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All knots are algebraic

S. AKBULUT and H. KING

In this paper we study local knottedness of real algebraic sets. For instance we show that any knot in S^3 is "algebraic" i.e.

THEOREM 0.1. *Given a knot $K \subset S^3$ there is a real algebraic set $Z \subset R^4$ with* Sing $(Z) = 0$ *so that for all sufficiently small $\varepsilon > 0$, $(\varepsilon S^3, \varepsilon S^3 \cap Z)$ is diffeomorphic to (S^3, K). (Here εS^3 is the sphere of radius ε around* 0.)

In fact we have more generally the following theorem:

THEOREM 0.2. *Suppose $U \subset S^{k-1}$ is a compact smooth submanifold with codimension ≥ 1 and trivial normal bundle. Then there is a real algebraic set $Z \subset R^k$ with* Sing $(Z) = 0$ *so that for all sufficiently small $\varepsilon > 0$, $(\varepsilon S^{k-1}, \varepsilon S^{k-1} \cap Z)$ is diffeomorphic to $(S^{k-1}, \partial U)$. (In fact, $\varepsilon \partial U$ will be isotopic in εS^{k-1} to $\varepsilon S^{k-1} \cap Z$ where $\varepsilon \partial U = \{x \in \varepsilon S^{k-1} \mid x/\varepsilon \in \partial U\}$.)*

To deduce Theorem 0.1 from Theorem 0.2 we may let U be a Seifert surface for the knot K. However, we give a direct proof of Theorem 0.1 in Section 2 since it is simpler and illustrates the main ideas in the proof of Theorem 0.2.

§1. Preliminary lemmas

Definitions are as in [1]. For instance a polynomial $p: R^n \to R$ is *overt* if $x \in R^n - 0$ implies $p^*(0, x) \neq 0$ where $p^*: R \times R^n \to R$ is the homogenization of p, ($p^*(t, x) = t^d p(x/t)$ where $d =$ degree of p). An algebraic set $V \subset R^n$ is *projectively closed* if $V = p^{-1}(0)$ for some overt polynomial p.

DEFINITION. Let W be a topological space, $M \subset W$. We say that M *compactly separates* W if there are closed sets W_0 and W_1 so that $W = W_0 \cup W_1$ and $M = W_0 \cap W_1$ and W_1 is compact.

We first give a few useful lemmas about algebraic sets.

LEMMA 1.1. *Suppose V and W are nonsingular algebraic sets, $W \subset V$ and* $\dim W = \dim V$. *Then $V - W$ is a nonsingular algebraic set.*

Proof. Lemma 1.6 of [1].

LEMMA 1.2 (Imprecise version). *The boundary of a submanifold with trivial normal bundle is isotopic to an algebraic set. This isotopy can fix nice subsets.*

LEMMA 1.2 (Precise statement). *Suppose W, L_i and V_i $i = 1, \ldots, k$ are algebraic sets, $U \subset$ Nonsing W is a smooth compact submanifold with trivial normal bundle, the V_i are pairwise disjoint, $L_i \subset (\text{Nonsing } V_i) \cap \partial U$ $i = 1, \ldots, k$ and $V_i \cap \partial U$ contains a neighborhood of L_i in V_i $i = 1, \ldots, k$. Then there are arbitrarily small isotopies of ∂U which fix $\bigcup_{i=1}^k L_i$ and take ∂U to a nonsingular algebraic set. If each L_i is projectively closed then we may take this algebraic set to be projectively closed also.*

Proof. We will prove this by induction on the codimension of U in W. In case U has codimension 0, then ∂U compactly separates W so the result is essentially Lemma 2.2 of [1]. (The statement of Lemma 2.2 assumes $k = 1$ but its proof works for arbitrary k.)

Now let us prove the lemma for arbitrary codimension of U. Let $m > 0$ be the codimension of U in Nonsing W. Then we have a smooth open imbedding $\beta : U \times R^m \to$ Nonsing W so that $\beta(U \times 0) = U$. By induction $\beta(\partial(U \times B^m))$ is isotopic fixing $\bigcup_{i=1}^k L_i$ to a nonsingular algebraic set Y. Also there is a smooth compact submanifold $M \subset$ Nonsing W so that U is a codimension 0 submanifold of ∂M, M has trivial normal bundle, ∂M intersects $\beta(\partial(U \times B^m))$ transversely and $\partial M \cap \beta(\partial(U \times B^m)) = \partial U$ (see Figure 1). Then by induction we may isotop ∂M fixing $\bigcup_{i=1}^k L_i$ to a nonsingular algebraic set Z. Then since these isotopies can be C^1 small, there is an isotopy of ∂U to $Y \cap Z$ fixing $\bigcup_{i=1}^k L_i$.

COROLLARY 1.3. *Let $K \subset R^2$ be a link (i.e. a union of disjoint imbedded circles). Then there is a small isotopy of R^3 taking K to a projectively closed nonsingular algebraic set.*

Proof. Let U be a Seifert surface for K, i.e. U is a smoothly imbedded compact surface with trivial normal bundle so that $\partial U = K$. The result now follows from Lemma 1.2 setting $W = R^3$, $L_i = \phi = V_i$ and $U = U$.

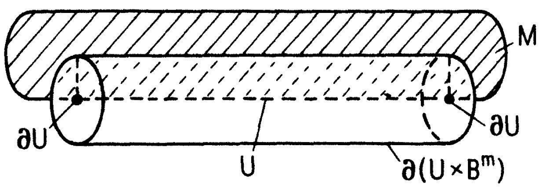


Fig. 1.

