Comment. Math. Helvetici 56 (1981) 163–178 0010–2571/81/002163–16$01.50+0.20/0

Characterization of unbounded spectral operators with spectrum in a half-line

SHMUEL KANTOROVITZ[1]

Abstract. Let T be a possibly unbounded linear operator in the Banach space X such that $R(t)=(t+T)^{-1}$ is defined on $\mathbf{R}^+$. Let $S=TR(I-TR)$ and let $B(.,.)$ denote the Beta function.

THEOREM 1.1. *T is a scalar-type spectral operator with spectrum in $[0,\infty)$ if and only if*

$$\sup\left\{B(k,k)^{-1}\int_0^\infty |x^*S^k(t)x|\,dt/t;\ \|x\|\leq 1,\ \|x^*\|\leq 1,\ k\geq 1\right\}<\infty.$$

A "local" version of this result is formulated in Theorem 2.2.

Introduction

A problem of fundamental importance in Mathematical Physics consists of the discovery of "practical" criteria for the self-adjointness of symmetric operators. Extensive work done in this general direction is efficiently summarized in Reed and Simon [6]. Selfadjointness has found a proper generalization to Banach space in the concept of scalar-type spectrality invented by N. Dunford in the early 1950's and developed since then by his students and followers. The monograph of N. Dunford and J. T. Schwartz [1; Part III] contains a detailed exposition of the subject and an updated bibliography up to 1968. We shall freely use the terminology of [1].

Let T be a possibly unbounded linear operator with domain $D(T)$ in a Banach space X. Suppose there exists a (strongly) countably additive spectral measure E on the Borel σ-field $\mathfrak{B}(\mathbf{R})$ such that

$$D(T)=\left\{x\in X \mid \text{strong}\lim_n \int_{-n}^{n} sE(ds)x \text{ exists}\right\}$$

---

[1] Research partially done while the author was a visiting member at the Forschungsinstitut für Mathematik, ETH, Zürich.

Research supported by the Israel Academy of Science and Bar-Ilan University Research Authority.

and

$$Tx = \lim_n \int_{-n}^{n} sE(ds)x, \qquad x \in D(T).$$

One then says that T is a (real) scalar-type spectral operator, with resolution of the identity E.

In [1, Chapters XIX and XX], the (scalar-type) spectrality of various perturbations $T = S + P$ is studied, and many deep results providing sufficient conditions are obtained, when S is either a relatively simple differential operator, or a selfadjoint operator in Hilbert space, or a multiplication operator. Accordingly, the methods are very specialized and have to overcome many technical difficulties. Even the statements of the theorems involve heavily technical assumptions (see for example Theorems XIX.4.16; XX.1.12, 2.21, 2.22, 4.9, etc. . . .). Methods run from direct efforts toward the construction of the resolution of the identity, to the Friedrichs method of similar operators, to the Kato-Kuroda method of wave operators and similarity. The authors remark that the material in their exposition is still somewhat "fragmentary", "and is presented in the hope that it may stimulate research in the topics treated".

Motivated by the fact that the selfadjointness analysis for *semibounded* symmetric operators in Hilbert space is enormously simpler than the general case, we consider in this paper the construction of a (scalar-type) spectrality criterion for operators whose resolvent set contains a half-line. Our main result (Theorem 1.1) provides a criterion in terms of the asymptotic behavior of adequate functions of the resolvent operator. We also obtain a "localized version" of this result (Theorem 2.2), in the spirit of [4]. Also in the spirit of previous work of ours [2, 3, 4, 5], we shall use transform methods – in this case, naturally, the Stieltjes transform. The following classical result of Widder's is the key to our analysis. Let M and D denote respectively the formal operators of multiplication

$$M: f(t) \to tf(t)$$

and differentiation $D: f \to f'$.

The Widder (formal) differential operators L_k are given by

$$L_k = c_k M^{k-1} D^{2k-1} M^k \qquad k = 1, 2, \ldots$$

where $c_1 = 1$ and $c_k = (-1)^{k-1}[k!(k-2)!]^{-1}$ for $k \geqslant 2$.

The STR (Stieltjes Transform Representation) *Theorem.*
Let f be a C^∞ *complex function on* $\mathbf{R}^+=(0,\infty)$ *such that*

$$K=\sup_{k\geq 1}\int_0^\infty |L_k f(t)|\,dt<\infty. \tag{$*$}$$

Then there exists a unique complex regular Borel measure μ *on* $\mathbf{R}^+$ *such that*

$$f(t)=\int_0^\infty (t+s)^{-1}\mu(ds)$$

$$\operatorname{var}\mu\leq K'$$

where $K'=2K+|A|$ *and* $A=\lim_{t\to 0+} tf(t)$. A $(*)$.

In [7; Theorem 16, p. 361], the STR theorem is stated (in more classical terminology) for *real f*. The complex ease is a trivial consequence, and the estimate for var μ follows easily from the proof in the cited reference. Note that the *existence* of the limit A follows from Condition $(*)$.

