subseteq F_{\nu+1}$, we deduce (4.4) in this case.

(ii) Suppose next that $|H| < \frac{1}{4}|I_0|$.

In this case we see that

$$\Big|\bigcup_V (3a_1 I)\Big| > \tfrac{1}{4}|I_0|.$$

Thus

$$\sum_V |I| \geq \frac{1}{12a_1}|I_0|. \tag{4.6}$$

Suppose that $I$ is an interval in $V_k$. Then by hypothesis $a_1 I$ contains a set $e(I)$ not meeting any interval of $S$ having length less than $a_3|I|$, i.e., no interval of $S$ having index greater than $n_k$ in view of (4.1), (4.3) and the fact that $I$ has index $n_k$. We note that the sets $e(I)$ corresponding to distinct intervals $I$ are disjoint, since the intervals $a_1 I$ are disjoint by (4.2), (4.5) and the fact $a_1 \geq 1$. Thus using (3.11) and (4.6) we deduce

$$\left|\bigcup_V e(I)\right| = \sum_V |e(I)| \geq a_1 a_2 \sum_V |I| \geq \frac{a_2}{12} |I_0|. \tag{4.7}$$

We have just seen that $e(I)$ corresponding to $I \in V_k$ meets no interval of $S$ having index greater than $n_k$. Next suppose that $e(I)$ meets an interval $I' \in S$ having index $n$, $n_0 < n < n_k$. It follows from (4.1) that $n \leq n_k - k_2$, so that

$$|I'| \geq 2^{k_2} |I|.$$

On the other hand, $I'$ cannot contain $I$ since otherwise $I'$ would meet $H_{k-1}$ and have index less than $n_k$, contrary to hypothesis. Thus $e(I) \cap I'$ must lie in one of two subintervals of $I'$, which adjoin the end points of $I'$ and have total length

$$(a_1 + 1)|I| \leq 2a_1 \cdot 2^{-k_2} |I'|.$$

Thus we see that the total length of the intersections of the $e(I)$ with intervals $I'$ of $S$ having index $n$, such that $n_0 < n(I') < n(I)$ is at most

$$\sum a_1 2^{1-k_2} |I'| \leq a_1 2^{1-k_2} |I_0|, \tag{4.8}$$

where $\sum$ is taken over all the maximal intervals $I'$ in $S$ of index greater than $n_0$ and containing some point $x \in I_0$. For these intervals $I'$ are disjoint and lie in $I_0$ and so have total length at most $|I_0|$. Thus if $e'(I)$ is the subset of $e(I)$ meeting no intervals of $S$ of index greater than $n_0$ and different from $n_k$, we deduce, using (4.7) and (4.8) that

$$\begin{aligned} |\bigcup e'(I)| &\geq |\bigcup e(I)| - a_1 \cdot 2^{1-k_2} |I_0| \\ &\geq \left(\frac{a_2}{12} - a_1 2^{1-k_2}\right) |I_0|. \end{aligned} \tag{4.9}$$

Using (4.3) and (4.9) we deduce that

$$|\bigcup e'(I)| \geq \frac{a_2}{24} |I_0|.$$

Also a point $x$ in $e'(I)$ lies in no interval of $S$ having index $n > n_0$, $n \neq n(I)$. Thus $x \notin F_{\nu+2}$. Hence in this case (4.4) holds, so that (4.4) is true in all cases.

We write

$$\theta = 1 - \frac{a_2}{24},$$

and deduce by induction that since $|F_1| \leq 1$, we have

$$|F_{2\nu}| \leq |F_{2\nu-1}| \leq \theta^{\nu-1}, \qquad \nu \geq 1.$$

Again

$$\sum_S |I| = \sum_1^\infty |F_\nu| = \sum_1^\infty (|F_{2\nu}| + |F_{2\nu-1}|) \leq 2 \sum_1^\infty \theta^{\nu-1} = \frac{2}{1-\theta}.$$

Thus with the hypotheses of Lemma 9, together with (4.1) we have

$$\sum_S |I| \leq \frac{2}{1-\theta} = \frac{48}{a_2}. \tag{4.10}$$

Also in the general case, when (4.1) is not satisfied we can divide the intervals of $S$ into at most $k_1 k_2$ subclasses each of which satisfies (4.10) so that we always have

$$\sum_S |I| \leq 48 k_1 k_2 / a_2 = K(a_1, a_2, a_3).$$

Using (4.2) and (4.3) we obtain

$$k_1 \leq 3a_1 + 1 \leq 4a_1,$$

$$k_2 \leq \frac{1}{\log 2} \log\left(\frac{50 a_1}{a_2 a_3}\right) + 1 < \frac{3}{2}\left(\log\left(\frac{a_1}{a_2 a_3}\right) + 5\right).$$

We deduce that

$$K \leq \frac{3}{2} \cdot 4 \cdot 48 \frac{a_1}{a_2}\left(5 + \log\left(\frac{a_1}{a_2 a_3}\right)\right) = 288 \frac{a_1}{a_2}\left(5 + \log \frac{a_1}{a_2 a_3}\right).$$

This proves (3.13).

Next it follows from Lemma 8, that if the class $S$ consists of the projections of the $q_k$, we may take

$$a_1 = 12{,}000, \qquad a_2 = \tfrac{1}{120}, \qquad a_3 = \tfrac{1}{100}.$$

Thus

$$\sum |q_k| \leq 288 \cdot 12{,}000 \cdot 120(5+\log(1.44\times 10^8))$$
$$<10^{10}.$$

Using (3.5) we obtain

$$\sum \delta(l_k) < 10^{15}.$$

This completes the proof of Lemma 9.

## 5. Proof of (1.2)

Let $\gamma_k$ now be the level curves of a univalent function in $\Delta$. We divide $\gamma_k$ into 3 subclasses. Consider first those $\gamma_k$, which lie entirely in

$$\Delta_1 = \Delta \cap (x > -\tfrac{1}{10}). \tag{5.1}$$

We consider the transformation

$$w = u + iv = \frac{i}{5}\left(\frac{1-z}{1+z}\right) + \frac{1}{2} = \frac{i}{5}\left(\frac{(1-|z|^2)-2iy}{|1+z|^2}\right) + \frac{1}{2}.$$