LEMMA 1.4. *Let W be a smooth n-dimensional manifold and $M \subset W$ a closed codimension 1 submanifold. Then M compactly separates W if and only if M represents 0 in $H_{n-1}(W, \partial W; \mathbf{Z}/2\mathbf{Z})$.*

Proof. Triangulate W so that M is a subpolyhedron. Then M represents 0 in $H_{n-1}(W, \partial W; \mathbf{Z}/2\mathbf{Z})$ if and only if in simplicial homology $[M] = \partial(\sum_{\sigma \in K} a_\sigma \cdot \sigma)$ where K is the set of n-simplices in W and $a_\sigma \in \mathbf{Z}/2\mathbf{Z}$ and only a finite number of a_σ are non-zero which happens if and only if $W = W_0 \cup W_1$ and $M = W_0 \cap W_1$ with W_1 compact where $W_i = \bigcup_{\substack{\sigma \in K \\ a_\sigma = i}} \mathrm{Cl}(\sigma)$.

§2. All knots are algebraic

This section is devoted to a proof of Theorem 0.1 which shows all knots are algebraic. Note that a very slight modification of the proof shows that all links are algebraic. In fact, the proof for links with an even number of components is a good deal easier since it is not necessary to add the extra one handle. Let us now proceed with the proof of Theorem 0.1.

Pick a point $z \in S^3 - K$ and a diffeomorphism $h: S^3 - z \to R^3$. We show below that there is a projectively closed nonsingular algebraic set $W \subset R^3$ and an algebraic subset $L \subset W$ so that the boundary of a smooth regular neighborhood of L in W is isotopic in R^3 to $h(K)$. Supposing this, let $W = p^{-1}(0)$ and $L = q^{-1}(0)$ for overt polynomials p and q. Let p^* and q^* be the homogenizations of p and q. Define an algebraic set

$$Z = \{(t, x) \in R \times R^3 \mid t^{2\deg(q)+1} = (q^*(t, x))^2 \quad \text{and} \quad p^*(t, x) = 0\}.$$

Then if $(t, x) \in Z$, $t \geq 0$. Also if $t = 0$ then $p^*(0, x) = 0$ so $x = 0$ by overtness of p. Hence $\varepsilon S^3 \cap Z = \varepsilon H^3 \cap Z$ where $\varepsilon H^3 = \{(t, x) \in R \times R^3 \mid t^2 + |x|^2 = \varepsilon^2$ and $t > 0\}$. If $\varphi_\varepsilon : R^3 \to \varepsilon H^3$ is the diffeomorphism $\varphi_\varepsilon(y) = (\varepsilon, \varepsilon y)/(1 + |y|^2)^{1/2}$ then

$$\varphi_\varepsilon^{-1}(\varepsilon H^3 \cap Z) = \{y \in R^3 \mid p(y) = 0$$

and

$$q^4(y)(1 + |y|^2) = \varepsilon^2\} = \{y \in W \mid q^4(y)(1 + |y|^2) = \varepsilon^2\}.$$

Since $q^4(y)(1 + |y|^2)$ is 0 only on L, [2] implies that for small enough ε, $\varphi_\varepsilon^{-1}(\varepsilon H^3 \cap Z)$ is the boundary of a smooth regular neighbourhood of L in W. Thus $\varepsilon S^3 \cap Z$ is isotopic in εS^3 to $\varphi_\varepsilon(h(K))$ so $(\varepsilon S^3, \varepsilon S^3 \cap Z)$ is diffeomorphic to (S^3, K) as we desired.

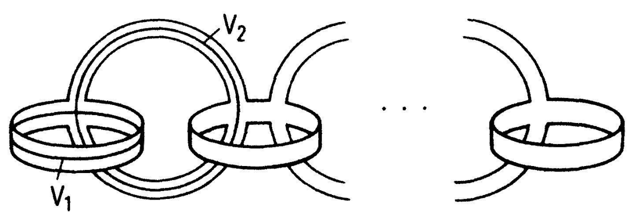


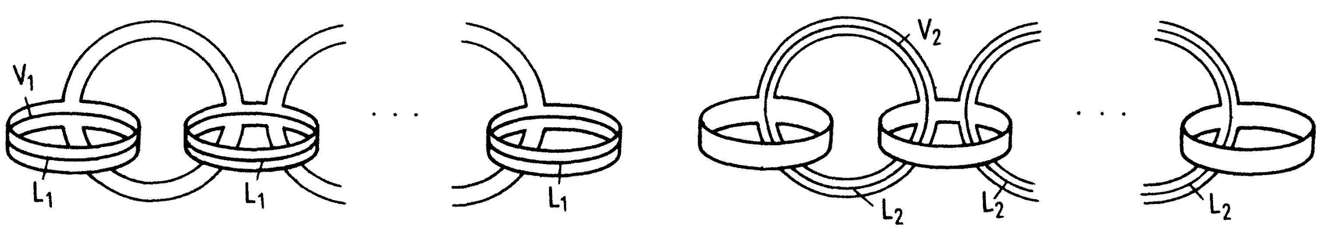
Fig. 2.

