

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 56 (1981)

Artikel: Many different disk knots with the same exterior.
Autor: Hitt, L.R. / Sumners, D.W.
DOI: <https://doi.org/10.5169/seals-43237>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Many different disk knots with the same exterior

L. R. HITT⁽¹⁾ and D. W. SUMNERS

§1. Introduction

Much of codimension-two knot theory is concerned with finding and computing topological invariants of knot exteriors in order to distinguish between the knots themselves. It is well-known ([G], [L-S], [B]) that there are at most two inequivalent smooth n -sphere knots with the same exterior ($n \geq 2$), and examples of two inequivalent n -knots with the same exterior have recently been discovered ([C-S], [Go]). We show that the corresponding theory for $(n+1)$ -disk knots is more complicated. Let Y denote the bounded exterior of a smooth $(n+1)$ -disk knot. The *indeterminacy index* $\zeta(Y)$ is the number of inequivalent $(n+1)$ -disk pairs having exteriors diffeomorphic to Y . We show that there exist disk knots with large indeterminacy indices (bigger than two, in particular). We then show that $\zeta(Y) \leq 2|\pi'|$, where $|\pi'|$ denotes the cardinality of π' , the commutator subgroup of $\pi = \pi_1(\partial Y)$. This yields as a corollary a new and easy proof of the well-known fact that $\zeta(X) \leq 2$, where X is the exterior of an n -sphere knot, and $\zeta(X)$ its indeterminacy index.

§2. The indeterminacy index

For convenience, we work in the smooth category (the same results hold in the locally flat PL situation). We let S^n and D^{n+1} denote the standard n -sphere and $(n+1)$ -disk, respectively. An *n -sphere knot* (or just *n -knot*) is the pair (S^{n+2}, kS^n) where $k : S^n \rightarrow S^{n+2}$ is an embedding. The *exterior* X of an n -knot is the complement in S^{n+2} of an open trivial 2-disk bundle neighborhood of the submanifold kS^n . An *$(n+1)$ -disk knot* is the pair (D^{n+3}, gD^{n+1}) where $g : D^{n+1} \rightarrow D^{n+3}$ denotes a proper embedding, one in which the submanifold gD^{n+1} intersects ∂D^{n+3} transversely in $g(\partial D^{n+1})$. We let Y denote the $(n+1)$ -disk knot exterior. Two knots are *equivalent* if there is a diffeomorphism of the ambient space throwing one submanifold onto the other (we disregard orienta-

¹ Research partially supported by the University of South Alabama Research Committee.

tions), and the *indeterminacy index* ζ is the number of inequivalent knots determined by a given knot exterior.

We will now produce examples to show that $\zeta(Y)$ can be large. The reason for this is that ∂Y contains the exterior X of the boundary sphere pair, and X can be very complicated. Recall the example of Kato [Ka 2, Theorem 4.9]:

Let $n \geq 3$, and M^{n+2} be a contractible manifold such that $\pi_1(\partial M)$ is the binary icosahedral group $G = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$ [Ke]. Let $Y^{n+3} = S^1 \times M^{n+2}$; we will show that Y is the exterior of at least three inequivalent $(n+1)$ -disk knots. Then by modifying the construction, we will show that the indeterminacy index of a disk knot exterior can be at least as large as six.

Let H be a group. A *weight element* of H is an element whose normal closure is all of H . The *automorphism class* of an element of H is the orbit of the element under the automorphism group of H . Two elements of H are *algebraically distinct* if they are in different automorphism classes.

We are interested in finding different automorphism classes of weight elements in the group $\pi_1(\partial Y) \cong Z \times G \cong \langle t, a, b \mid a^5 = b^3 = (ab)^2, ta = at, tb = bt \rangle$ where Z denotes the infinite cyclic group generated by t . An element of the form $t^n g$, for $g \in G$, is a weight element of $Z \times G$ if and only if t^n is a weight element of Z and g is a weight element of G , which forces $n = \pm 1$. To determine the weight elements of G , note that $\{1\} \triangleleft \{1, (ab)^2\} \triangleleft G$ is a composition series for G , since $\langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$ is a presentation of the simple group A_5 . The center of G is $C(G) = \{1, (ab)^2\}$, the cyclic group of order 2. Any element of G which is not in $C(G)$ is a weight element of G . The set of algebraically distinct weight elements of G is $\{a, a^2, b, b^2, ab\}$. That they are algebraically distinct follows from their different orders: 10, 5, 6, 3 and 4, respectively.