Clearly $\Delta$ corresponds to the upper half-plane $H$ in the $w$ plane and the subset (5.1) maps into the unit square $R_0$ given by (3.1) with $(u, v)$ instead of $(x, y)$. The level curves $\gamma_k$ in $\Delta_1$ correspond to level curves $l_k$ in $R_0$. Also

$$\left|\frac{dw}{dz}\right| = \frac{2}{5\,|1+z|^2} > \tfrac{1}{10} \quad \text{in} \quad \Delta_1,$$

and so

$$\left|\frac{dz}{dw}\right| \leq 10$$

in the image of $\Delta_1$. We deduce that

$$\delta(\gamma_k) \leq 10\delta(l_k).$$

Using Lemma 9, we obtain

$$\sum_1 \delta(\gamma_k) \le 10^{16}. \tag{5.2}$$

Similarly if $\Delta_2 = \Delta \cap (x < \frac{1}{10})$ and $\sum_2$ denotes the sum over those $\gamma_k$ which lie entirely in $\Delta_2$ we obtain

$$\sum_2 \delta(\gamma_k) \le 10^{16}. \tag{5.3}$$

Consider now the remaining level curves $\gamma_k$. Each of them must contain an arc $\tilde{\gamma}_k$ with end points on $x = \mp\frac{1}{10}$ in $\Delta$ and hence meets the imaginary axis at finitely many points $iy_k$ with $|y_k| < 1$. We choose the least such $y_k$, and enumerate the $\gamma_k$ in order of increasing $y_k$. We proceed to prove that if $y_k \le 0$ then

$$y_{k+1} - y_k > \tfrac{1}{70}. \tag{5.4}$$

Suppose that (5.4) is false for some $k$. We note that $\tilde{\gamma}_k$ separates $iy_{k+1}$ from the arc.

$$-\tfrac{1}{10} < x < \tfrac{1}{10}, \qquad y = -\sqrt{1 - x^2} \tag{5.5}$$

in $-\frac{1}{10} < x < \frac{1}{10}$. Let $\Delta_k$ be the disk

$$|z - iy_k| < \tfrac{1}{10}$$

and let $\Delta_k'$ be that component of $\Delta_k \setminus \tilde{\gamma}_k$ which contains $iy_{k+1}$. Then $\Delta_k'$ cannot contain any point of the arc (5.5). Thus since $y_k \le 0$, we deduce that $\Delta_k' \subseteq \Delta$. Hence if $\tilde{\tilde{\gamma}}_k = \tilde{\gamma}_k \cap \Delta_k$, we deduce that

$$\omega(iy_{k+1}, \tilde{\tilde{\gamma}}_k, \Delta_k') \le \omega(iy_{k+1}, \tilde{\tilde{\gamma}}_k, \Delta) \le \omega(iy_{k+1}, \gamma_k, \Delta) \le \tfrac{1}{2}$$

in view of Lemma 1. On the other hand $\tilde{\tilde{\gamma}}_k$ contains an arc $\eta_k$ joining $z = iy_k$ to the boundary of $\Delta_k$ and so we deduce that

$$\omega(iy_{k+1}, \eta_k, \Delta_k) \le \omega(iy_{k+1}, \tilde{\tilde{\gamma}}_k, \Delta_k) = \omega(iy_{k+1}, \tilde{\tilde{\gamma}}_k, \Delta_k') \le \tfrac{1}{2}.$$

Now we apply Lemma 2 and deduce that

$$y_{k+1} - y_k \ge \tfrac{1}{10} \tan^2 \frac{\pi}{8} > \tfrac{1}{70}.$$

Thus (5.4) is true after all and less than 70 different $\gamma_k$ can meet the negative imaginary axis. Thus there are at most $140\gamma_k$ in our remaining group and if $\sum_3$ denotes the sum over these, we have

$$\sum_3 \delta(\gamma_k) < 300.$$

On combining this with (5.2) and (5.3) we deduce (1.2).

## 6. Preliminary reductions

We now embark on the proof of (1.3). We confine ourselves to the following special case to which the general result can easily be reduced. We assume that $\Omega$ is the interior of an analytic Jordan curve $\Gamma$ and that $I$ is a segment $[b_1, b_2]$ of the real axis in $\Omega$ whose endpoints $b_1$, $b_2$ lie on $\Gamma$. We denote by $\gamma$ the image of $I$ under the conformal map

$$z = F(w) = f^{-1}(w)$$

of $\Omega$ onto $\Delta$ and shall show that $\gamma = \gamma_k$ satisfies (1.3). In this part we work with the geometry of $\Omega$ rather than that in $H$ or $\Delta$. If $w$ is a point of $\Omega$ we write

$$d(w) = \inf_{\zeta \in \Gamma} |w - \zeta| \tag{6.1}$$

for the distance from $w$ to $\Gamma$. We start by constructing a function $\phi(u)$ which is comparable to $d(u)$ on $I$ but behaves in a smooth manner. We shall then dissect the interval $I$ into a sequence of intervals $I_{j,k}$, and with each $I_{j,k}$ we associate an arc $\Gamma_{j,k}$ of $\Gamma$, such that length of the image of $I_{j,k}$ by $F(w)$ is comparable with that of $\Gamma_{j,k}$. The $\Gamma_{j,k}$ will be disjoint and so their images have total length at most $2\pi$ and from this (1.3) will follow.

We set $w = u + iv$.

By our construction the line $u = c$ meets $\Gamma$ for $b_1 \le c \le b_2$ and we write

$$\phi_0(u) = \inf\{|v| \,|\, u + iv \in \Gamma\}, \qquad b_1 \le u \le b_2. \tag{6.2}$$

Further we define

$$\phi(u) = \inf_{u_1 \in I} \{\phi_0(u_1) + \tfrac{1}{12}|u_1 - u|\}. \tag{6.3}$$

Clearly $\phi(u) \geq 0$, with equality only at the endpoints $b_1, b_2$ of $I$. A point $u \in I$ will be called a *spike-point* if

$$\phi(u) = \phi_0(u).$$

Evidently the endpoints $b_1, b_2$ are spike points.

In order to establish our results we proceed to subdivide I into intervals bounded by spike points. We prove first

LEMMA 10. *If $u$ is a point of $I$ and $d(u)$, $\phi(u)$ are defined as above then*

$$d(u) \geq \tfrac{24}{25}\phi(u).$$

Suppose that $w_1 = u_1 + iv_1$ is a point of $\Gamma$ such that

$$|w_1 - u| = d(u).$$

Then

$$\begin{aligned}\phi(u) &\leq \phi_0(u_1) + \tfrac{1}{12}|u_1 - u| \leq |v_1| + \tfrac{1}{12}|u_1 - u| \\ &\leq [1 + (\tfrac{1}{12})^2)^{1/2}|w_1 - u| \leq \tfrac{25}{24}d(u).\end{aligned}$$

This proves Lemma 10.

LEMMA 11. *We have for $u_1, u_2 \in I$*

$$|\phi(u_1) - \phi(u_2)| \leq \tfrac{1}{12}|u - u_2|.$$

Suppose that $u_3 \in I$. Then

$$\phi_0(u_3) + \tfrac{1}{12}|u_1 - u_3| \leq \phi_0(u_3) + \tfrac{1}{12}|u_2 - u_3| + \tfrac{1}{12}|u_1 - u_2|.$$

Taking lower bounds for varying $u_3$ we obtain

$$\phi(u_1) \leq \phi(u_2) + \tfrac{1}{12}|u_1 - u_2|.$$

Interchanging $u_1$ and $u_2$ we obtain

$$\phi(u_2) \leq \phi(u_1) + \tfrac{1}{12}|u_1 - u_2|$$

and so we deduce Lemma 11.