It remains to show the existence of W and L. Let $U \subset R^3$ be a connected Seifert surface for $h(K)$, that is U is an orientable compact surface with $\partial U = h(K)$. For a technical reason we must add a special 1-handle to U. In particular, let $V \subset R^3$ be some translation of the torus {points in R^3 of distance 1 from the circle $x=0$, $y^2+z^2=4\} = \{(x, y, z) \in R^3 \mid (x^2+y^2+z^2+3)^2 = 16(y^2+z^2)\}$ so that V is disjoint from U. We have a meridian V_1 and longitude V_2 of V corresponding to $\{z=0,\ x^2+(y-2)^2=1\}$ and $\{x=0,\ y^2+z^2=1\}$. Notice V, V_1 and V_2 are nonsingular projectively closed algebraic sets. We let U' be a connected sum of U and V, i.e. we run a. tube from U to V, being careful to connect the tube to V somewhere in $V-(V_1 \cup V_2)$. Hence $V_1 \subset U'$ and $V_2 \subset U'$ and the germ of U' at $V_1 \cup V_2$ is the germ of V at $V_1 \cup V_2$.

Let U'' be U' with a disc deleted from its interior. If we took U'' out of R^3 and unknotted it, it would look like Figure 2.

We will consider two transversely intersecting submanifolds of U'', L_1 and L_2 as shown in Figure 3. We want V_i to be a component of L_i $i=1, 2$. Notice that $U''-(L_1 \cup L_2)$ is diffeomorphic to $\partial U'' \times [0, 1)$. Also, $U'-((L_1-V_1) \cup L_2)$ is diffeomorphic to $\partial U' \times [0, 1)$.

We now let W' be the double of U'', i.e. since U'' has trivial normal bundle we may find a diffeomorphism $g: R^3 \to R^3$ so $g(U'') \cap U'' = \partial U''$ and $g(U'') \cup U''$ is a smooth submanifold W' of R^3. Note that $g(L_1) \cup (L_2 - V_2)$ is a union of disjoint circles. Thus by Corollary 1.3 we may isotop $g(L_1) \cup (L_2 - V_2)$ to a nonsingular projectively closed algebraic subset of R^3 so we may as well assume (after isotoping U'' a bit) that $g(L_1) \cup L_2$ is a nonsingular algebraic set. Note that W' compactly separates R^3 so by Lemma 1.2 we may isotop W', fixing $V_1 \cup L_2 \cup g(L_1)$ to a nonsingular projectively closed algebraic set W. Let $f: W' \to W$ be the time one map of this isotopy. Since $L_1 \cup g(L_1)$ compactly separates W', we may



(a) (b)

Fig. 3.

assume by Lemma 1.2 that $f(L_1 \cup g(L_1))$ is a nonsingular algebraic set. (We might have to do a little isotopy of W to itself to do this.) But then $f(L_1 \cup g(L_1)) - (V_1 \cup g(L_1)) = f(L_1 - V_1)$ is a nonsingular algebraic set by Lemma 1.1. Hence $f(L_2 \cup (L_1 - V_1)) = L_2 \cup f(L_1 - V_1)$ is an algebraic subset L of W. Notice that a regular neighborhood of L in W is isotopic to a regular neighborhood of $L_2 \cup (L_1 - V_1)$ in U'' which is isotopic to $\partial U' = h(K)$. So we are done.

§3. Obtaining nice spines

An important feature in the proof of Theorem 0.1 was that a Seifert surface for the knot had a spine of transversely intersecting circles. This section is devoted to the corresponding result we will need to prove Theorem 0.2. We will show that after adding one-handles to U we will have a spine of codimension one spheres and circles in general position. This point is used to great advantage in other papers of ours. We are indebted to Lowell Jones for discovering Lemma 3.2 for us.

EXAMPLE 3.1. At this point we need examples of nonsingular algebraic sets. Let

$$V = \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \,\middle|\, 16 \sum_{i=2}^{n+1} x_i^2 = \left(3 + \sum_{i=1}^{n+1} x_i^2\right)^2\right\},$$

$$V_1 = \{(x_1, x_2, 0, 0, \ldots, 0) \in \mathbf{R}^{n+1} \mid x_1^2 + (2 - x_2)^2 = 1\},$$

$$V_2 = \left\{(0, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \,\middle|\, \sum_{i=2}^{n+1} x_i^2 = 1\right\}.$$

A moment of examination shows that V, V_1 and V_2 are nonsingular projectively closed algebraic sets, V_1 is a circle, V_2 is an $n-1$ sphere and V_1 and V_2 intersect transversely in V. V is diffeomorphic to $S^1 \times S^{n-1}$ and can be described geometrically as the set of points in $\mathbf{R}^{n+1}$ of distance 1 from the $n-1$ sphere of radius 2 in $0 \times \mathbf{R}^n \subset \mathbf{R}^{n+1}$.

LEMMA 3.2. *Let W be a compact connected smooth manifold with boundary. Then there is a finite collection D_α, $\alpha \in A$ of imbedded discs in* int W *so that*

(1) *The boundaries $S_\alpha = \partial D_\alpha$, $\alpha \in A$ are in general position.*
(2) *W minus a finite disjoint collection of discs is a smooth regular neighborhood of $\bigcup_{\alpha \in A} S_\alpha$ in W.*

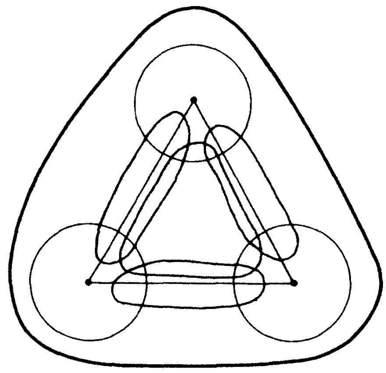


Fig. 4.