We conclude this introduction with a list of notation. The (normed) dual space of the Banach space X is denoted by X^*. The norm-closed unit balls of X and X^* are X_1 and X_1^* respectively. $B(X)$ denotes the Banach algebra of all bounded linear operators in X. For $s, t>0$, $B(s,t)$ stands for the Beta function

$$B(s,t)=\Gamma(s)\Gamma(t)\Gamma(s+t)^{-1}$$

$\mathfrak{B}(\mathbf{R}^+)$ is the Borel σ-field of $\mathbf{R}^+$, and $\|\cdot\|_1$ denotes the $L^1(\mathbf{R}^+, dt/t)$-norm (*attention*! the measure is dt/t!). Strong limits mean limits in the given norm of X.

1. Operators with spectrum in a half-line

Since spectrality is preserved by the map $T\to\alpha I+\beta T$ $(\alpha,\beta\in\mathbf{C})$, we may restrict our discussion, without loss of generality, to the case where the half-line of the title is $[0,\infty)$.

Let R denote the resolvent of $-T$, that is $R(t)=(t+T)^{-1}$. Thus, R is defined on $\mathbf{R}^+$, and so is the $B(X)$-valued function

$$S=TR(I-TR). \tag{1.1}$$

For any weakly measurable function $F:\mathbf{R}^+\to B(X)$, we let

$$|||F|||=\sup\{\|x^*F(\cdot)x\|_1;\ x\in X_1,\ x^*\in X_1^*\}. \tag{1.2}$$

Clearly, $|||\cdot|||$ is a semi-norm on the vector space $\mathfrak{F}$ (under pointwise operations) of all such functions F with $|||F|||<\infty$, and

$$\|x^*F(\cdot)x\|_1 \leq |||F|||\,\|x\|\,\|x^*\| \tag{1.3}$$

for all $F\in\mathfrak{F}$, $x\in X$ and $x^*\in X^*$.

Since $S(t)=tR(t)[I-tR(t)]$, the functions S^k $(k=1,2,\dots)$ are of class C^∞ on $\mathbf{R}^+$; in particular, $|||S^k|||$ is well-defined (finite or infinite). We are now prepared to state our spectrality criterion for T, which bears some analogy with the Hille-Yosida criterion for semi-group generation.

THEOREM 1.1 *Let T be a possibly unbounded operator in the reflexive Banach space X, whose resolvent set contains the half-line $(-\infty,0)$. Then T is spectral of scalar type with spectrum in $[0,\infty)$ if and only if*

$$|||S^k|||\leq MB(k,k) \qquad k=1,2,\dots \tag{1.4}$$

for some constant M.

Proof. Necessity. Since $TR(t)=I-tR(t)$ for $t\notin\sigma(T)$, therefore

$$S(t)=t[I-tR(t)]R(t)=tTR(t)^2. \tag{1.5}$$

Let T be spectral of scalar type with spectrum in $[0,\infty)$, and let E denote its resolution of the identity. By (1.5), we have for $t>0$ and $k=1,2,\dots$

$$S^k(t)=t^kT^k(t+T)^{-2k}=t^k\int_0^\infty s^k(t+s)^{-2k}E(ds)=\int_0^\infty (s/t)^k(1+s/t)^{-2k}E(ds).$$

Therefore, for all $x\in X_1$ and $x^*\in X_1^*$,

$$\|x^*S^kx\|_1\leq\int_0^\infty\int_0^\infty (s/t)^k(1+s/t)^{-2k}\,|x^*E(ds)x|\,dt/t.$$

If we interchange the order of integration and substitute $u=s/t$ in the inner integral, the right hand side becomes

$$\int_0^\infty\int_0^\infty u^k(1+u)^{-2k}\,du/u\,|x^*E(ds)x|=B(k,k)\operatorname{var}(x^*Ex)\leq B(k,k)\operatorname{var}E<\infty.$$

Applying Tonelli's theorem, we conclude that

$$|||S^k||| \leq \text{var } E \cdot B(k, k) \qquad k = 1, 2, \ldots. \tag{1.6}$$

Observe that the reflexivity assumption on X is not needed for the necessity part of the theorem.

Sufficiency. Let L_k $(k = 1, 2, \ldots)$ be the Widder formal differential operators mentioned in the introduction. A straightforward calculation using Leibnitz' rule shows that

$$L_k = c'_k \sum_{j=0}^{k} \Gamma(k+j)^{-1} \binom{k}{j} M^{k+j-1} D^{k+j-1} \tag{1.7}$$

where $c'_1 = 1$ and $c'_k = (-1)^{k-1} B(k-1, k+1)$ for $k \geq 2$. Fix $x \in X$ and $x^* \in X^*$, and consider the C^∞ function on $\mathbf{R}^+$

$$f(t) = x^* R(t) x \qquad t \in \mathbf{R}^+.$$

Since

$$D^{k+j-1} f = (-1)^{k+j-1} \Gamma(k+j) x^* R^{k+j} x,$$

we have

$$L_k f(t) = c''_k t^{-1} x^* (tR)^k \sum_{j=0}^{k} \binom{k}{j} (-tR)^j x = c''_k t^{-1} x^* S^k(t) x \tag{1.8}$$

where $c''_1 = 1$ and $c''_k = B(k-1, k+1)^{-1}$ for $k \geq 2$.