Lemmas 10 and 11 are not true if we use $\phi_0$ in place of $\phi$.

LEMMA 12. *The set of spike points is a closed non-empty subset of* $I$.

Since $b_1, b_2$ are spike points, the set is certainly non-empty. Next we note that since $\Gamma$ is closed the function $\phi_0$ defined by (6.2) is lower semi-continuous. Also we see by Lemma 11 that $\phi(u)$ is continuous. Thus $h(u)=\phi_0(u)-\phi(u)$ is a lower semicontinuous nonnegative function. To see that $h(u)\geq 0$ we just set $u_1=u$ in (6.3). Hence the set where $h(u)\leq 0$ is closed and this is the set of spike points.

It follows from Lemma 12, that the complement in $I$ of the set of spike points consists of a finite or countable set of open intervals $J$. In each of these intervals $\phi(u)$ has a particularly simple form.

LEMMA 13. *Suppose that* $a, a'$ *are spike points in* $I$ *such that* $a<a'$, *and that the interval* $(a, a')$ *contains no spike points. Then for* $a<u<a'$,

$$\phi(u)=\min\{\phi_0(a)+\tfrac{1}{12}(u-a),\ \phi_0(a')+\tfrac{1}{12}(a'-u)\}.$$

Since $a$ is a spike point we have for $a_1<a$, $a_1\in I$,

$$\phi_0(a_1)+\tfrac{1}{12}(a-a_1)\geq\phi_0(a).$$

Thus for $a<u<a'$, we deduce that

$$\begin{aligned}\phi_0(a_1)+\tfrac{1}{12}(u-a_1)&=\phi_0(a)+\tfrac{1}{12}(u-a)+\phi_0(a_1)-\phi_0(a)+\tfrac{1}{12}(a-a_1)\\&\geq\phi_0(a)+\tfrac{1}{12}(u-a).\end{aligned}\tag{6.4}$$

Similarly if $a_1>a'$, we deduce that

$$\phi_0(a_1)+\tfrac{1}{12}(a_1-u)\geq\phi_0(a')+\tfrac{1}{12}(a'-u).\tag{6.5}$$

Next we note that in the definition (6.3) of $\phi(u)$ we may allow $u_1$ to range only over spike points in $I$. For since $\phi_0(u)$ is lower semicontinuous the infimum in (6.3) is attained for some $u_1$ in $I$. If $u_1$ is not a spike point we can find $u_2$ such that

$$\phi_0(u_2)+\tfrac{1}{12}|u_2-u_1|<\phi_0(u_1).$$

Thus

$$\begin{aligned}\phi_0(u_2)+\tfrac{1}{12}|u-u_2|&\leq\phi_0(u_2)+\tfrac{1}{12}|u_2-u_1|+\tfrac{1}{12}|u-u_1|\\&<\phi_0(u_1)+\tfrac{1}{12}|u-u_1|=\phi(u)\end{aligned}$$

and this contradicts the definition of $\phi(u)$. Hence if $a<u<a'$ there exists a spike point $u_1$, such that

$$\phi(u)=\phi_0(u_1)+\tfrac{1}{12}|u_1-u|. \tag{6.6}$$

In view of (6.4) and (6.5) we may suppose that $a\le u_1\le a'$, so that $u_1=a$ or $u_1=a'$, since $(a, a')$ contains no spike points. Using Lemma 11, we see that $u_1$ is that one of $a$, $a'$ which gives the smaller value of $\phi(u)$ in (6.6).

## 7. A dissection of the interval $I$

Suppose first that the interval $I$ contains no spike point other than the end points $b_1, b_2$. In this case we write $a_1=b_1$, $a_2=b_2$ and deduce from Lemma 13 that

$$\phi(u)=\tfrac{1}{12}\min(a-a_1, a_2-a), \qquad a_1<a<a_2. \tag{7.1}$$

Thus we deduce from Lemma 10 that in this case

$$\tfrac{2}{25}\min(a-a_1, a_2-a)\le d(a), \qquad a_1<a<a_2. \tag{7.2}$$

Suppose next that $I$ contains at least one spike point $a_0$, such that $b_1<a_0<b_2$. Having chosen $a_0$ we define other spike points $a_j$ inductively as follows. If $a_j$ has been defined, $j\ge 0$, we define $a_{j+1}$ to be the smallest spike point such that

$$a_{j+1}\ge a_j+6\phi(a_j). \tag{7.3}$$

We deduce from (6.3)

$$\phi_0(b_2)+\tfrac{1}{12}|b_2-a_j|\ge\phi(a_j)$$

i.e.

$$b_2-a_j\ge 12\phi(a_j),$$

so that $a_{j+1}\le b_2$. Thus either at some stage $a_{j+1}=b_2$, in which case we stop the procedure, or else the $a_j$ are defined for all positive $j$ and

$$a_j\to b_2, \quad \text{as} \quad j\to+\infty.$$

Similarly if, for some nonpositive $j$, $a_j$ has been defined we define $a_{j-1}$ to be the largest spike point such that

$$a_{j-1} \leq a_j - 6\phi(a_j). \tag{7.4}$$

If $a_j > b_1$, we deduce again that $a_{j-1} \geq b_1$.

The relevant properties of our subdivision are given in

LEMMA 14. *The interval $I$ can be divided into a finite or countable set of subintervals $[a_j, a_{j+1}]$, where the $a_j$ are spike points with the following properties*

$$a_{j+1} - a_j \geq 4 \max\{\phi(a_j), \phi(a_{j+1})\}. \tag{7.5}$$

*Further, if $d(a)$ denotes the distance of $a$ from the boundary $\Gamma$ of $D$, we have for $a_j < a < a_{j+1}$*

$$d(a) \geq \tfrac{2}{5} \min\{\max[\phi(a_j), \tfrac{1}{5}(a-a_j)], \max[\phi(a_{j+1}), \tfrac{1}{5}(a_{j+1}-a)]\}. \tag{7.6}$$

If $a_j = b_1$, $a_{j+1} = b_2$, (7.5) is trivial and (7.6) follows from (7.2). Thus we may assume that $a_0$ is a spike point in $I$ and that the remaining $a_j$ are defined by (7.3) and (7.4). We concentrate on (7.3) and $j \geq 0$ for definiteness. The case $j < 0$ is similar.

We first prove (7.5). Suppose first that $\phi(a_{j+1}) \leq \tfrac{3}{2}\phi(a_j)$. Then (7.5) follows from (7.3). On the other hand if

$$\phi(a_j) < \tfrac{2}{3}\phi(a_{j+1}) \quad \text{so that} \quad |\phi(a_{j+1}) - \phi(a_j)| > \tfrac{1}{3}\phi(a_{j+1})$$

we deduce from Lemma 11 that

$$|a_j - a_{j+1}| \geq 12\,|\phi(a_{j+1}) - \phi(a_j)| > 4\phi(a_{j+1}).$$

Thus (7.5) holds in all cases.