Proof. Take a smooth triangulation of W and let X be the subpolyhedron consisting of all simplices which do not touch ∂W. There will at the end be one disc D_α for each simplex of X. One way to describe these discs is as follows. For a simplex σ of dimension k, let E_σ be the union of all simplices in a $k+1$-th barycentric subdivision of σ which do not touch $\partial\sigma$. Now let D_σ be the union of all simplices in a $k+2$-th barycentric subdivision of W which intersect E_σ. Of course we must take the subdivisions in such a manner that D_σ is smooth. Figure 4 illustrates the case where X is a 2-simplex.

Let T be the closure of some component of $W - U_{\alpha\in A} S_\alpha$. If T contains a component Y of ∂W, then T is homeomorphic to $Y\times[0, 1]$, since $\bigcup_{\alpha\in A} D_\alpha$ is a regular neighborhood of X and W is a regular neighborhood of X, so $W - \bigcup_{\alpha\in A} D_\alpha$ is diffeomorphic to $\partial W\times[0, 1]$ and W is a regular neighborhood of $\bigcup_{\alpha\in A} D_\alpha$. If our component T lies in $\bigcup_{\alpha\in A} D_\alpha$ then it is of the form $\bigcap_{\alpha\in A'} D_\alpha - \bigcup_{\alpha\notin A'} \operatorname{int} D_\alpha$ for some nonempty $A'\subset A$. Let $A'=\{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ where α_j is a face of α_{j+1} for all j. We can do this because $D_\alpha \cap D_{\alpha'} \neq \varnothing$ implies α is a face of α' or vice versa. Then note that $\bigcap_{\alpha\in A'} D_\alpha - \bigcup_{\alpha\notin A'} \operatorname{int} D_\alpha$ collapses to $E_{\alpha_k} \cap (\bigcap_{\alpha\in A'} D_\alpha - \bigcup_{\alpha\notin A'} \operatorname{int} D_\alpha)$ which is always a disc in the simplex α_k.

Thus a smooth regular neighborhood of $\bigcup_{\alpha\in A} S_\alpha$ is obtained by deleting a disc from each component of $\bigcup_{\alpha\in A} D_\alpha - \bigcup_{\alpha\in A} S_\alpha$. (We in fact delete one disc for each simplex in the first barycentric subdivision of X.)

LEMMA 3.3. *Suppose W is a compact connected n-dimensional manifold with nonempty boundary. Then there is a compact manifold $W'\subset W\times R$ with trivial normal bundle in $W\times R$ and a finite collection of closed submanifolds in general position S_α, $\alpha\in A$, R_β and T_β, $\beta\in B$ and a collection of pairwise disjoint imbeddings $\varphi_\beta: R^{n+1} \to W\times R$ so that*

(a) *$\partial W'$ is isotopic to $\partial W\times 0$ in $W\times R$.*
(b) *W' is a smooth regular neighborhood of $\bigcup_{\alpha\in A} S_\alpha \cup \bigcup_{\beta\in B} R_\beta$ in W'.*
(c) *For each $\alpha\in A$ there is a $B_\alpha\subset B$ so that $S_\alpha \cup \bigcup_{\beta\in B_\alpha} T_\beta$ is a manifold compactly separating W'.*

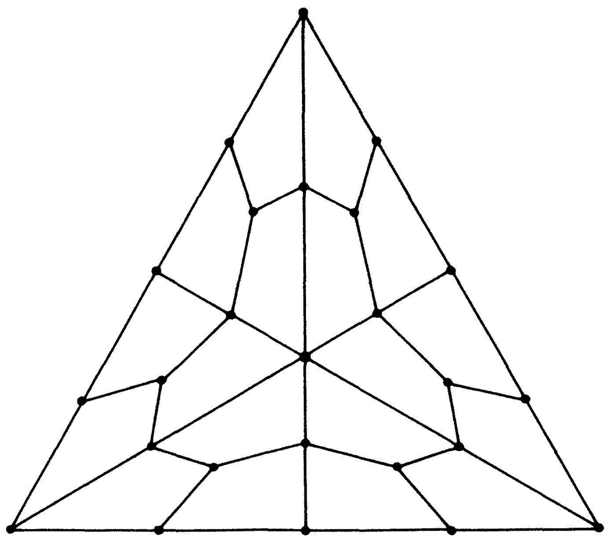


Fig. 5.

(d) $T_\beta = \varphi_\beta(V_2)$ *and* $R_\beta = \varphi_\beta(V_1)$ *and the germ of* W' *at* $T_\beta \cup R_\beta$ *is the germ of* $\varphi_\beta(V)$ *at* $T_\beta \cup R_\beta$ *where* V_1, V_2 *and* V *are as in Example* 3.1.

(*see Figure* 11).

Proof. By Lemma 3.2 we have a collection D_α, $\alpha \in A$ of n-discs in W satisfying the conclusion of Lemma 3.2. Let K be the one-complex whose vertices are components of $\bigcup_{\alpha\in A} D_\alpha - \bigcup_{\alpha\in A} S_\alpha$ and so that there is a one-simplex between two vertices if and only if there is an $\alpha' \in A$ so that the components of $\bigcup_{\alpha\in A} D_\alpha - \bigcup_{\alpha\in A} S_\alpha$ corresponding to the two vertices lie in the same component of $\bigcup_{\alpha\in A} D_\alpha - \bigcup_{\alpha\in A-\alpha'} S_\alpha$. (This K is a subcomplex of the second barycentric subdivision of the X in the proof of Lemma 3.2. For instance, for the two simplex of Figure 4, K is as in Figure 5.) Notice that K is connected, for if p and q are any two points of $\bigcup_{\alpha\in A} D_\alpha - \bigcup_{\alpha\in A} S_\alpha$ we may draw a smooth path in $\bigcup_{\alpha\in A} D_\alpha$ from p to q in general position with the collection of S_α. This path gives us a path in K for whenever the path crosses some S_α, it passes between two components of $\bigcup D_\alpha - \bigcup S_\alpha$ which have a one simplex between them.