Therefore, by (1.3) and (1.4),

$$\begin{aligned} \int_0^\infty |L_k f(t)| \, dt &= c''_k \, \|x^* S^k x\|_1 \leq c''_k \, |||S^k||| \, \|x\| \, \|x^*\| \\ &\leq M \, \|x\| \, \|x^*\| \qquad k = 1, 2, \ldots, \end{aligned} \tag{1.9}$$

since $c''_1 \, |||S||| \leq M$ and for $k \geq 2$

$$c''_k \, |||S^k||| \leq MB(k, k)/B(k-1, k+1) = M(k-1)/k \leq M.$$

By the STR theorem, it follows that there exists a unique complex regular Borel measure $\mu = \mu(\cdot \mid x, x^*)$ on $\mathbf{R}^+$ such that

$$x^*R(t)x = \int_0^\infty (t+s)^{-1}\mu(ds \mid x, x^*), \; (t>0, x \in X, x^* \in X^*). \tag{1.10}$$

In particular,

$$\sup_{t>0} |x^*tR(t)x| = \sup_{t>0} \left| \int_0^\infty t(t+s)^{-1}\mu(ds \mid x, x^*) \right| \leq \operatorname{var} \mu < \infty.$$

By the Uniform Boundedness Theorem, it follows that

$$H = \sup_{t>0} \|tR(t)\| < \infty. \tag{1.11}$$

Hence, for our function f, $|tf(t)| \leq H \|x\| \|x^*\|$ for all $t>0$, and therefore $|A| \leq H\|x\| \|x^*\|$ (with notation as in the STR theorem). We conclude that

$$\operatorname{var} \mu(\cdot \mid x, x^*) \leq M' \|x\| \|x^*\| \tag{1.12}$$

where $M' = 2M + H$.

The uniqueness of the representation (1.10) implies that for each fixed $\delta \in \mathfrak{B}(\mathbf{R}^+)$, $\mu(\delta \mid .\,,.)$ is a bilinear form on $X \times X^*$. By (1.12), the form is bounded (with bound M'), and since X is reflexive, there exists a unique function $E : \mathfrak{B}(\mathbf{R}^+) \to B(X)$ such that

$$\mu(\cdot \mid x, x^*) = x^*E(\cdot)x \tag{1.13}$$

for all $x \in X$ and $x^* \in X^*$.

We shall now verify that E is a spectral measure on $\mathfrak{B}(\mathbf{R}^+)$. By (1.13) and (1.12), E is weakly, hence strongly countably additive, and $\|E(\delta)\| \leq M'$ for all $\delta \in \mathfrak{B}(\mathbf{R}^+)$.

Let $U \in B(X)$ be such that $UD(T) \subset D(T)$ and $TUx = UTx$ for all $x \in D(T)$ (i.e., $TU \supset UT$, which is the usual definition of commutativity with an unbounded operator T). Equivalently, $UR(t) = R(t)U$ for all $t>0$. By (1.10), for all $t>0$, $x \in X$ and $x^* \in X^*$,

$$\int_0^\infty (t+s)^{-1}\mu(ds \mid x, U^*x^*) = [U^*x^*]R(t)x = x^*R(t)Ux = \int_0^\infty (t+s)^{-1}\mu(ds \mid Ux, x^*).$$

By the uniqueness statement of the STR theorem, it follows that

$$\mu(\delta \mid x, U^*x^*) = \mu(\delta \mid Ux, x^*)$$

for all $\delta \in \mathfrak{B}(\mathbf{R}^+)$, $x \in X$ and $x^* \in X^*$.

Hence, by (1.13),

$$UE(\delta) = E(\delta)U \qquad \delta \in \mathfrak{B}(\mathbf{R}^+). \tag{1.14}$$

Taking in particular $U = R(u)$ $(u > 0)$, we obtain that $E(\delta)D(T) \subset D(T)$ for all $\delta \in \mathfrak{B}(\mathbf{R}^+)$. For $t > u > 0$, we have by the First Resolvent Equation

$$tR(t)R(u) = t(t-u)^{-1}R(u) - [tR(t)](t-u)^{-1}.$$

Fixing u and letting $t \to \infty$, it follows from (1.11) that $tR(t)R(u) \to R(u)$ in the uniform operator topology. By (1.10) and the Lebesgue dominated convergence theorem, we have for all $u > 0$, $x \in X$, and $x^* \in X^*$

$$\begin{aligned} x^*R(u)x &= \lim_{t \to \infty} x^*tR(t)R(u)x \\ &= \lim_{t \to \infty} \int_0^\infty t(t+s)^{-1}\mu(ds \mid R(u)x, x^*) \\ &= \mu(\mathbf{R}^+ \mid R(u)x, x^*) = x^*E(\mathbf{R}^+)R(u)x. \end{aligned}$$