Next we prove (7.6). Suppose first that

$$a_j < a \leq a_j + 6\phi(a_j).$$

Then Lemma 11 shows that

$$|\phi(a) - \phi(a_j)| \leq \tfrac{1}{12}(a - a_j) \leq \tfrac{1}{2}\phi(a_j).$$

Thus in this case

$$\phi(a) \geq \tfrac{1}{2}\phi(a_j) \geq \tfrac{1}{12}(a - a_j). \tag{7.7}$$

Using Lemma 10, we deduce (7.6).

Suppose next that $a_j + 6\phi(a_j) < a < a_{j+1}$. In this case it follows from (7.3) that if $b$ is the largest spike point such that $b \leq a$, we have

$$a_j \leq b < a_j + 6\phi(a_j).$$

Also the interval $(b, a_{j+1})$ contains no spike point and so by Lemma 13, we have

$$\phi(a) = \min\{\phi(b) + \tfrac{1}{12}(a - b),\ \phi(a_{j+1}) + \tfrac{1}{12}(a_{j+1} - a)\}. \tag{7.8}$$

In view of (7.7) applied to $b$ instead of $a$, we have

$$\phi(b) \geq \tfrac{1}{2}\phi(a_j) \geq \tfrac{1}{12}(b - a_j).$$

Thus

$$\phi(b) + \tfrac{1}{12}(a - b) \geq \tfrac{1}{2}\phi(a_j) + \tfrac{1}{12}(a - b) \geq \tfrac{1}{12}(a - a_j).$$

Hence (7.8) yields in this case

$$\phi(a) \geq \min\{\max[\tfrac{1}{2}\phi(a_j),\ \tfrac{1}{12}(a - a_j)],\ \max[\tfrac{1}{2}\phi(a_{j+1}),\ \tfrac{1}{12}(a_{j+1} - a)]\}.$$

Because of (7.7), this inequality also holds for $a_j < a < a_{j+1}$. Using Lemma 10, we deduce (7.6). This completes the proof of Lemma 14.

Having obtained the points $a_j$ satisfying the conditions of Lemma 14, we now proceed to a further subdivision as follows. Let $(a_j, a_{j+1})$ be one of the intervals defined in Lemma 14. We write

$$a_{j,0} = \tfrac{1}{2}(a_j + a_{j+1}). \tag{7.9}$$

We then define $a_{j,k}$ for positive $k$ inductively by

$$a_{j,k+1} = \tfrac{1}{2}(a_{j+1} + a_{j,k}), \tag{7.10}$$

provided that $a_{j,k+1}$, so defined, satisfies

$$a_{j,k+1} \leq a_{j+1} - \phi(a_{j+1}). \tag{7.11}$$

Otherwise we set this value of $k+1$ equal to $k_2$ and define

$$a_{j,k_2} = a_{j+1}.$$

If $a_{j+1} = b_2$, so that $\phi(a_{j+1}) = 0$, the process continues indefinitely and we set $k_2 = \infty$. Thus (7.10) defines $a_{j,k+1}$ for $0 < k+1 < k_2$.

Similarly we define $a_{j,k}$ for negative $k$ inductively by

$$a_{j,k-1} = \tfrac{1}{2}(a_j + a_{j,k}). \tag{7.12}$$

The process continues as long as $a_{j,k-1}$ so defined satisfies

$$a_{j,k-1} \geq a_j + \phi(a_j).$$

Otherwise we set $k-1 = k_1$, and define

$$a_{j,k_1} = a_j. \tag{7.13}$$

In this way (7.9)–(7.12) define $a_{j,k}$ for $k_1 < k < k_2$. We deduce from (7.5) that

$$-\infty \leq k_1 \leq -2 \tag{7.14}$$

and

$$2 \leq k_2 \leq +\infty \tag{7.15}$$

We define

$$I_{j,k} = (a_{j,k}, a_{j,k+1}), \qquad k_1 + 1 < k < k_2 - 2. \tag{7.16}$$

If $k_1$ is finite we set

$$I_{j,k} = (a_{j,k-1}, a_{j,k+1}), \qquad k = k_1 + 1, \tag{7.17}$$

and if $k_2$ is finite we set

$$I_{j,k} = (a_{j,k}, a_{j,k+2}), \qquad k = k_2 - 2. \tag{7.18}$$

Thus $I_{j,k}$ is defined for $k_1<k<k_2-1$, i.e. certainly for $k=-1,0$. Also the $I_{j,k}$ cover $I$ apart from isolated points.

## 8. An association of arcs $\Gamma_{j,k}$ of $\Gamma$ with $I_{j,k}$

Suppose that $k_1<k\leq 0$. We construct an arc of the circle

$$|w-a_j|=|a_{j,k}-a_j| \tag{8.1}$$

starting from the point $a_{j,k}$. If $a_j$ is the left end point of $I$ we start anticlockwise into the upper half plane. Otherwise $\phi_0(a_j)>0$ and one of the two points

$$a_j \pm i\phi_0(a_j) \tag{8.2}$$

lies on $\Gamma$. If $a_j+i\phi_0(a_j)$ lies on $\Gamma$, we start the arc of (8.1) by moving in the anticlockwise sense into the upper half plane; otherwise we move in the clockwise sense into the lower half plane. In either case we continue along the circle (8.1) until we first meet a point of $\Gamma$ which we denote by $b_{j,k}$. Since $\Gamma$ contains points inside or on the circle (8.1) namely one of the points (8.2), and since $a_{j,k}$ lies inside $\Gamma$ the point $b_{j,k}$ certainly exists.

If $k_1<k<0$, the points $b_{j,k}$, $b_{j,k+1}$ determine two arcs of $\Gamma$. We choose that arc $\Gamma_{j,k}$ which we reach first when going along the circles

$$|w-a_j|=r, \qquad |a_{j,k}-a_j|<r<|a_{j,k+1}-a_j|$$

from the point $a_j+r$ in the anticlockwise or clockwise sense according as $a_j+i\phi_0(a_j)$ does or does not lie on $\Gamma$. Thus we have associated with each interval $I_{j,k}$ defined by (7.15) or (7.16) an arc $\Gamma_{j,k}$ of $\Gamma$ if $k_1<k<0$. It follows from (7.13) that at least one such $k$ exists. The corresponding intervals $I_{j,k}$ cover the interval $[a_j, a_{j,0}]$.