Now pick a vertex v of K so that for some $\alpha' \in A$, v lies in the same component of $W - \bigcup_{\alpha\in A-\alpha'} S_\alpha$ as some component $*$ of $W - \bigcup_{\alpha\in A} D_\alpha$. Now define a one complex K^* to be the complex K with one additional vertex $*$ and one additional 1-simplex which lies between $*$ and v. For each vertex u of K let B_u be a smooth disc in the component u of $\bigcup D_\alpha - \bigcup S_\alpha$. Denote $U = W - \bigcup_{u\in K} \operatorname{int} B_u$. Then for each vertex of K^* we have $\kappa(u)$, an associated component of ∂U. Namely $\kappa(u) = \partial B_u$ if $u \in K$ and $\kappa(*) = \partial^* \subset \partial W$.

Notice that U is a smooth regular neighborhood of $\bigcup_{\alpha\in A} S_\alpha$. Let C be a maximal tree in K^*, (that is C is a contractible subcomplex containing all the vertices of K^*). Let B be the set of one-simplices of C. For each $\beta \in B$ we may

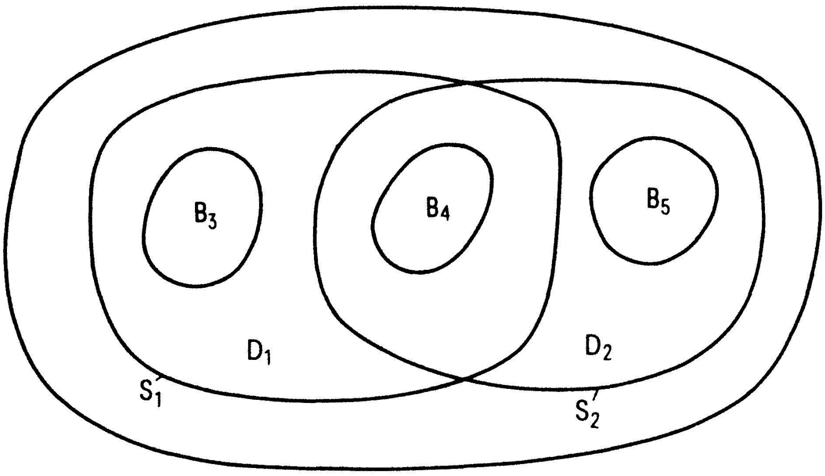


Fig. 6.

associate a smooth arc A_β in U so that

(1) The A_β's are pairwise disjoint.
(2) If β has vertices u and u' then $A_\beta \cap \partial U = \{p_\beta, p'_\beta\} = \partial A_\beta$ with $p_\beta \in u$ and $p'_\beta \in u'$ and A_β is transverse to ∂U.
(3) A_β is in general position with the collection S_α, $\alpha \in A$ and $A_\beta \cap \bigcup_{\alpha \in A} S_\alpha =$ one point q_β.

For instance, suppose W is a 2-disc and $A = \{1, 2\}$ so the discs D_α and B_u are as in Figure 6.

Then K and K^* are as in Figure 7 and $C = K^*$ because K^* is contractable. Denote the simplices of C as in Figure 7.

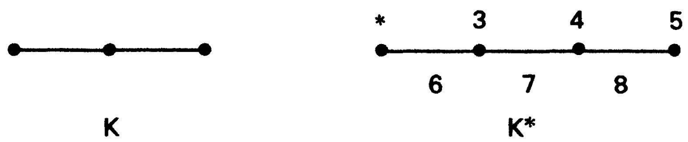


Fig. 7.

We now show the A_β in Figure 8. Pick an imbedding $g:(\bigcup A_\beta)\times \mathbf{R}^{n-1} \to U$ with $g^{-1}(\partial U) = \bigcup (\partial A_\beta)\times \mathbf{R}^{n-1}$ and $g(x, 0) = x$ for all x. Now we define $W' \subset W \times \mathbf{R}$ by

$$W' = U\times 0 \cup g(\bigcup \partial A_\beta \times B^{n-1})\times[0, 1] \cup g(\bigcup A_\beta \times B^{n-1})\times 1$$

with the corners rounded. We define $R_\beta \subset W'$ by

$$R_\beta = \partial A_\beta \times [0, 1] \cup A_\beta \times \{0, 1\}.$$

Thus, W' is obtained from U by attaching 1-handles to the boundary of U.

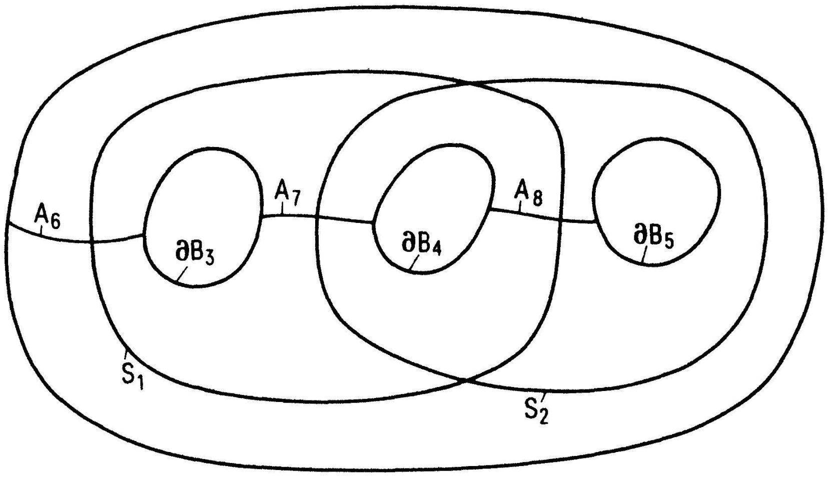


Fig. 8.