Hence, by (1.14) with $U = R(u)$, $R(u)E(\mathbf{R}^+) = R(u)$. Applying $u + T$ to both sides (on the left), we conclude that $E(\mathbf{R}^+) = I$. For $t, u > 0$, $t \neq u$, the First Resolvent Equation implies that

$$\begin{aligned} \int_0^\infty (t+s)^{-1}\mu(ds \mid R(u)x, x^*) &= x^*R(t)R(u)x \\ &= (t-u)^{-1} \int_0^\infty [(u+s)^{-1} - (t+s)^{-1}]\mu(ds \mid x, x^*) \\ &= \int_0^\infty (t+s)^{-1}(u+s)^{-1}\mu(ds \mid x, x^*). \end{aligned}$$

The uniqueness claim in the STR theorem gives

$$\mu(\delta \mid R(u)x, x^*) = \int_0^\infty (u+s)^{-1}\chi_\delta(s)\mu(ds \mid x, x^*) \tag{1.15}$$

for all $u > 0$, $\delta \in \mathfrak{B}(\mathbf{R}^+)$, etc. . . .

On the other hand, by (1.13), (1.14) with $S=R(u)$, and (1.10),

$$\mu(\delta \mid R(u)x, x^*) = x^*R(u)E(\delta)x$$

$$= \int_0^\infty (u+s)^{-1}\mu(ds \mid E(\delta)x, x^*). \tag{1.16}$$

Comparing the two relations, we obtain (again by uniqueness!)

$$\mu(\sigma \mid E(\delta)x, x^*) = \int_0^\infty \chi_\sigma(s)\chi_\delta(s)\mu(ds \mid x, x^*) = \mu(\sigma \cap \delta \mid x, x^*)$$

for all $\sigma, \delta \in \mathfrak{B}(\mathbf{R}^+)$, etc. ..., that is, by (1.13),

$$E(\sigma)E(\delta) = E(\sigma \cap \delta)$$

for all $\sigma, \delta \in \mathfrak{B}(\mathbf{R}^+)$.

In conclusion, E is a (strongly) countably additive spectral measure on $\mathfrak{B}(\mathbf{R}^+)$, which sends $D(T)$ into itself and commutes with every $U \in B(X)$ which commutes with t. By (1.13), we may rewrite (1.10) in the form

$$R(t) = \int_0^\infty (t+s)^{-1}E(ds) \qquad (t>0) \tag{1.17}$$

where the integral makes sense in $B(X)$ as described for example in [1, Part II; pp. 891–892].

From (1.17) and [1, Part III; Theorems XVIII.2.17 and XVIII.2.11(h)] we may easily conclude that T is spectral of scalar type with resolution of the identity E and spectrum in $[0, \infty)$.

However the presentation can be made direct and selfcontained with only a small additional effort.

By (1.17), for all $t>0$,

$$\int_0^\infty s(t+s)^{-1}E(ds) = \int_0^\infty [1-t(t+s)^{-1}]E(ds)$$

$$= I - tR(t) = TR(t). \tag{1.18}$$

For all $n>0$, $y \in X$, and $x^* \in X^*$, we have by (1.13) and (1.16)

$$x^* \int_0^n sE(ds)R(t)y = \int_0^n s\mu(ds \mid R(t)y, x^*)$$

$$= \int_0^n s(t+s)^{-1}\mu(ds \mid y, x^*) = x^* \int_0^n s(t+s)^{-1}E(ds)y$$

Therefore, by (1.15),

$$\int_0^n sE(ds)R(t)y = \int_0^n sR(t)E(ds)y = \int_0^n s(t+s)^{-1}E(ds)y \tag{1.19}$$

for all $n, t>0$ and $y \in X$.

Suppose now that $x \in D(T)$. Then $x = R(t)y$ for $t>0$ fixed and suitable $y \in X$. By (1.19),

$$\int_0^n sE(ds)x = \int_0^n s(t+s)^{-1}E(ds)y \xrightarrow[n\to\infty]{} \int_0^\infty s(t+s)^{-1}E(ds)y$$

(strong convergence).

By (1.18), the above limit equals $TR(t)y = Tx$, that is

$$Tx = \text{strong} \lim_{n\to\infty} \int_0^n sE(ds)x, \qquad x \in D(T). \tag{1.20}$$

On the other hand, if $x \in X$ is such that the limit on the right of (1.20) exists, then denoting this limit by z, we have for any $t>0$

$$\begin{aligned} R(t)z &= \text{strong} \lim_{n\to\infty} \int_0^n sR(t)E(ds)x \\ &= \text{strong} \lim_{n\to\infty} \int_0^n s(t+s)^{-1}E(ds)x = \int_0^\infty s(t+s)^{-1}E(ds)x \\ &= x - tR(t)x, \end{aligned}$$

where we used (1.19) and (1.18). Hence $x = R(t)[z+tx] \in D(T)$. This shows that

$$D(T) = \left\{ x \in X \mid \text{strong} \lim_{n\to\infty} \int_0^n sE(ds)x \text{ exists} \right\},$$

and together with (1.20), this completes the proof of the scalar-type spectrality of T. Since the (uniquely determined) resolution of the identity E is supported by $[0,\infty)$, the spectrum of T is necessarily contained in $[0,\infty)$. Q.E.D.