We proceed in an exactly analogous manner with the intervals $I_{j,k}$, $0\leq k<k_2-1$. We go along the circle

$$|w-a_{j+1}|=|a_{j,k}-a_{j+1}| \tag{8.3}$$

where $0\leq k<k_2$ starting at the point $a_{j,k}$ until we meet $\Gamma$ at $b'_{j,k}$. We then associate with the interval $I_{j,k}$, $0\leq k<k_2-1$, one of the arcs $[b'_{j,k}, b'_{j,k+1}]$ of $\Gamma$, which we denote by $\Gamma_{j,k}$. We move along circles $|w-a_{j+1}|=r$, $|a_{j,k+1}-a_{j+1}|<r<|a_{j,k}-a_{j+1}|$ in the clockwise sense if $a_{j+1}+i\phi(a_{j+1})$ lies on $\Gamma$ and the anticlockwise sense otherwise and $\Gamma_{j,k}$ contains the first point of $\Gamma$ we meet in this way. In general $\bigcup \Gamma_{j,k}$ is a proper subset of $\Gamma$.

It follows from the construction that our interval $I$ has been subdivided into a finite or countable set of subintervals $I_{j,k}$ which are associated with arcs $\Gamma_{j,k}$ of $\Gamma$, and no arc of $\Gamma$ is associated with more than one distinct interval of $I$. We complete this section by proving that in this association distinct arcs are disjoint except for endpoints. We show in the next section that the length of the image of $I_{j,k}$ is not much greater than that of $\Gamma_{j,k}$. From these two facts (1.3) will follow.

LEMMA 15. *The arcs $\Gamma_{j,k}$ defined as above are pairwise disjoint except for endpoints.*

Let $\beta_{j,k}$ denote the circular arc from $a_{j,k}$ to $b_{j,k}$, or $b'_{j,k}$ defined as above. We show that distinct arcs $\beta_{j,k}$ are disjoint except for endpoints.

Let $\beta$ be one of these circular arcs starting at $a = a_{j,k}$. Then $\beta$ can contain a semicircle $s$ starting at $a$ only if the other endpoint $a'$ of $s$ lies outside $I$. For if $a'$ lies in $I$, then the segment $aa'$ together with $s$ constitutes a closed Jordan curve $c$ in $\Omega$. If $a_j$ is the midpoint of $aa'$, then our construction ensures that one of the points $a_j \mp i\phi(a_j)$ lies on $\Gamma$ and so is in the same halfplane as $c$ and so by (7.11) inside $c$, which is impossible since $\Omega$ is simply connected.

Suppose now that $\beta_1, \beta_2$ are two distinct circular arcs starting at $P_1, P_2$ and first meeting at a point $P$ of $\Omega$. Consider the curve $\gamma$ formed by going along $\beta_1$ from $P_1$ to $P$ then along $\beta_2$ from $P$ to $P_2$ and returning along the segment $P_1, P_2$. By construction $\gamma$ lies in $\Omega$ and the arcs $P_1P$, $PP_2$ have only endpoints in common. Neither $\beta_1$ not $\beta_2$ can meet the segment $P_1P_2$ again since $\beta_1, \beta_2$ have no points in $I$ other than $P_1, P_2$ respectively. Thus $\gamma$ is a Jordan curve in $\Omega$.

We shall show that $\gamma$ contains a point of $\Gamma$ in its interior and this leads to a contradiction since $\Omega$ is simply connected. Suppose that $P_1$ lies to the left of $P_2$. It is not possible for the centres of both circular arcs to lie outside the segment $P_1P_2$, for if the centres are on the same side, the circles are concentric and distinct[1] and can certainly not meet, and if they are on opposite sides, $P_1P_2$ is the shortest distance between the circles. Suppose then that at least one centre, say $a_j$, the centre of the arc $P_1P$, lies on $P_1P_2$. Suppose also that $z_j = a_j + i\phi(a_j)$ lies on $\Gamma$. We distinguish a number of cases.

(i) Suppose first that neither $\beta_1$ nor $\beta_2$ contains a semicircle. Then $\beta_1$, $\beta_2$ both lie in the upper half plane. We show that in this case the segment $(a_j, z_j)$ lies inside $\gamma$. Suppose first that $P_2$ lies inside the circle $s_1$, of which $\beta_1$ is an arc. Then the

[1] If $P = a_j$ is the centre of the arc at $P_2 = a_{j,k}$ and $P$ lies to the left of $P_1$, then, by (8.1), $P_1 = a_{j,k'}$, where $k' < k < 0$.

centre of the circle having $\beta_2$ as an arc must lie to the right of $a_j$ and so to the right of $P_2$ since otherwise $\beta_2$ would lie inside $s_1$. Hence the segment $[a_j, z_j]$ does not meet $\beta_2$ and so $z_j$ lies inside $\gamma$.

Thus $P_2$ must lie outside $s_1$ and so does the whole arc $\beta_2$, since if $\beta_2$ went inside $s_1$, $\beta_2$ would contain a semi-circle. Hence in this case the interior of $\gamma$ includes all points in the upper half plane and inside $s_1$ and so in particular $z_j$ because of (7.11).

(ii) Suppose that $\beta_1$ contains a semi-circle, but $\beta_2$ does not. Then $\beta_2$ lies entirely in the lower half-plane and again the interior of $\gamma$ contains all points in the upper half-plane and inside $s_1$, and in particular $z_j$.

(iii) Suppose that $\beta_2$ contains a semi-circle but that $\beta_1$ does not. If the centre $a_k$ of the circle $s_2$ containing $\beta_2$ lies to the right of $P_2$, then the whole of $s_2$ lies to the right of $P_2$, and so cannot meet the segment $[a_j, z_j]$. Thus in this case $z_j$ again lies inside $\gamma$. If on the other hand $a_k$ lies on $P_1$, $P_2$ then we have the case (ii) with $P_1$, $P_2$ interchanged. Finally $a_k$ cannot lie to the left of $P_1$, since otherwise $\beta_1, \beta_2$ would be arcs of concentric circles which cannot meet. For all circular arcs starting from a point between $a_k$ and $P_2$ have centre $a_k$.

(iv) If $\beta_1, \beta_2$ both contain semicircles, they must reduce to semi-circles, since otherwise they would have two distinct points of intersection. In this case $\beta_1, \beta_2$ meet at $P$, which is to the right of $P_2$ and again the point $z_j$ lies inside $\gamma$.

Thus in all cases $\beta_1, \beta_2$ can have at most end points in common, since otherwise $\gamma$ contains a point of $\Gamma$ in its interior, which contradicts the fact that $\gamma$ lies inside $\Gamma$.

Suppose now that $\Gamma_{j,k}$ is an arc corresponding to an interval $I_{j,k}$. Let $\beta_k$, $\beta_{k+1}$ be the arcs of the circles (8.1) if $k<0$, or (8.3) if $k\geq 0$, to the points $a_{j,k}$, $a_{j,k+1}$. Then $\beta_k$, $\beta_{k+1}$, and the interval $[a_{j,k}, a_{j,k+1}]$ determine a crosscut $\delta_{j,k}$ in $\Omega$. In view of what we have just proved distinct crosscuts $\delta_{j,k}$ may have a common arc $\beta_k$ or a common point $a_{j,k}$, but cannot cross each other. Thus if $D_{j,k}$ is the interior of the Jordan curve formed by $\delta_{j,k}$ and $\Gamma_{j,k}$ then two distinct domains $D_{j,k}$ are disjoint.

