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$H^p(\mathbf{R}^n)$ is equidistributed with $L^p(\mathbf{R}^n)$[1]

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Abstract. Let $0<p<\infty$. Let H^p $(\mathbf{R}^n)$ be the real variable Hardy spaces defined by Stein and Weiss. Let $L^p(\mathbf{R}^n)$ be the usual Lebesgue space. It is shown that for $f \in L^p$ there is an $\tilde{f} \in H^p$ with the distribution functions of $|f|$ and $|\tilde{f}|$ identical and $\|\tilde{f}\|_{H^p} \approx \|f\|_{L^p}$. The converse is trivially true.

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For $0<p<\infty$, let

$$L^p(\mathbf{R}^n)=\left\{f: \int_{\mathbf{R}^n} |f(x)|^p\, dx \equiv \|f\|_{L^p}^p<\infty\right\}.$$

Fix $\varphi \in C_c^\infty(\mathbf{R}^n)$, $\int \varphi(x)\, dx=1$. Let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz space, $\mathcal{S}'(\mathbf{R}^n)$ the Schwartz distributions, let $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$, and define

$$f^*(x)=\sup_{\varepsilon>0} \varphi_\varepsilon * f(x), \qquad f \in \mathcal{S}'.$$

Let $\|f\|_{H^p} \equiv \|f^*\|_{L^p}$ for $f \in \mathcal{S}'$ and $H^p(\mathbf{R}^n)=\{f \in \mathcal{S}': \|f\|_{H^p}<\infty\}$. The space $H^p(\mathbf{R}^n)$ has a number of important equivalent characterizations, for which see [3].

Elements of the space of *distributions* $H^p(\mathbf{R}^n)$ may be represented by integration against $L^p(\mathbf{R}^n)$ functions satisfying certain *moment conditions* (see Section 1). It is with this in mind that all ensuing statements about H^p should be read.

Note that $\chi_{[0,1]}$ on $\mathbf{R}$ cannot represent an H^p function, $0<p\le 1$. So not all L^p functions represent H^p functions. Calderón–Zygmund operators are bounded on H^p, but not on L^p, $0<p\le 1$.

If $f:\mathbf{R}^n \to \mathbf{C}$ is measurable, let $m_f(\lambda)=|\{x:|f(x)|>\lambda\}|$, $\lambda>0$, where $|\ \ |$ denotes Lebesgue measure. Abusing terminology slightly, let us say that two functions f_1, f_2 are *equidistributed* if $m_{f_1}(\lambda)=m_{f_2}(\lambda)$ for all λ. Two function spaces X_1, X_2 on $\mathbf{R}^n$ are said to be equidistributed if to every $f_1 \in X_1$ there corresponds an $f_2 \in X_2$ so that f_1 and f_2 are equidistributed and *vice versa.* The main result of this paper is that

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THEOREM A. *The spaces* $H^p(\mathbf{R}^n)$ *and* $L^p(\mathbf{R}^n)$ *are equidistributed,* $0<p<\infty$. *Indeed there are universal constants* $C=C(p,n)$ *so that each* $f\in L^p$ *is equidistributed with an* $f'\in H^p$ *satisfying* $1/C\le\|f\|_{L^p}/\|f'\|_{H^p}\le C$.

The proof of this result is an application of the atomic theory of the H^p spaces. The result emphasizes that the distinction between H^p and L^p for a given p is strictly a moment condition and does not involve size. The second inequality in the theorem is trivial with $C=1$. So it is the first inequality that we prove.

This work was motivated by a question of Colin Bennett. John Garnett independently discovered Theorem A for $q=1$, $n=1$.