Therefore we have ta, ta^2, tb, tb^2 , and tab as weight elements of $Z \times G$. However ta and ta^2 are in the same automorphism class in $Z \times G$, as are tb and tb^2 (e.g., the automorphism θ , induced by $\theta(t) = t(ab)^2$, $\theta(a) = a^7$, and $\theta(b) = ba^8b$ sends ta to ta^2). So our list of possibly algebraically distinct weight elements is shortened to ta, tb , and tab . That these three elements are algebraically distinct follows from the fact that the center of $Z \times G$ is $Z \times \{1, (ab)^2\}$, so $Z \times G$ modulo its center is $A_5 \cong \langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$. But the center is a characteristic subgroup, so any automorphism of $Z \times G$ induces one on A_5 . Since a, b , and ab have different orders in A_5 , their counterparts in $Z \times G$ must be algebraically distinct.

Let $\{\sigma_i \mid 1 \leq i \leq 3\}$ denote smooth embeddings of S^1 in ∂Y representing the homotopy class in ∂Y of each of the above weight elements of $Z \times G$. Choose a trivialization of the normal bundle of each σ_i , and attach 2-handles to form the manifolds $Y \cup_{\sigma_i} h^2$. The cocore or transverse disk of each 2-handle is an $(n+1)$ -disk, and $(Y \cup_{\sigma_i} h^2, \text{cocore}(h^2)) \approx (D^{n+3}, g_i d^{n+1})$, where $g_i : D^{n+1} \rightarrow D^{n+3}$ is a

proper smooth embedding. This is because $Y \cup_{\sigma_i} h^2$ is contractible, with simply-connected boundary, and $n + 3 \geq 6$. However, no two of the three disk pairs $(Y \cup_{\sigma_i} h^2, g_i D^{n+1})$ are equivalent, because any diffeomorphism of pairs between them would restrict to a diffeomorphism on Y , inducing an isomorphism on $\pi_1(\partial Y)$ taking one of the weight elements of $Z \times G$ to another, or its inverse.

In [S], it is shown that $(n + 1)$ -disk pairs ($n \geq 2$) can be constructed with an arbitrarily prescribed Alexander polynomial in a single dimension p ($2 \leq p \leq n$), and trivial Alexander polynomials elsewhere. Moreover, these disk pairs have the property that $\pi_1(Y) \cong \pi_i(\partial Y) \cong \pi_i(S^1)$ for $i < p$. Thus, by taking the boundary connected sum of the above examples with these disk pairs, one obtains infinitely many distinct $(n + 1)$ -disk exteriors, each with indeterminacy index $\zeta \geq 3$. This proves

THEOREM 2.1. *For each $n \geq 3$, there exist infinitely many homeomorphically distinct $(n + 1)$ -disk knot exteriors Y_i , each with indeterminacy index $\zeta(Y_i) \geq 3$.*

Remark. The analogue of Theorem 2.1 for $n = 2$ can be done in the topological category (non-PL embeddings). One takes $Y = S^1 \times (c * \Sigma^3)$, where $c * \Sigma^3$ is the cone on Σ^3 , the Poincaré' 3-sphere. Then Y is a topological manifold [Ca], and arguments of Scharlemann [Sc] can be used to prove that the various handle attachments give rise to different non-PL disk pairs (D^5, gD^3) .

We can modify the above construction to increase the lower bound for the indeterminacy index. Consider the group

$$G \times G \times G = \langle a, b, c, d, e, f \mid a^5 = b^3 = (ab)^2, c^5 = d^3 = (cd)^2, e^5 = f^3 = (ef)^2, \\ ac = ca, ad = da, bc = cb, bd = db, ae = ea, af = fa, be = eb, bf = fb, \\ ce = ec, cf = fc, de = ed, df = fd \rangle.$$

Now $G \times G \times G$ is finitely presented, and $H_1(G \times G \times G) = H_2(G \times G \times G) = 0$, so by Kervaire [Ke], for $n \geq 4$ there exists a contractible manifold M^{n+2} with $\pi_1(\partial M) \cong G \times G \times G$. As before, $Y = S^1 \times M$, and $\pi_1(\partial Y) \cong Z \times G \times G \times G$. Since the center of $Z \times G \times G \times G$ is the product of the center of each factor, we see that $Z \times G \times G \times G$ modulo its center is $A_5 \times A_5 \times A_5$. Then, as before, $tacf$, $tacef$, $tbdef$, $tace$, tbd , and $tabcdef$ are all algebraically distinct since their projections modulo the center have orders 15, 10, 6, 5, 3, and 2, respectively. We have the following