In fact otherwise one of these domains would lie inside the other. However our construction ensures that the interval $I$ lies outside all the $D_{j,k}$, since none of the $\beta_k$ meet $I$ again. Hence points near $I_{j,k}$ inside $D_{j,k}$ are exterior to all the other domains $D_{j',k'}$. Thus $D_{j,k}$ cannot lie in $D_{j',k'}$. Now it follows that two distinct arcs $\Gamma_{j,k}$ are disjoint except for end points and this proves Lemma 15.

## 9. Images of intervals and associated arcs

Suppose that $I_{j,k}$ is an interval of the real axis and that $\Gamma_{j,k}$ is the associated arc of $\Gamma$. We assume for definiteness that $k<0$. If $k\geq 0$ the argument is similar. We recall from (7.9) to (7.11) that

$$a_{j,k} = a_j + 2^{(k-1)}(a_{j+1} - a_j), \qquad k_1 < k \le 0. \tag{9.1}$$

We denote by $\omega(w)$ the harmonic measure of $\Gamma_{j,k}$ with reference to the full open set $\Omega$. We write

$$r = 2^{k-1/2}(a_{j+1} - a_j) \tag{9.2}$$

and prove

LEMMA 16. *If* $w_1 = a_j + 2^{\delta} r$, *where* $-\frac{7}{16} \le \delta \le \frac{7}{16}$, *then* $\omega(w_1) > \exp\{-30.9\}$.

Let $\beta$ be the arc of the circle (8.1), joining $a_{j,k}$ to an endpoint $\beta_{j,k}$ of $\Gamma_{j,k}$. Let $\beta'$ be the corresponding arc of (8.1) with $k+1$ instead of $k$, which joins $a_{j,k+1}$ to the other endpoint $\beta_{j,k+1}$ of $\Gamma_{j,k}$. Then $\beta$, $\Gamma_{j,k}$, $\beta'$ and the segment $[a_{j,k}, a_{j,k+1}]$ form a subdomain $D_{j,k}$ of $\Omega$. We now define a domain $D' = D'_{j,k}$ as follows. We define $\theta_0$ by

$$\sin(\tfrac{1}{2}\theta_0) = \tfrac{1}{25}. \tag{9.3}$$

Then it follows from (7.6) of Lemma 14, that the sectorial region

$$2^{-1/2} r < |w - a_j| < 2^{1/2} r, \ |\arg(w - a_j)| < \theta_0 \tag{9.4}$$

does not meet $\Gamma$ and so lies inside $\Omega$. Next it follows from our construction that $\Gamma_{j,k}$ contains an arc $\eta_{j,k}$ joining the circles $|w - a_j| = 2^{\mp 1/2} r$ in the annulus

$$2^{-1/2} r < |w - a_j| < 2^{1/2} r. \tag{9.5}$$

We now define $D'$ to be a subdomain of the annulus (9.5) determined by such an arc $\eta_{j,k}$ and one of the rays $\arg(w - a_j) = \mp\theta_0$ and having the arcs $\beta$, $\beta'$ as part of its boundary. In other words if $\beta$, $\beta'$ start off in the clockwise sense we choose the ray $\arg(w - a_j) = +\theta_0$ and otherwise the ray $\arg(w - a_j) = -\theta_0$. We continue along $|w - a_j| = 2\mp^{1/2} r$ until we meet the first arc $\eta_{j,k}$.

We note that $D'$ constructed as above lies in the annulus (9.5) and contains the region (9.4). Let $\omega'(w)$ be the harmonic measure of $\eta_{j,k}$ at $w$ w.r.t. $D'$, and let $D_1$ be that component of $D' \cap \Omega$ which contains the segment $(a_{j,k}, a_{j,k+1})$. Then

$$\omega(w_1) \ge \omega'(w_1), \qquad w_1 \in D_1. \tag{9.6}$$

To see this let $\zeta$ be any boundary point of $D_1$. If $\zeta$ lies on one of the circles $|w-a_j|=2^{\mp 1/2}r$ or on the ray $\arg(w-a_j)=\mp\theta_0$, we have

$$\omega'(\zeta)=0\leq\omega(\zeta)$$

Any other boundary point $\zeta$ of $D_1$ lies on $\Gamma_{j,k}$ so that

$$1=\omega(\zeta)\geq\omega'(\zeta).$$

Thus $\omega(w)-\omega'(w)\geq 0$ on the boundary of $D_1$ and so in the interior of $D_1$ by the maximum principle. This proves (9.6).

A further application of the maximum principle now shows that $\omega'(w_1)$ assumes its lower bound for variable $\eta_{j,k}$ in the (limiting) case, when $D'$ reduces to the subdomain

$$-\theta_0<\arg(w-a_j)<2\pi-\theta_0, \qquad 2^{-1/2}r<|w-a_j|<2^{1/2}r \tag{9.7}$$

of the annulus (9.5) and $\eta_{j,k}$ to the arc $\arg(w-a_j)=2\pi-\theta_0$. It remains to estimate the corresponding harmonic measure. To do this we use the invariance of harmonic measure and set

$$s=\sigma+i\tau=\log(w-a_j). \tag{9.8}$$

We write

$$\sigma_0=\log r,$$

and note that the cut annulus (9.7) corresponds by (9.8) to the rectangle

$$\Delta_1:\sigma_0-\tfrac{1}{2}\log 2<\sigma<\sigma_0+\tfrac{1}{2}\log 2, \qquad -\theta_0<\tau<2\pi-\theta_0.$$

We have to estimate the harmonic measure w.r.t. $\Delta_1$ of $\tau=2\pi-\theta_0$ at $s_1=\log w_1=\sigma_0+\delta\log 2$.

By the maximum principle this harmonic measure is greater than that of the two rays

$$\tau\geq 2\pi-\theta_0, \qquad \sigma=\sigma_0\mp\tfrac{1}{2}\log 2 \tag{9.9}$$

w.r.t. the half strip

$$\Delta_2:-\theta_0<\tau<+\infty, \qquad \sigma_0-\tfrac{1}{2}\log 2<\sigma<\sigma_0+\tfrac{1}{2}\log 2$$

at $s_1$. This latter harmonic measure is calculated explicitly as follows. We set

$$z=\exp\left\{\frac{\pi i(s+i\theta_0-\sigma_0)}{\log 2}\right\}=x+iy.$$

This maps $\Delta_2$ onto the semidisk

$$T_2: |z|<1, \qquad x>0,$$

the pair of rays (9.9) onto the segment

$$x=0, \qquad |y|\le\eta=\exp(-2\pi^2/\log 2), \tag{9.10}$$

and $s_1$ onto

$$z_1=\exp\left\{\frac{-\pi\theta_0}{\log 2}+i\delta\pi\right\}=x_1+iy_1=r_1e^{i\delta\pi},$$

say. The harmonic measure of the segment (9.10) at $z=x+iy$ w.r.t. $T_2$ is

$$\omega_1(z)=\frac{1}{\pi}\left\{\tan^{-1}\frac{y+\eta}{x}+\tan^{-1}\frac{\eta-y}{x}-\tan^{-1}\frac{x\eta}{1-\eta y}-\tan^{-1}\frac{x\eta}{1+\eta y}\right\}.$$