§1. Proof of Theorem A

The atomic characterization of H^p proceeds as follows. Let $0<p\le 1$. Let

$$\mathcal{P}_k=\{\text{polynomials on } \mathbf{R}^n \text{ of degree not exceeding } k\},\qquad k=0,1,2,\ldots.$$

A measurable function $a:\mathbf{R}^n\to\mathbf{C}$ is said to be a *p-atom* if

$$a \text{ is supported on a ball } \overline{B(x,r)}=\{|y-x|\le r\} \tag{1.1}$$

$$|a|\le|B(x,r)|^{-1/p} \tag{1.2}$$

$$\int a(x)p(x)\,dx=0\qquad \forall p\in\mathcal{P}_{[(n/p)-n]}. \tag{1.3}$$

THEOREM 1.1 ([1], [2], [4]). *Let* $0<p\le 1$. *Let* $f\in\mathcal{S}'(\mathbf{R}^n)$. *Then* $f\in H^p(\mathbf{R}^n)$ *if and only if there is a sequence* $\{a_i\}_{i=1}^{\infty}$ *of p atoms and a sequence* $\{\lambda_i\}_{i=1}^{\infty}\subseteq\mathbf{C}$ *with* $f=\sum\lambda_i a_i$ *in the sense of distributions and*

$$(1/C)\cdot\sum|\lambda_i|^p\le\|f\|_{H^p}^p\le C\sum|\lambda_i|^p.$$

Here $C=C(n,p)$ *is a universal constant.*

Remark. Since, for $0<p\le1$,

$$\int\Big|\sum_i\lambda_i a_i(x)\Big|^p\,dx\le\sum_i|\lambda_i|^p\int|a_1(x)|^p\,dx\le\sum|\lambda_i|^p\le C\|f\|_{H^p}^p,$$

it is possible to represent elements of H^p by L^p functions.

In order to prove Theorem A, it is enough to consider $0<p\leq 1$ and to prove that every $f\in L^p$ can be rearranged so that it is manifestly a linear combination of p atoms $\{a_i\}$ with coefficients $\{\lambda_i\}\in l^p$ satisfying $\|\{\lambda_i\}\|_{l^p}\approx\|f\|_{L^p}$.

Now the main technical result which is required for the proof of Theorem A is

PROPOSITION 1.2. *Let* $[a, b]\subseteq\mathbf{R}^1$ and let $f:[a, b]\to[0,\infty)$, $f=0$ *off* $[a, b]$, $f\in L^1(\mathbf{R})$. *Let* $0\leq k\in\mathbf{Z}$. *There is a measurable function* $\tilde{f}$ *on* $[0,(b-a)]$, *equidistributed with* f, *so that* $\int\tilde{f}(x)p(x)\,dx=0$ for all $p\in\mathcal{P}_k$.

Proposition 1.2 will be proved in Section 2. Taking it for granted, let us complete the proof of Theorem A. In order to simplify notation, the details will be given in $\mathbf{R}^2$ only. Fix $0<p\leq 1$. Let $f:\mathbf{R}^2\to\mathbf{C}$, $f\in L^p$. Assume for now that $|\text{supp} f|=1$. Let Nf be the non-increasing rearrangement of f (see [7]). So $\text{supp}\, Nf=[0, 1]$.

For $j\in\mathbf{Z}$, let $I_j=\{x:2^j\leq Nf<2^{j+1}\}$. Then $[0,1]=\cup I_j$, each I_j is an interval with endpoints $a_j\leq b_j$, and

$$\cdots a_1\leq b_1=a_0\leq b_0=a_{-1}\leq b_{-1}=\cdots.$$

Let $k=[(2/p)-2]$ and for each j apply Proposition 1.2 to $(Nf)|_{I_j}$ and $\mathcal{P}_k$. This yields, for each j, a function f^j on $[0,|I_j|]$ which is L^2 orthogonal to $\mathcal{P}_k$.

For each j, let $l_j=|I_j|\equiv b_j-a_j$. Write

$$0=\alpha_1^j<\beta_1^j=\alpha_2^j<\beta_2^j=\cdots=\alpha_{M_j}^j<\beta_{M_j}^j=1$$

where for each $i=1,\ldots,M_j-1$, $\beta_i^j-\alpha_i^j=l_j$, and $\beta_{M_j}^j-\alpha_{M_j}^j\leq l_j$. For each j and $1\leq i\leq M_j$, apply Proposition 1.2 to the function 1 on $[\alpha_i^j,\beta_i^j]$ and $\mathcal{P}_k$. Call the resulting function h_i^j on $[0,(\beta_i^j-\alpha_i^j)]$. Now define