2. Local spectral analysis

As in the preceding section, T is a possibly unbounded operator in the reflexive Banach space X, whose resolvent set contains a half-line, which we assume to be $(-\infty, 0)$ without loss of generality. We shall use the function $S:\mathbf{R}^+ \to B(X)$ in order to obtain a so-called "local" spectral analysis for any such operator T (cf. [4] for an analogous treatment of *bounded* operators).

For any $x \in X$, we let

$$\begin{aligned} |||x|||_0 &= \sup\{B(k, k)^{-1} \|x^*S^k x\|_1;\ x^* \in X_1^*,\ k = 1, 2, \ldots\}, \\ |||x||| &= \max\{\|x\|, |||x|||_0\}, \end{aligned} \tag{2.1}$$

and

$$Z = \{x \in X;\ |||x||| < \infty\}. \tag{2.2}$$

The functional $|||\cdot|||$ is a norm on the linear manifold Z, and in fact $|||x||| \geq \|x\|$ on Z. An easy calculation shows for example that if $x \in D(T)$ is an eigenvector of T corresponding to an eigenvalue $\lambda \in \mathbf{R}$, then $|||x||| = \|x\|$, so that Z contains all such eigenvectors.

If $U \in B(X)$ commutes with T (that is $UT \subset TU$), then Z is U-invariant, and for all $x \in Z$

$$|||Ux||| \leq \|U\|\ |||x|||. \tag{2.3}$$

It is also clear that

$$\|x^*S^k x\|_1 \leq \|x^*\|\ |||x|||\ B(k, k) \tag{2.4}$$

for all $x \in Z$, $x^* \in X^*$, and $k = 1, 2, \ldots$.

PROPOSITION 2.1. *$(Z, |||\cdot|||)$ is a Banach space.*

Proof. Let $\{x_n\} \subset Z$ be $|||\cdot|||$-Cauchy. Since $|||\cdot||| \geq \|\cdot\|$, $\{x_n\}$ converges to some $x \in X$ in the given norm. For each $k \geq 0$ and $x^* \in X_1^*$, $x^*S^k(\cdot)x_n \to x^*S^k(\cdot)x$ pointwise as $n \to \infty$, and by (2.4),

$$B(k, k)^{-1} \|x^*S^k x_n\|_1 \leq |||x_n||| \leq K. \qquad \text{(for all } n\text{)}.$$

By Fatou's lemma, it follows that $|||x||| \leq K$, i.e., $x \in Z$. Now, given $\varepsilon > 0$, let n_0 be such that $|||x_n - x_m||| < \varepsilon$ for $n > m > n_0$. Since $x^*S^k(x - x_m) = \lim_{n\to\infty} x^*S^k(x_n - x_m)$ pointwise, we have again by Fatou's lemma and (2.4) $B(k, k)^{-1} \|x^*S^k(x - x_m)\|_1 \leq \varepsilon$ for all $k \geq 1$, $x^* \in X_1^*$, and $m > n_0$, that is, $|||x - x_m||| \leq \varepsilon$ for $m > n_0$. ∎

For $t \in \mathbf{R}^+$, the linear manifold $R(t)Z = Z_0$ is contained in $Z \cap D(T)$ (since

$R(t)$ commutes with T), and is independent of t. Indeed, for $s, t > 0$ and $z \in Z$, we have by the first resolvent equation

$$R(t)z = R(s)[(s-t)R(t)z + z] \in R(s)Z,$$

hence $R(t)Z \subset R(s)Z$, and the equality follows by symmetry.

We denote by $\mathbf{T}(Z)$ the set of all (linear) operators with domain Z and range contained $\mathbf{Z}$. For the usual operations, $\mathbf{T}(Z)$ is an algebra with the identity $I \mid Z$. We are now ready to state our local version of Theorem 1.1.

THEOREM 2.2. *There exists an algebra homomorphism E of $\mathfrak{B}(\mathbf{R}^+)$ (as a Boolean algebra) into $\mathbf{T}(Z)$ with the following properties* (1)–(4):

(1) $E(\mathbf{R}^+) = I \mid Z$, *and for each $x \in Z$, $E(\cdot)x$ is a countably additive measure on $\mathfrak{B}(\mathbf{R}^+)$ (necessarily bounded and strongly countably additive)*:

(2) $E(\delta)$ *commutes with every $U \in B(X)$ which commutes with T (for each $\delta \in \mathfrak{B}(\mathbf{R}^+)$)*;

(3) $Z_0 = \{x \in Z \mid \text{strong } \lim_{n\to\infty} \int_0^n sE(ds)x$ *exists and belongs to* $Z\}$;

(4) $Tx = \text{strong } \lim_{n\to\infty} \int_0^n sE(ds)x$ *for all* $x \in Z_0$.