For clearly $\omega_1$ is harmonic and bounded in $T_1$, and $\omega_1(z)=1$ on the segment (9.10) and $\omega_1(z)=0$, elsewhere on the boundary of $T_1$. To see this when $|z_1|=1$, we use the fact that the triangles $0\eta z$ and $0z\eta^{-1}$ are similar in this case. Using the addition formula for the inverse tangent we obtain finally

$$\omega_1(z_1)=\frac{1}{\pi}\tan^{-1}\left\{\frac{2\eta x_1(1+\eta^2)(1-r_1^2)}{(r_1^2-\eta^2)(1-\eta^2r_1^2)+4x_1^2\eta^2}\right\}$$

since $\eta<r_1<1$. In our case we have, using (9.3),

$$r_1=\exp\left(\frac{-\pi\theta_0}{\log 2}\right), \quad \text{where} \quad 0.08<\theta_0<0.081,$$

while $\eta<10^{-10}$. Also $x_1=r_1\cos\delta\pi\ge r_1\cos 7\pi/16=r_1\sin(\pi/16)$.

$$\omega(z_1)>0.999\frac{2\eta x_1(1-r_1^2)}{\pi r_1^2}\ge 0.999\frac{2\eta}{\pi}\left(\frac{1}{r_1}-r_1\right)\sin\frac{\pi}{16}$$

$$>\frac{\pi}{16}\frac{2\eta}{\pi}\cdot\frac{2\pi\theta_0}{\log 2}>\exp\{-30.9\}.$$

This completes the proof of Lemma 16.

We must extend Lemma 16 to obtain a bound for $\omega(w)$ on the whole of $I_{j,k}$. This is

LEMMA 17. We have in $I_{j,k}$

$$\omega(w_1) \geq A_3 = e^{-37}.$$

We write

$$w_1 = a_j + t.$$

It follows from (7.6) that if

$$\phi(a_j) \leq t \leq \tfrac{1}{2}(a_{j+1} - a_j) \tag{9.11}$$

then

$$d(w_1) \geq \frac{2t}{25}.$$

Also the function $\omega(w)$ is positive and harmonic in the disk $|w - w_1| < d(w_1)$. Thus Harnack's inequality [3, p. 64] yields

$$\left|\frac{d}{dt}\,\omega(w_1)\right| < \frac{2}{d(w_1)}\,\omega(w_1) \tag{9.12}$$

i.e.

$$\left|\frac{d}{dt}\log \omega(w_1)\right| \leq \frac{25}{t}$$

hence if $w_1 = a_j + t_1$, $w_1' = a_j + t_1'$ are two points in the range (9.11) we have

$$|\log \omega(w_1) - \log \omega(w_1')| \leq 25\,|\log t_1' - \log t_1|.$$

We suppose that

$$t_1 = 2^{7/16} r < t_1' \leq 2^{1/2} r \quad \text{or} \quad 2^{-1/2} r \leq t_1' < t_1 = 2^{7/16} r$$

and apply Lemma 16 to $w_1$. Thus we obtain

$$\log \omega(w_1') \geq -\{30.9+\tfrac{25}{16}\log 2\} > -32, \qquad a_{j,k} \leq w_1' \leq a_{j,k+1}. \tag{9.13}$$

This yields Lemma 17 if $k_1+1<k<0$.

If $k=k_1+1$, we have to estimate $\omega(w_1)$ also on the interval $I'=[a_j, a_{j,k}]$. In this case it follows from our construction that

$$a_{j,k} \leq a_j + 2\phi(a_j).$$

Thus if $w_1$ is any point on $I'$ it follows from Lemmas 10 and 11 that

$$d(a) \geq \tfrac{4}{5}\phi(a_j).$$

We deduce from (9.12) in this case that

$$\left|\frac{d}{dw_1}\log \omega(w_1)\right| \leq \frac{2.5}{\phi(a_j)} \quad \text{on} \quad I'.$$

Thus if $w_1$ is any point on $I'$ and $w_1' = a_{j,k}$ is the right endpoint of $I'$, we deduce that

$$\log \omega(w_1) \geq \log \omega(w_1') - 2\phi(a_j)\frac{2.5}{\phi(a_j)} \geq \log \omega(w_1') - 5.$$

On combining this with (9.13) we deduce that the inequality of Lemma 17 holds on $I'$ also and so on all of $I_{j,k}$. Thus Lemma 17 is proved whenever $k<0$. The case $k \geq 0$ is similar and our proof is complete.

## 10. Proof of (1.3)

We need a final estimate.

LEMMA 18. *If $l$, $\lambda$ are the lengths of the images of an interval $I'=I_{j,k}$ and the associated arc $\Gamma'=\Gamma_{j,k}$ respectively in the $z$ plane then*

$$\frac{l}{\lambda} \leq \frac{1}{\pi A_3}(5+25\log 2).$$

Suppose that $w_0$ is any point on $I'$ and let $z_0=\rho e^{i\theta}$ be the image of $w_0$ in $\Delta$. Let $\gamma=[\phi_1, \phi_1+\lambda]$ be the image of $\Gamma'$ on $|z|=1$. Then, since harmonic measure is invariant under conformal mapping,

$$\omega(w_0, \Gamma', \Omega)=\omega(z_0, \gamma, \Delta)=\frac{1}{2\pi}\int_{\phi_1}^{\phi_1+\lambda}\frac{(1-\rho^2)\,d\phi}{1-2\rho\cos(\theta-\phi)+\rho^2}$$
$$\leq\frac{\lambda}{2\pi}\frac{1+\rho}{1-\rho}.$$

Thus

$$1-\rho\leq\frac{\lambda}{\pi\omega(w_0, \Gamma', \Omega)}\leq\frac{\lambda}{\pi A_3}$$

in view of Lemma 17.