Each R_β is A_β union the core of one of these 1-handles. See Figure 9 for the W' constructed via Figures 6, 7 and 8.

It is easy to see (b) from the fact that U is a smooth regular neighborhood of $\bigcup_{\alpha\in A} S_\alpha$. Likewise, (a) is easily seen since $\partial W'$ is obtained from ∂W by "growing" the tree C, see Figure 10.

We now construct the T_β's. For each $\beta\in B$, let $Y_\beta=$ the component of $\partial W'-g(q_\beta\times S^{n-2})\times 1$ which does not contain points of $\partial W\times 0$ (i.e. Y_β is the limb we get by sawing the branch β). Notice that each Y_β is an open $n-1$ disc. Let $h:\partial W'\times[0,1)\to W'-(\bigcup_{\alpha\in A} S_\alpha\cup\bigcup_{\beta\in B} R_\beta)$ be a collar, i.e. an open imbedding so that $h(x,0)=x$ for all $x\in\partial W'$. By extending collars we may assume that $h((g(x,y),t),s)=(g(x,(1-s)y),t)$ for all $y\in S^{n-2}$ and $(x,t)\in\bigcup_\beta \partial A_\beta\times[0,1]\cup\bigcup_\beta A_\beta\times 1$. We may put an order $<$ on B by $\beta'<\beta$ if $Y_{\beta'}\subsetneq Y_\beta$. Now let $\gamma:B\to(0,1)$ be any $1-1$ order preserving function. Then define $T_\beta=h(Y_\beta\times(1-\gamma(\beta)))\cup g(q_\beta\times\gamma(\beta)B^{n-1})\times 1$ after smoothing out the corner. (See Figure 11.)

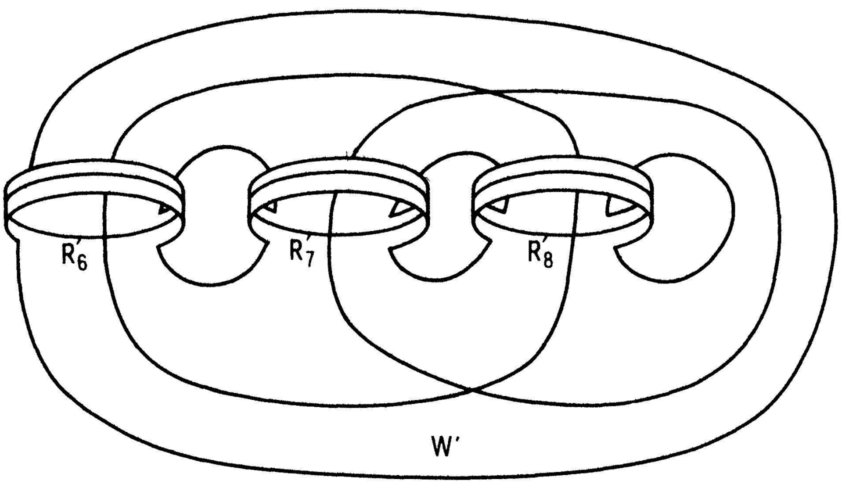


Fig. 9.

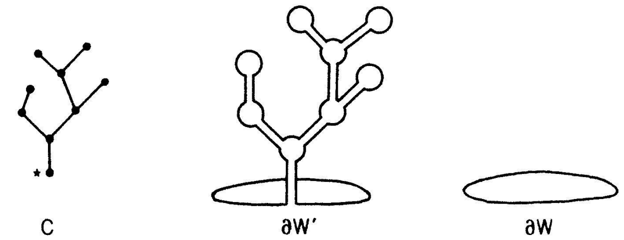


Fig. 10.

Now we show that (c) is satisfied. For any $\alpha \in A$, let $B_\alpha = \{\beta \in B \mid R_\beta \cap S_\alpha \neq \varnothing\}$. Then we claim $S_\alpha \cup \bigcup_{\beta \in B_\alpha} T_\beta$ represents 0 in $H_{n-1}(W', \mathbf{Z}/2\mathbf{Z})$. Lemma 1.4 then insures that $S_\alpha \cup \bigcup_{\beta \in B_\alpha} T_\beta$ is a manifold compactly separating W'. Let E be the set of vertices of K. For each $\beta \in B$ let

$$E_\beta = \{u \in E \mid Y_\beta \cap \kappa(u) \text{ is nonempty}\}$$

(recall $\kappa(u)$ is a component of ∂U) and for each $\alpha \in A$ let

$$E_\alpha = \{u \in E \mid \text{so that } u \subset D_\alpha\}.$$

Then if [ ] denotes the homology class in $H_{n-1}(W', \mathbf{Z}/2\mathbf{Z})$ we easily see that

$$[T_\beta] = \sum_{u \in E_\beta} [\kappa(u)] \quad \text{and} \quad [S_\alpha] = \sum_{u \in E_\alpha} [\kappa(u)].$$