COROLLARY 2.2. *For each $n \geq 4$, there exist infinitely many homeomorphically distinct $(n + 1)$ -disk knot exteriors Y_i , each with indeterminacy index $\zeta(Y_i) \geq 6$.*

§3. An upper bound for the indeterminacy index

Now that we have seen that in some cases the lower bound of ζ can be large, we are interested in finding upper bounds. Along these lines, we have the following

THEOREM 3.1. *Let Y^{n+3} be an $(n+1)$ -disk knot exterior ($n \geq 2$). Then $\zeta(Y) \leq 2|\pi'|$, where $|\pi'|$ denotes the cardinality of the commutator subgroup π' of $\pi = \pi_1(\partial Y)$.*

Proof. Consider the disk pair (D^{n+3}, gD^{n+1}) . Choose a trivialization $G : D^2 \times D^{n+1} \rightarrow N(gD^{n+1})$ of the tubular neighborhood of the submanifold; thus $G(\{0\} \times y) = g(y)$ for $y \in D^{n+1}$. We have that the exterior $Y = D^{n+3} - G(\overset{\circ}{D}^2 \times D^{n+1})$. Regarding $N(gD^{n+1})$ as a 2-handle attached to Y via the meridian attaching curve $G(\partial D^2 \times \{0\})$, we have $(D^{n+3}, gD^{n+1}) \approx (Y \cup_G h^2, \text{cocore}(h^2))$. We now wish to study the number of different ways it is possible to attach a 2-handle to Y to produce D^{n+3} . We first count the maximum number of possible isotopy classes in ∂Y of attaching curves for a 2-handle which produce a contractible manifold after handle attachment is performed. If $\pi = \pi_1(\partial Y)$, and π' is the commutator subgroup of π , we have the short exact sequence

$$1 \rightarrow \pi' \rightarrow \pi \rightarrow Z \rightarrow 1. \tag{3.2}$$

Denoting the generator of the infinite cyclic multiplicative group by t , we have a semi-direct product structure for π , and once a splitting for (3.2) is chosen, we can write each element $x \in \pi$ uniquely as $x = t^a g$ where a is an integer and $g \in \pi'$. By abuse of notation, let $t^a g$ represent an embedding of S^1 in the same homotopy class, and choose a trivialization of its normal bundle. In order for $Y \cup_{t^a g} h^2$ to be acyclic, we must have $a = \pm 1$, because $H_1(Y; Z)$ is infinite cyclic on the generator t . In order for $Y \cup_{t^a g} h^2$ to be contractible, $i_*(t^a g)$ must be a weight element of $\pi_1(Y)$, where $i_* : \pi_1(\partial Y) \rightarrow \pi_1(Y)$ is the inclusion homomorphism. In order for $\partial(Y \cup_{t^a g} h^2)$ to be simply-connected, $t^a g$ must be a weight element of $\pi_1(\partial Y)$. The upper bound we are aiming at is very crude, coming just from the homology condition ($a = \pm 1$), so we are in fact counting the ways it is possible to complete Y to obtain an integral homology disk. The set of elements of $\pi_1(\partial Y)$ producing acyclic manifolds upon handle attachment is $\{t^{\pm 1} g \mid g \in \pi'\}$. But since the sign of the exponent of t in an element of $\pi_1(\partial Y)$ is reversed by changing the orientation of the attaching curve of h^2 (or equivalently, reversing the orientation on the cocore D^{n+1}), the set of elements of π corresponding to possibly different manifold pairs is $\{tg \mid g \in \pi'\}$, a set of the cardinality of π' . Now since we are in the

dimension range $(n+2) \geq 4$ for ∂Y , homotopy of embedded one-spheres gives rise to isotopy, so the number of possible isotopy classes of attaching curves in ∂Y giving rise to acyclic manifolds is bounded above by $|\pi'|$. Now, given a representative of an isotopy class of attaching curves in ∂Y , there are precisely two ways to attach the 2-handle h^2 , corresponding to the $\pi_1(SO) = Z_2$ ways of choosing a trivialization of the normal bundle of the curve. Hence the number of possible handle attachments yielding acyclic manifolds is bounded above by $2|\pi'|$.