On the other hand if $z=F(w)$ maps $\Omega$ onto $\Delta$, then $F(w)$ maps $|w-w_0|<d(w_0)$ into $\Delta$ and now we deduce from Schwarz's Lemma that

$$|F'(w_0)|\leq\frac{1-|z_0|^2}{d(w_0)}\leq\frac{2(1-\rho)}{d(w_0)}\leq\frac{2\lambda}{\pi A_3\, d(w_0)}.$$

Thus

$$l=\int_{I'}|F'(w_0)|\,|dw_0|\leq\frac{2\lambda}{\pi A_3}\int_{I'}\frac{|dw_0|}{d(w_0)}. \tag{10.1}$$

Now we again use (7.6). If $I'$ is the interval $[a_{j,k}, a_{j,k+1}]$ where $k_1+1<k<0$, we set

$$w_0=a_{j,k}+t,$$

and deduce from Lemma 14 that

$$d(w_0)\geq\tfrac{2}{25}t.$$

Thus

$$\int_{I'}\frac{|dw_0|}{d(w_0)}\leq\frac{25}{2}\int_{I'}\frac{dt}{t}=\frac{25}{2}\log 2.$$

If $k=k_1+1$ we must add to $[a_{j,k}, a_{j,k+1}]$ the interval $[a_j, a_{j,k}]$. In this case $t\le 2\phi(a_j)$, $d(w_0)\ge\frac{4}{5}\phi(a_j)$ and

$$\int_{a_j}^{a_{j,k}}\frac{|dw_0|}{d(w_0)}\le\frac{5}{4\phi(a_j)}\int_0^{2\phi(a_j)}dt\le\frac{5}{2}.$$

Thus in all cases

$$\int_{I'}\frac{|dw_0|}{d(w_0)}\le\frac{5}{2}+\frac{25}{2}\log 2.$$

Hence (10.1) yields

$$\frac{l}{\lambda}\le\frac{1}{\pi A_3}(5+25\log 2).$$

This proves Lemma 18.

Now (1.3) follows at once. For the length $L$ of the level curve is the sum of the lengths $l_{j,k}$ of the images of the $I_{j,k}$. This yields

$$L\le\sum l_{j,k}\le\frac{1}{\pi A_3}(5+25\log 2)\sum\lambda_{j,k}\le\frac{1}{A_3}(10+50\log 2)<10^{18}$$

which is (1.3).

## 11. Proof of Theorem 1

We can now put our various results together. We need a final Lemma.

LEMMA 19. *Suppose that $E$ is a set in $\Delta$. Then there exists a bilinear function $z=L(Z)$ mapping $|Z|<1$ into $\Delta$ and a set $E'$ in $|Z|<1$ onto $E$, such that*

$$|L'(Z)|<2\delta(E)\quad on\quad E'. \tag{11.1}$$

Suppose that the upper bound of $|z|$ in the closure $\bar{E}$ of $E$ is $\rho$. We suppose without loss of generality in this case that $\bar{E}$ contains $z=\rho$ so that $\bar{E}$ lies in

$$|\rho-z|\le\delta, \tag{11.2}$$

where $\delta=\delta(E)$. We define $z=L(Z)$ by

$$z=\rho\frac{Z+tr}{t+rZ}, \qquad Z=t\frac{z-\rho r}{\rho-rz}, \quad \text{where} \quad r=\frac{\rho}{\rho+\delta}, \qquad \frac{2\rho}{2\rho+\delta}<t\leq 1.$$

If $\rho=1$, we choose $t=1$, so that $z=L(Z)$ is a bilinear map of $|Z|<1$ onto $|z|<1$, and the inverse image $E'$ of $E$ lies in $|Z|<1$. If $\rho<1$, we choose $t$ just less than 1. Then $|Z|\leq t$ corresponds to $|z|\leq\rho$ and so if $t$ is sufficiently near 1, $|Z|<1$ corresponds to a subset of $|z|<1$, which contains $|z|\leq\rho$ and so $E$.

Then if $z\in E$, so that (11.2) holds, we compute the derivative of $L^{-1}$ and find

$$|L'(Z)|=\frac{|\rho-rz|^2}{t\rho(1-r^2)}=\frac{|\rho-r\rho+r(\rho-z)|^2}{t\rho(1-r^2)}\leq\frac{(\rho(1-r)+\delta r)^2}{t\rho(1-r^2)}=\frac{4\delta\rho}{t(2\rho+\delta)}<2\delta,$$

which proves (11.1). This proves Lemma 19.

We recall that we have proved (1.3). We now apply this result to

$$g(Z)=f\{L(Z)\} \tag{11.3}$$

where $E$ is a single level curve of $f(z)$ and $E'=L^{-1}(E)$. Then $E'$ is part of a level curve of $g(Z)$ and so

$$|E'|\leq A_2.$$

In view of (11.1) we deduce that

$$|E|\leq 2\delta\,|E'|\leq 2\delta A_2.$$

where $\delta=\delta(E)$ and this is (1.4). Using (1.2) we deduce (1.5), for the length of any level set $\gamma$.

Finally suppose that $E$ is part of a level set of $f(z)$ and that $\delta(E)=\delta$. We again employ the subsidiary function $g(Z)$, given by (11.3) and define $E'=L^{-1}(E)$. Since $E'$ is part of a level set $\gamma$ of $g(z)$, we can now apply (1.5) and deduce that

$$|E'|\leq|\gamma|\leq 2A_1A_2.$$

Since also $E=L(E')$ we deduce from (11.1) that

$$|E|\leq 2\delta\,|E'|\leq 4A_1A_2\delta(E).$$

This proves (1.1) with

$$A_0 = 4A_1A_2 < 10^{35},$$

as stated.

We have assumed throughout that $\Omega$ is an analytic Jordan domain and that a level set $E$ is the inverse image of the real axis by $F(z)$. If $E$ is the inverse image by $F(z)$ of a circle or straight line $L$, we can find a bilinear map $W = \phi(w)$ which maps $L$ onto the real axis so that $E$ is also the inverse image of the real axis by $f(z) = \phi\{F(z)\}$.

Next if $F(z)$ is a general univalent function and $E$ is the inverse image of the real axis, suppose first that the image of $E$ does not cover the whole real axis but leaves out a point $w_0$. Then $(F(z) - w_0)^{-1}$ is a regular univalent function with the same level set $E$. Thus we may assume that $F(z)$ is regular and univalent in this case. We now apply (1.2) and (1.3) to the level sets $E_\rho$ of $F(\rho z)$, where $0 < \rho < 1$. Clearly $F(\rho z)$ maps $\Delta$ onto an analytic Jordan domain Also $|E_\rho|$ tends to $|E|$ as $\rho \to 1$, so that (1.2) and (1.3) also hold for $F(z)$.

Finally if the image of $E$ covers the whole real axis, then $E$ must consist of a single closed Jordan curve in $\Delta$. Let $\rho$ be the upper bound of $z$ on $E$. Then $E$ has at least one point on $|z| = \rho$.

Let $z_1 = \rho e^{i\theta}$ be such a point, and write $t = \frac{1}{2}(1+\rho)$, $z_0 = (\rho - t)e^{i\theta}$ and

$$f(z) = F(z_0 + tz).$$

Then the level set $E'$ of $f$ consists of a single curve going from $e^{i\theta}$ to $e^{i\theta}$ in $|z| < 1$ and having length $|E|/t > |E|$. Thus we may apply (1.3) to $E'$ and obtain

$$|E| < |E'| \leq A_2$$

in this case also. Thus (1.3) holds in all cases. Also (1.2) is trivial in this case, since $E$ is connected. Thus (1.2) and (1.3) hold in all cases and so do (1.4), (1.5) and (1.1). This completes the proof of Theorem 1.

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