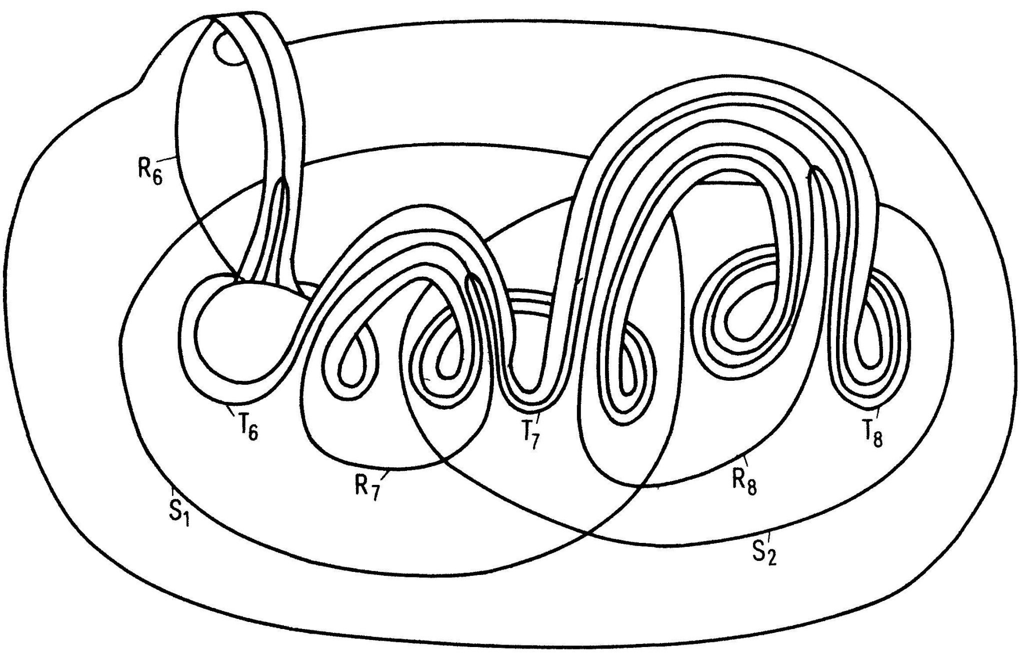


Fig. 11.

Now for $u \in E$, let C_u be the unique subcomplex of C which is homeomorphic to a line with one end u and the other end $*$. The 1-simplices of C_u are those β for which $u \in E_\beta$. Let $B_{\alpha u} = \{\beta \in B_\alpha \mid u \in E_\beta\}$ and notice that $B_{\alpha u}$ is just the set of 1-simplices of C_u one of the vertices of which is in E_α and the other vertex of which is not in E_α. Hence if $u \notin E_\alpha$, $B_{\alpha u}$ has even cardinality and if $u \in E_\alpha$ then $B_{\alpha u}$ has odd cardinality. Thus we have

$$\left[S_\alpha \cup \bigcup_{\beta \in B_\alpha} T_\beta\right] = \sum_{u \in E_\alpha} [\kappa(u)] + \sum_{u \in E} (\#B_{\alpha u})[\kappa(u)] = 0$$

where $\#B_{\alpha u}$ is the cardinality of $B_{\alpha u}$.

It remains to show the existence of the φ_β's. Notice that each R_β bounds a 2-disc $R'_\beta = A_\beta \times [0, 1]$ in $W \times R$. Likewise, each T_β bounds an n-disk T'_β in $W \times R$ which is a union of B_u's and part of the 1-handles, more precisely it is given as follows: Let $f: W' \times R \to W \times R$ be an imbedding so that $f(x, 0) = x$ all $x \in W'$ and $R'_\beta \cap f(W' \times (0, 1))$ is empty and $f((x, 0), t) = (x, -t)$ for $(x, t) \in U \times [0, 1]$. Let

$$H_\beta = h((Y_\beta \cap \partial U) \times [0, 1 - \gamma(\beta)]) \cup \text{closure}\,(h((Y_\beta - \partial U) \times [1 - \gamma(\beta), 1))).$$

We now define

$$T'_\beta = f(T_\beta \times [0, \gamma(\beta)] \cup H_\beta \times \gamma(\beta)) \cup \bigcup_{u \in E_\beta} B_u \times -\gamma(\beta),$$

i.e. we push out a little bit and then fill in the holes we originally took out of W to make U (see Figure 12).

Now $R'_\beta \cup T'_\beta$ is collapsible so its smooth regular neighborhood is a ball, which we make the image of φ_β, then φ_β exists since R'_β and T'_β can be isotoped inside the ball to a standard pair of discs.

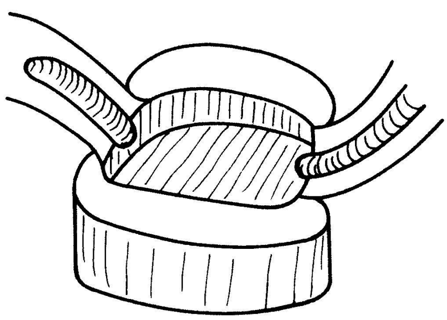


Fig. 12.