$$\begin{aligned}\tilde{f}(x_1,x_2)&\equiv\sum_{j=-\infty}^{\infty}\sum_{i=1}^{M_j}\chi_{[0,l_j]}\Big(x_1-\sum_{m=-\infty}^{j-1}l_m\Big)\cdot f^j\Big(x_1-\sum_{m=-\infty}^{j-1}l_m\Big)\\&\quad\times\chi_{[0,(\beta_i^j-\alpha_i^j)]}\Big(x_2-\sum_{p=1}^{i-1}(\beta_i^j-\alpha_i^j)\Big)h_i^j\Big(x_2-\sum_{p=1}^{i-1}(\beta_i^j-\alpha_i^j)\Big)\\&\equiv\sum_{j=-\infty}^{\infty}\sum_{i=1}^{M_j}\lambda_{ij}a_{ij}(x_1,x_2)\end{aligned}$$

where $\lambda_{ij}=2^j(l_j)^{2/p}$. Then $\tilde{f}$ is equidistributed with f since Nf is. Also, each a_{ij} is a p-atom. For each a_{ij} satisfies (i) a_{ij} is supported in a box of size $\leq l_j\times l_j$, (ii) each a_{ij} satisfies $|a_{ij}|\sim(l_j)^{-2/p}\sim|B(0,l_j)|^{-1/p}$, and (iii) each a_{ij} is orthogonal to $\mathcal{P}_k$ by Fubini's Theorem. In the verification of (iii) we have used the fact that if $p\in\mathcal{P}_k$

and $h \in R$ is fixed then the function $x \mapsto p(x+h)$ is in $\mathcal{P}_k$. Finally,

$$\sum_{i,j} |\lambda_{ij}|^p \le \sum_j j_M 2^{jp} l_j^2 = \sum_j (2^{jp} l_j)(j_M l_j) \le 2 \|f\|_{L^p}^p.$$

Likewise, $\sum |\lambda_{ij}|^p \ge \|f\|_{L^p}^p/4$. So $\tilde{f} \in H^p(\mathbf{R}^2)$ by Theorem 1.1, and $\|\tilde{f}\|_{H^p} \approx \|f\|_{L^p}$.

This completes the proof of Theorem A in case $|\text{supp } f| = 1$. For the general case, write $f = \sum f_j$ where each f_j, except possibly one, has support of measure 1 and the odd one has measure not exceeding 1. Then each f_j gives rise to an H^p function $\tilde{f}_j$ on $[0, 1] \times [0, 1]$. Let

$$\tilde{f}(x) = \sum_j \tilde{f}_j(x_1 - 4j, x_2 - 4j). \qquad \square$$

§2. Proof of Proposition 1.2

The proposition proceeds from some rather more technical lemmas about polynomials and about L^1 functions.

If $f : \mathbf{R} \to \mathbf{C}$, $h \in \mathbf{R}$, let

$$\Delta_h f(x) \equiv f(x+h) - f(x).$$

LEMMA 2.1. *If* $0 < k \in \mathbf{Z}$, $h \in \mathbf{R}$, *and* $p \in \mathcal{P}_k$, *then* $\Delta_h p \in \mathcal{P}_{k-1}$ (*where* $\mathcal{P}_{-1} = \{0\}$).

Proof. Apply the binomial theorem. $\square$

LEMMA 2.2. *Let* $h_1, \ldots, h_{k+1}$ *be non-zero real numbers with* $|h_j| > 2\,|h_{j+1}|$, $j = 2, 3, \ldots, k+1$. *Then for* $p \in \mathcal{P}_k$,

$$\Delta_{h_{k+1}}(\Delta_{h_k}(\ldots(\Delta_{h_1} p)\ldots)) \equiv 0, \tag{2.2.1}$$

and the expression on the left side of (2.2.1) *may be written as*

$$\sum_{j=1}^{2^{k+1}} \varepsilon_j p(x + a_j) \tag{2.2.2}$$

with $\varepsilon_j = \pm 1$ *for each* j *and the* a_j *distinct.*

Proof. This follows from 2.1 by induction. □

Remark. By choosing the h_j in Lemma 2.2 adroitly, we may arrange that the a_j are equally spaced, with any preselected distance d_0 between successive a_j's. By renaming the a_j's if necessary, and by the translation invariance of Lemma 2.2, we may suppose that

$$0 > a_1 > \cdots > a_{2^{k+1}}$$

where $a_{j-1} - a_j = d_0$, $j = 2, \ldots, 2^{k+1}$.