Proof. Fix $x \in Z$ and $x^* \in X^*$, and let f be as in Section 1. By (1.8) and (2.4),

$$\int_0^\infty |L_k f(t)|\, dt \leq \|x^*\|\, |||x|||c_k'' B(k, k) \leq \|x^*\|\, |||x|||$$

for all $k \geq 1$. By the STR theorem, there exists a unique complex regular Borel measure $\mu(\cdot \mid x, x^*)$ such that (1.10) is valid for all $x \in Z$ and $x^* \in X^*$. As before, we obtain that for each $x \in Z$,

$$H_x = \sup_{t>0} \|tR(t)x\| < \infty, \tag{2.5}$$

and

$$\operatorname{var} \mu(\cdot \mid x, x^*) \leq M_x \|x^*\| \qquad (x \in Z, x^* \in X^*) \tag{2.6}$$

where $M_x = 2\, |||x||| + H_x$.

As before, the uniqueness of the representation (1.10) and the reflexivity of X imply the existence of a unique function $E(\cdot)x : \mathfrak{B}(\mathbf{R}^+) \to X$ (for each fixed $x \in Z$) such that $\mu(\delta \mid x, x^*) = x^* E(\delta)x$ for all $\delta \in \mathfrak{B}(\mathbf{R}^+)$ and $x^* \in X^*$. Clearly, $E(\delta)$ is a linear operator with domain Z, and $\|E(\delta)x\| \leq M_x$ for all $\delta \in \mathfrak{B}(\mathbf{R}^+)$ and $x \in Z$. For fixed $x \in Z$, $E(\cdot)x$ is weakly, hence strongly countably additive.

If $U \in B(X)$ commutes with T, we already observed that $UZ \subset Z$, and the commutativity $UE(\delta) \subset E(\delta)U$ is verified as (1.14) ($\delta \in \mathfrak{B}(\mathbf{R}^+)$). Fix $u>0$. For each $x \in Z$ and $x^* \in X^*$,

$$\begin{aligned} x^*R(u)E(\mathbf{R}^+)x &= x^*E(\mathbf{R}^+)R(u)x = \mu(\mathbf{R}^+ \mid R(u)x, x^*) \\ &= \lim_{t\to\infty} \int_0^\infty t(t+s)^{-1}\mu(ds \mid R(u)x, x^*) \\ &= \lim_{t\to\infty} x^*tR(t)R(u)x = x^*R(u)x, \end{aligned}$$

where we used the first resolvent equation and (2.5). Since $R(u)$ is one-to-one, we conclude that $E(\mathbf{R}^+) = I \mid Z$.

We prove now that for all $\delta \in \mathfrak{B}(\mathbf{R}^+)$ and $x \in Z$,

$$|||E(\delta)x||| \le M_x. \tag{2.7}$$

In particular, $E(\delta) \in \mathbf{T}(Z)$ for each $\delta \in \mathfrak{B}(\mathbf{R}^+)$.

Clearly, (1.15) is valid for all variables as before and x varying in Z. Briefly

$$\mu(ds \mid R(u)x, x^*) = (u+s)^{-1}\mu(ds \mid x, x^*) \tag{2.8}$$

($u>0$, $x \in Z$, $x^* \in X^*$), and inductively

$$\mu(ds \mid R(u)^k x, x^*) = (u+s)^{-k}\mu(ds \mid x, x^*),$$

for $k = 0, 1, 2, \ldots$. Hence

$$\mu(ds \mid p(R(u))x, x^*) = p((u+s)^{-1})\mu(ds \mid x, x^*)$$

for any polynomial p. In particular

$$\begin{aligned} \mu(ds \mid S^k(u)x, x^*) &= u^k(u+s)^{-k}[1-u(u+s)^{-1}]^k\mu(ds \mid x, x^*) \\ &= (us)^k(u+s)^{-2k}\mu(ds \mid x, x^*) \end{aligned} \tag{2.9}$$

($u>0$, $x \in Z$, $x^* \in X^*$: $k = 1, 2, \ldots$).

Since $S^k(u)$ commutes with $R(t)$ (that is, with T), it commutes with $E(\delta)$, and therefore, by (2.9)

$$\begin{aligned} x^*S^k(u)E(\delta)x &= x^*E(\delta)S^k(u)x = \mu(\delta \mid S^k(u)x, x^*) \\ &= \int_\delta (us)^k(u+s)^{-2k}\mu(ds \mid x, x^*). \end{aligned}$$

Hence, by Tonelli's theorem and (2.6)

$$\begin{aligned}\|x^*S^k(u)E(\delta)x\|_1 &\leq \int_\delta \int_0^\infty (us)^k(u+s)^{-2k}du/u\,|\mu|\,(ds\mid x, x^*)\\ &= \int_\delta \int_0^\infty t^k(1+t)^{-2k}dt/t\,|\mu|\,(ds\mid x, x^*)\\ &= B(k,k)\,|\mu|\,(\delta\mid x, x^*) \leq B(k,k)M_x\,\|x^*\|.\end{aligned}$$

Therefore $|||E(\delta)x||| \leq M_x$.