§4. Generalized knots are algebraic

This section is devoted to a proof of Theorem 0.2. We may assume each component of U has non-empty boundary and after applying an isotopy to S^{k-1} we may also assume that U lies in just one hemisphere $H^{k-1}=\{(x_1,\ldots,x_k)\in S^{k-1}\mid x_k>0\}$. Let $n<k-1$ be the dimension of U. Let $\pi:R^k\to R^{k-1}$ be projection onto the first $k-1$ coordinates and let $W=\pi(U)$. Then W is a submanifold of R^{k-1} diffeomorphic to U. Let $h:W\times\mathbf{R}\times\mathbf{R}^{k-2-n}\to R^{k-1}$ be an imbedding so $h(x,0,0)=x$ for all $x\in W$. By Lemma 3.3 there is a $W'\subset W\times R$ and S_α, $\alpha\in A$ and R_β, T_β and φ_β satisfying the conclusions of Lemma 3.3. Notice that $h(W'\times 0)$ has trivial normal bundle in R^{k-1} and $h(\partial W'\times 0)$ is isotopic to ∂W. Pick injective linear transformations $\lambda_\beta:R^{n+1}\to R^{k-1}$ for each $\beta\in B$ so that $\lambda_\beta(V)\cap\lambda_{\beta'}(V)$ is empty for $\beta\neq\beta'$. In addition, if $n+1=k-1$ we require λ_β to preserve orientation if $h\varphi_\beta$ preserves orientation and we require λ_β to reverse orientation if $h\varphi_\beta$ reverses orientation. Then since any two orientation preserving imbeddings of a disc are isotopic, we may (after isotoping R^{k-1}) assume that $h(\varphi_\beta(x),0)=\lambda_\beta(x)$ for all x near V in R^{n+1}. Let $U''=h(W'\times 0)$ and $L=\bigcup_{\beta\in B}\lambda_\beta(V_1\cup V_2)$. Notice that each $\lambda_\beta(V)$ and $\lambda_\beta(V_i)$ is a nonsingular projectively closed algebraic set, $L\subset U''$ and the germ of U'' at L is the germ of $\bigcup\lambda_\beta(V)$ at L.

Now let Y be a smooth closed n dimensional submanifold of R^{k-1} so that $U''\subset Y$ and Y bounds a compact smooth submanifold of R^{k-1} with trivial normal bundle. For instance, if $h':U''\times R\times R^{k-2-n}\to R^{k-1}$ is a trivialization of the normal bundle of U'', let Y be $h'(\partial(U''\times[0,1]\times 0))$ with corners smoothed.

By Lemma 1.2 there is a projectively closed nonsingular algebraic set $X\subset R^{k-1}$ so that $L\subset X$ and Y is isotopic to X fixing L. Let $g:Y\to X$ be the time one map of this isotopy.

Notice that for each $\alpha\in A$, $g(S_\alpha)\cup\bigcup_{\beta\in B_\alpha}\lambda_\beta(V_2)$ compactly separates X. Hence by isotoping a little more we may assume by Lemma 1.2 that each $g(S_\alpha)\cup\bigcup_{\beta\in B_\alpha}\lambda_\beta(V_2)$ is a nonsingular algebraic set. But then by Lemma 1.1 each $g(S_\alpha)$ is a nonsingular algebraic set.

Pick polynomials $p:R^{k-1}\to R$ and $q:R^{k-1}\to R$ so that p is overt, $X=p^{-1}(0)$ and $q^{-1}(0)=\bigcup_{\alpha\in A}g(S_\alpha)\cup\bigcup_{\beta\in B}\lambda_\beta(V_1)$. Let p^* and q^* be the homogenizations of p and q.

Define an algebraic set $Z\in R^{k-1}\times R$ by

$$Z=\{(x,t)\in R^{k-1}\times R\mid t^{2\deg(q)+1}=(q^*(t,x))^2\text{ and }p^*(t,x)=0\}.$$

Notice that $(x,t)\in Z$ implies $t\geq 0$ and if $t=0$ then $x=0$ also since p is overt. Hence $\varepsilon S^{k-1}\cap Z=\varepsilon H^{k-1}\cap Z$ where $\varepsilon H^{k-1}=\{(x_1,\ldots,x_k)\in\varepsilon S^{k-1}\mid x_k>0\}$. If

$\varphi_\varepsilon : \mathbf{R}^{k-1} \to H^{k-1}$ is the diffeomorphism $\varphi_\varepsilon(y) = (\varepsilon y, \varepsilon)/(1+|y|^2)^{1/2}$ then $\varphi_\varepsilon^{-1}(\varepsilon H^{k-1} \cap Z) = \{y \in \mathbf{R}^{k-1} \mid p(y) = 0,\ q^4(y)(1+|y|^2) = \varepsilon^2\}$ which for small enough $\varepsilon > 0$ is the boundary of an algebraic regular neighborhood of $q^{-1}(0)$ in X which by [2] is isotopic to ∂W. Hence $\varepsilon S^{k-1} \cap Z$ is isotopic in εS^{k-1} to $\varepsilon \partial U$.

Since [2] has not yet appeared, we indicate to the reader how a proof would go for this special case. Norice that $q^{-1}(0)$ is a union of nonsingular algebraic sets in general position, so we could have taken q to be a product of polynomials q_i such that for small $\varepsilon > 0$; $X \cap q_i^{-1}([-\varepsilon, \varepsilon])$ is a tubular neighborhood in X of the nonsingular algebraic set $q_i^{-1}(0)$. Now (using the curve selection lemma, say) one sees that for small ε; $\{y \in X \mid \Pi q_i^4(y)(1+|y|^2) \le \varepsilon\}$ is the union of tubular neighborhoods of these nonsingular algebraic sets, rounded off where they intersect; i.e. it is a smooth regular neighborhood of $q^{-1}(0)$. Since ∂W is the boundary of a smooth regular neighborhood of $q^{-1}(0)$, uniqueness of smooth regular neighborhoods gives an isotopy from ∂W to $\varphi_\varepsilon^{-1}(\varepsilon H^{k-1} \cap Z)$ as we desired.

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