DEFINITION 2.3. If $d_0 > 0$, $0 \leq k \in \mathbf{Z}$, and $a_1, \ldots, a_{2^{k+1}}$, $\varepsilon_1, \ldots, \varepsilon_{2^{k+1}}$ are selected according to Proposition 2.2 and the subsequent remark, let $T_k^{d_0} f(x) \equiv \sum_{j=1}^{2k+1} \varepsilon_j f(x + a_j)$, any $f : \mathbf{R} \to C$.

LEMMA 2.4. *Let* $f : [0, 1] \to [0, \infty)$, $f \in L^1(\mathbf{R})$. *Let* $0 < M \in \mathbf{Z}$. *There exists a function* $f_M : [0, 1/M] \to [0, \infty)$ *such that* f *is equidistributed with* $g(x) = \sum_{j=1}^{M} f_M(x - j/M)$. *In particular,* $m_f(\lambda) = M \cdot m_{f_M}(\lambda)$ *for every* $\lambda > 0$.

Proof. Let $f_M(x) = f(Mx)$. □

Proof of Proposition 1.2. Assume without loss of generality that $a = 0$, $b = 1$. Let other notation be as in the statement of the proposition. Apply Lemma 2.4 with $M = 2^{k+1}$. The resulting function f_M is supported on $[0, 2^{-k-1}]$. Let $d_0 = 2^{-k-1}$. Using Definition 2.3 let $\tilde{f} = T_k^{d_0} f_M$. Then $\tilde{f}$ is equidistributed with f and $\operatorname{supp} \tilde{f} \subseteq [0, 1]$. Then if $p \in \mathcal{P}_k$ we have, letting $R(x) = -x$, that

$$\int p(x)\tilde{f}(x)\, dx = \int p(x)(T_k^{d_0} f_M)(x)\, dx$$

$$\text{(ch. of variable)} = \int R(T_k^{d_0} Rp)(x) f_M(x)\, dx.$$

But $Rp \in \mathcal{P}_k$ so $T_k^{d_0} Rp = 0$ whence the last line is 0 as desired. □

§3. Concluding remarks

The proof we have given of Theorem A uses the structure of $\mathbf{R}^n$ rather decisively. With considerable additional technical difficulty, a proof of the same kind appears to work for the case when H^p is the non-isotopic Kähler H^p on the boundary of the unit ball $B \subseteq C^n$ (see [4]) and L^p is the usual L^p space with respect to rotationally invariant measure on ∂B. It would be interesting to know to what extent Theorem A, or a modification thereof, holds for the H^p spaces defined on certain spaces of homogeneous type ([2], [5]).

REFERENCES

[1] COIFMAN, R., *A real variable characterization of* H^p, Studia Math. *51* (1974), 269–274.
[2] COIFMAN, R. and WEISS, G., *Extensions of Hardy spaces and their use in analysis*, BAMS *38* (1977), 569–645.
[3] FEFFERMAN, C. and STEIN, E. M., H^p *Spaces of several variables*, Acta Math. *129* (1972), 137–193.
[4] GARNETT, J. and LATTER, R., *The atomic decomposition for Hardy spaces in several complex variables*, Duke Math. Jour. *45* (1978), 815–846.
[5] LATTER, R., Thesis, UCLA, 1977.
[6] STEIN, E. M., *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.
[7] STEIN, E. M. and WEISS, G., *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, 1971.

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Added in Proof: Results related to, but distinct from, Theorem A, have recently appeared in "Hardy Spaces and Rearrangements," *Trans. A.M.S. 261* (1980), 211–233.