Using (1.15) and (1.16) (with $x \in Z$), we obtain that for all $\sigma, \delta \in \mathfrak{B}(\mathbf{R}^+)$, $E(\sigma)E(\delta) = E(\sigma \cap \delta)$ in $\mathbf{T}(Z)$.

We verify finally Properties (3) and (4). Denote the set on the right of (3) by Z_1. Let $x \in Z_0$, that is $x = R(u)y$ for $u > 0$ and suitable $y \in Z$. By (2.8),

$$\begin{aligned}\int_0^n sE(ds)x &= \int_0^n sE(ds)R(u)y = \int_0^n s(u+s)^{-1}E(ds)y\\ &\xrightarrow[n\to\infty]{} \int_0^\infty s(u+s)^{-1}E(ds)y \qquad \text{(strongly)}\\ &= \int_0^\infty [1-u(u+s)^{-1}]E(ds)y = [I-uR(u)]y \quad (\in Z)\\ &= TR(u)y = Tx.\end{aligned}$$

Thus $Z_0 \subset Z_1$ and (4) is valid for $x \in Z_0$. On the other hand, if $x \in Z_1$, denote the limit in (3) by $z \in Z$. As in the proof of Theorem 1.1., we obtain that for any $t > 0$,

$$x = R(t)[z + tx] \in R(t)Z = Z_0.$$

Therefore $Z_0 = Z_1$ and the proof is complete.

Remarks. 1. One has

$$Z_0 = \{x \in D(T) \cap Z \mid Tx \in Z\}. \tag{2.10}$$

Indeed, if $x \in Z_0$, then $x = R(t)z$ for $t > 0$ and suitable $z \in Z$; therefore $x \in D(T) \cap Z$ and $Tx = TR(t)z = [I - tR(t)]z \in Z$. On the other hand, if $x \in D(T) \cap Z$ and $Tx \in Z$, then $x = R(t)y$ with $y = tx + Tx \in Z$, i.e., $x \in Z_0$. By (2.10), Z_0 is the natural domain of the restriction $T \mid Z$.

2. Let $\mathbf{B}(\mathbf{R}^+)$ denote the Banach algebra of all bounded Borel functions on

$\mathbf{R}^+$, with pointwise operations and supremum norm. For $x \in Z$ and $h \in \mathbf{B}(\mathbf{R}^+)$, $h(T)x = \int_0^\infty h(s)E(ds)x$ belongs to Z by (2.9). Hence $h(T) \in \mathbf{T}(Z)$ for each $h \in \mathbf{B}(\mathbf{R}^+)$. Using the density of simple Borel functions in $\mathbf{B}(\mathbf{R}^+)$, (2.7), (2.6), and the relation $E(\sigma)E(\delta) = E(\sigma \cap \delta)$ (in $\mathbf{T}(Z)$), one deduces easily that the map $h \to h(T)$ is an algebra homomorphism of $\mathbf{B}(\mathbf{R}^+)$ into $\mathbf{T}(Z)$, which is continuous in the following sense: if $h_n \to h$ in $\mathbf{B}(\mathbf{R}^+)$, then for each $x \in Z$, $h_n(T)x \to h(T)x$ with respect to the $|||\cdot|||$-norm. This "operational calculus" may then be extended to unbounded Borel functions h in the "usual" way.

Let $h_n = h$ if $|h| \leq n$ and $h_n = 0$ otherwise. Define

$$D(h(T)) = \{x \in Z \mid \text{strong} \lim_{n\to\infty} h_n(T)x \text{ exists and belongs to } Z\} \tag{2.11}$$

and

$$h(T)x = \text{strong} \lim_{n\to\infty} h_n(T)x, \qquad x \in D(h(T)). \tag{2.12}$$

By Theorem 2.2 and Remark 1 above, $h(T) = T \mid Z$ for $h(s) = s$. The details of the analysis of this operational calculus will be omitted. One point however should be stressed. If p is any polynomial of degree k, and $p(T)$ has the usual meaning, the "natural" domain of $p(T) \mid Z$ is the set

$$D(p(T) \mid Z) = \{x \in D(T^k) \mid T^j x \in Z, 0 \leq j \leq k\}.$$

As in Remark 1, one verifies easily that this domain coincides with $R(t)^k Z$ (for any $t > 0$).