COROLLARY 3.3. *Suppose that Y^{n+3} ($n \geq 2$) is an $(n+1)$ -disk knot exterior, and that $\pi_1(\partial Y) = Z$. Then $\zeta(Y) \leq 2$, and the two possibly different disk pairs are obtained, each from the other, by re-attaching the 2-handle corresponding to the normal bundle over the submanifold via the non-trivial element of $\pi_1(SO)$.*

Corollary 3.3 yields an easy proof of the well-known result that there are at most two inequivalent n -knots with the same exterior:

COROLLARY 3.4. ([B], [L-S], [Ka 1], [Sw]). *Let X^{n+2} ($n \geq 3$) be an n -sphere exterior. Then $\zeta(X) \leq 2$. Moreover, if $(X \cup_{\gamma} (D^2 \times S^n), \{0\} \times S^n)$ denotes a sphere pair obtained by sewing $D^2 \times S^n$ onto X via some trivialization of the S^n -bundle over the meridian curve $\gamma = S^1 \times \{*\} \subset \partial X$, then the possibly different sphere pair is $(X \cup_{\bar{\gamma}} (D^2 \times S^n), \{0\} \times S^n)$, where $\bar{\gamma}$ denotes the same meridian curve with different trivialization of the S^n -bundle (i.e., $D^2 \times S^n$ is sewn in with a $\pi_1(SO)$ -twist).*

Proof. There is a one-to-one correspondence between n -sphere knots and n -disk knots with unknotted boundary $(n-1)$ -sphere pair, obtained by removing an unknotted disk pair (the neighborhood of a point on the submanifold) from the sphere pair to obtain the required disk pair. An n -sphere knot and its corresponding n -disk knot have the same exterior X . But $\partial X \approx S^1 \times S^n$, and $\pi_1(\partial X) = Z$, so by Corollary 2.4, $\zeta(X) \leq 2$. That is, X (thought of as a disk exterior) determines at most two inequivalent disk pairs. Therefore, thinking of it as a sphere pair exterior, then $\zeta(X) \leq 2$ as well.

§4. Some questions

1. Given a positive integer N , does there exist an $(n+1)$ -disk exterior Y with $\zeta(Y) \geq N$?
2. Is there an $(n+1)$ -disk exterior Y with $\zeta(Y) = +\infty$?
3. If X is an n -sphere exterior and $\pi_1(X) = Z$, must it follow that $\zeta(X) = 1$?

REFERENCES

- [B] W. BROWDER, *Diffeomorphisms of 1-connected manifolds*, Trans. Amer. Math. Soc. 128 (1967), 155–163.
- [Ca] J. W. CANNON, *Shrinking cell-like decompositions of manifolds: codimension three*, Ann. of Math. 110 (1979), 83–112.
- [C-S] S. CAPELL and J. L. SHANESON, *There exist inequivalent knots with the same complement*, Ann. of Math. 103 (1976), 349–353.
- [G] H. GLUCK, *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. 104 (1962), 308–333.
- [Go] C. M. GORDON, *Knots in the 4-sphere*, Comment. Math. Helv. 51 (1976), 585–596.
- [Ka 1] M. KATO, *A concordance classification of PL homeomorphisms of $S^p \times S^q$* , Topology 8 (1969), 371–383.
- [Ka 2] M. KATO, *Higher dimensional PL knots and knot manifolds*, Jour. Math. Soc. Jap. 21 (1969), 458–480.
- [Ke] M. KERVAIRE, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. 144 (1969), 67–72.
- [L-S] R. K. LASHOF and J. S. SHANESON, *Classification of knots in codimension two*, Bull. Amer. Math. Soc. 75 (1969), 171–175.
- [Sc] M. SCHARLEMANN, *Stably trivial non-PL knotted ball pairs*, Notices Amer. Math. Soc. 23 (1976), pg. A-22, Abstract #76T-G5.
- [S] D. W. SUMNERS, *Homotopy torsion in codimension two knots*, Proc. Amer. Math. Soc. 24 (1970), 229–240.
- [Sw] G. A. SWARUP, *A note on higher-dimensional knots*, Math. Zeit. 121 (1971), 141–144.

University of South Alabama
Mobile, Al 36688

Florida State University
Tallahassee, FL 32306

Received November 22, 1979/November 30, 1980