Let $x \in D(p(T) \mid Z)$, that is $x = R(t)^k z$ for some $z \in Z$. Then, since $p(s)(t+s)^{-k} \in \mathbf{B}(\mathbf{R}^+)$,

$$p_n(T)x = \int_0^\infty p_n(s)(t+s)^{-k}E(ds)z \xrightarrow[n\to\infty]{} \text{(strongly)}$$

$$\int_0^\infty p(s)(t+s)^{-k}E(ds)z \in Z,$$

i.e. $x \in D(p(T))$ in the sense of (2.11). Also, writing $p(s) = \sum_{j=0}^k \alpha_j s^j$, the last integral is equal to

$$\sum_{j=0}^k \alpha_j \int_0^\infty [s(t+s)^{-1}]^j (t+s)^{-(k-j)} E(ds)z$$

$$= \sum_{j=0}^k \alpha_j \int_0^\infty [1 - t(t+s)^{-1}]^j (t+s)^{-(k-j)} E(ds)z$$

$$= \sum_{j=0}^k \alpha_j [I - tR(t)]^j R(t)^{k-j} z = \sum_{j=0}^k \alpha_j T^j R(t)^k z$$

$= p(T)x$ (the "classical" meaning of $p(T)$), where we used the multiplicativity of the operational calculus with *bounded* functions. Thus the definition of $p(T)$ according to (2.11) and (2.12) coincides with the usual definition on $D(p(T) \mid Z)$.

3. Some corollaries

If T is an operator with real spectrum, we may apply Theorem 1.1 to T^2, since $\sigma(T^2) \subset [0, \infty)$. Let

$$S(t) = tR(t)[I - tR(t)], \qquad t > 0$$

where

$$R(t) = (t + T^2)^{-1}.$$

COROLLARY 3.1. *Let T be a possibly unbounded linear operator with real spectrum. Then T has a scalar-type spectral square if and only if*

$$\sup_{k \geq 1} B(k, k)^{-1} |||S^k||| < \infty. \tag{3.1}$$

An equivalent formulation is given below.

COROLLARY 3.2. *Let T be a possibly unbounded linear operator with real spectrum. Then there exists a scalar-type spectral operator $|T|$ with $\sigma(|T|) \subset [0, \infty)$ such that $|T|^2 = T^2$ if and only if Condition (3.1) is valid.*

Proof. The necessity of Condition (3.1) is a trivial consequence of Corollary 3.1, since the square of a scalar-type spectral (s.t.s) operator is s.t.s. Conversely, Condition (3.1) implies that T^2 is s.t.s. with spectrum in $[0, \infty)$.

If E denotes its resolution of the identity, define $|T|$ as the operator corresponding to the function $f(s) = s^{1/2}$ on $\mathbf{R}^+$ in the operational calculus (o.c.) associated with E (cf. [1; Definition 10, p. 2238]).

By Theorem 17 in [1; p. 2244], $|T|$ is s.t.s. with spectrum in $[0, \infty)$, and $|T|^2 \subset T^2$ by [1; Theorem 11(f), p. 2238]. However, if $x \in D(T^2)$, then $x = R(t)y$ for $t > 0$ and suitable y. Hence

$$\int_0^{n^2} s^{1/2} E(ds)x = \int_0^{n^2} s^{1/2}(t+s)^{-1} E(ds)y \xrightarrow[n \to \infty]{} \int_0^{\infty} s^{1/2}(t+s)^{-1} E(ds)y,$$

since the integrand is bounded on $\mathbf{R}^+$. Therefore $x \in D(|T|)$ and $|T|x$ is given by the above limit. Next, by the multiplicativity of the o.c. with bounded functions,

$$\int_0^{n^2} s^{1/2}E(ds)\,|T|x = \int_0^{n^2} s^{1/2}E(ds)\int_0^{\infty} s^{1/2}(t+s)^{-1}E(ds)y$$

$$= \int_0^{n^2} s(t+s)^{-1}E(ds)y \xrightarrow[n\to\infty]{} \int_0^{\infty} s(t+s)^{-1}E(ds)y,$$

that is $|T|x \in D(|T|)$, showing that $x \in D(|T|^2)$. Hence $|T|^2 = T^2$. Q.E.D.

REFERENCES

[1] DUNFORD, N. and SCHWARTZ, J. T., *Linear operators*, Parts II and III, Wiley-Interscience, New York (1963 and 1971).
[2] KANTOROVITZ, S., *On the characterization of spectral operators*, Trans. Amer. Math. Soc. *111* (1964), 152–181.
[3] ——, *Classification of operators by means of their operational calculus*, Trans. Amer. Math. Soc. *115* (1965), 194–224.
[4] ——, *The semi-simplicity manifold of arbitrary operators*, Trans. Amer. Math. Soc. *123* (1966), 241–252.
[5] ——, *Characterization of C^n-operators*, Indiana University Math. J. *25* (1976), 119–133.
[6] REED, M. and SIMON, B., *Methods of modern mathematical physics*, II. Fourier Analysis, Self-adjointness. IV. Analysis of operators. Academic Press, New York (1975 and 1978).
[7] WIDDER, D. V., *The Laplace transform*, Princeton Univ. Press, Princeton, N.J. (1941).

Temple University
Philadelphia, Pennsylvania 19122 USA.

Bar-Ilan University
Ramat-Gan
Israel

Received June 16, 1980