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An elementary formula for the Fenchel-Nielsen twist

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A deformation of hyperbolic Riemann surfaces was investigated in the Fenchel-Nielsen manuscript. The surface is cut along a simple closed hyperbolic geodesic. The two sides are separated, rotated relative to each other and glued. The purpose of this note is to present a formula for the first variation under the twist deformation of the length of a closed geodesic. The formula is local in nature and entails the angles at the intersections of the geodesic and the cut.

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1. A variational formula

Let $f_t: R_0 \to R_t$ be a smooth deformation. Denote by g the hyperbolic line element on R_t . Choose a closed curve γ on R_0 and denote by l(t) the length of the unique g geodesic freely homotopic to $f_t(\gamma)$ on R_t . Then

$$\frac{d}{dt}l(t) = \int_{\gamma_0} \frac{d}{dt} g_t \tag{1.1}$$

where $g_t = f_t^* g$ and γ_0 is the g geodesic on R_0 .

Proof. Consider the lift \tilde{f}_t of f_t to the universal covers, each represented as the upper half plane. Now \tilde{f}_t is smooth in t and z, thus for A a deck transformation corresponding to γ_0 the Möbius transformation $A_t = \tilde{f}_t A \tilde{f}_t^{-1}$ varies smoothly in t. The endpoints on \mathbb{R} of the axis of A_t (the fixed points of A_t) are algebraic in the matrix entries of A_t . Consequently the axes of the A_t vary smoothly in t. Now $\tilde{\gamma}_t = \tilde{f}_t^{-1}$ (axis of A_t) form a smooth 1-parameter family of curves. Denote by γ_t the projection to R_0 of $\tilde{\gamma}_t$. Then

$$\frac{d}{dt} l(t) = \frac{d}{dt} \int_{\gamma_t} g_t = \lim_{t \to 0} \frac{1}{t} \left(\int_{\gamma_t} g_t - \int_{\gamma_0} g_0 \right)$$
$$= \lim_{t \to 0} \frac{1}{t} \left(\int_{\gamma_t} g_t - \int_{\gamma_0} g_t + \int_{\gamma_0} g_t - \int_{\gamma_0} g_0 \right).$$

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Expanding in t, $g_t = g_o + t\dot{g} + O(t^2)$ where \dot{g}^2 is a symmetric 2-tensor. We then have

$$\frac{d}{dt}l(t) = \lim_{t \to 0} \frac{1}{t} \left(\int_{\gamma_t} g_0 - \int_{\gamma_0} g_0 \right) + \lim_{t \to 0} \left(\int_{\gamma_t} \dot{g} - \int_{\gamma_0} \dot{g} \right) + \int_{\gamma_0} \frac{d}{dt} g_t.$$

The curve γ_0 is a periodic geodesic and consequently the first variation of its length is zero; the second limit vanishes by continuity of the integral.

2. The Fenchel-Nielsen twist

Let $z \in H$, the upper half plane and $\theta = \arg z$. Choose $\phi(\theta)$ smooth with compact support in $(0, \pi)$, $\varphi \ge 0$ and $\int_0^{\pi} \varphi \, d\theta = \frac{1}{2}$. Define $\Phi(\theta) = \int_0^{\theta} \varphi \, d\theta$. The formula

$$w = z \exp(2t\Phi(\theta)) \tag{2.1}$$

defines a quasiconformal self map of H realizing the Fenchel-Nielsen twist deformation for the cyclic group stabilizing the imaginary axis. The boundary values of w are independent of the particular choice of φ . We recall that the boundary values determine the deformation as a point of the Teichmüller space. Now

$$w_z = \exp(2t\Phi)(1-it\varphi), \qquad w_{\bar{z}} = \exp(2t\Phi)(it\varphi)$$

and the Beltrami differential of w is

$$\mu = w_{\bar{z}}/w_z = \frac{it\varphi}{1 - it\varphi} \frac{z}{\bar{z}}$$

The first variation of μ with respect to t is the tangent vector to the deformation.

Let Γ be a finitely generated Fuchsian group containing a hyperbolic element A stabilizing the imaginary axis. We assume A is primitive and that the imaginary axis projects to a simple closed geodesic. The infinitesimal Fenchel-Nielsen twist deformation of Γ about A is

$$\mu_{\Gamma} = \sum_{B \in \langle A \rangle \setminus \Gamma} B^* \mu,$$

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for $\langle A \rangle$ the cyclic group generated by A, [2]. We emphasize that μ_{Γ} as a tangent to the deformation space is independent of the particular choice of φ .

3. The cosine formula

Let $C \in \Gamma$ correspond to a closed geodesic γ on $R = H/\Gamma$. The first variation of the length of γ under a Fenchel-Nielsen twist about the geodesic α corresponding to A is

$$\sum_{p \in \{\alpha\} \cap \{\gamma\}} \cos \beta_p.$$

The sum is over the intersection of the geodesics α and γ ; β_p is the angle between the tangents at p of α and γ . The angle β_p is defined by the segment between 0 and p on the imaginary axis and that segment of the axis of C in the right half plane. The angle is well defined; note the expression $\cos \beta_p$ is anti-symmetric in the roles of α and γ . In particular for a twist about γ the contribution at p to the derivative of the length of α is $\cos (\pi - \beta_p)$.

We begin the proof by considering the solution w_t of the Beltrami equation $w_{\bar{z}} = \mu_{\Gamma} w_z$. The solution was constructed in [2]. Indeed, in a neighborhood of the imaginary axis w_t coincides with the map defined by (2.1). Now in the notation of section 1

$$g = |dw|/\text{Im } w$$
 and
 $g_t = |w_z| |dz + \mu_\Gamma d\bar{z}|/\text{Im } w$
 $= |1 - it\varphi| |dz + \mu_\Gamma d\bar{z}|/\text{Im } z$.

Our calculations only require an expression for g_t in a neighborhood of the imaginary axis, where $\mu_{\Gamma} = it\varphi e^{2i\theta} + O(t^2)$. Furthermore, φ will approximate $\frac{1}{2}$ the Dirac delta at $\pi/2$; consequently

$$g_t = |dz - it\varphi \, d\bar{z}|/\text{Im } z + O(t^2, \text{ support } \varphi)$$

$$= ((dx - t\varphi \, dy)^2 + (dy - t\varphi \, dx)^2)^{1/2}/\text{Im } z + O(t^2, \text{ support } \varphi).$$

Now differentiating

$$\frac{d}{dt}((dx-t\varphi\,dy)^2+(dy-t\varphi\,dx)^2)^{1/2}/\text{Im }z|_{t=0}=-2\varphi\,dx\,dy/(dx^2+dy^2)^{1/2}\,\text{Im }z.$$

We are now ready to apply formula (1.1). Let s be the arc length parameter along γ . Assume increasing s corresponds to moving to the left. The tangent of γ

at p is

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right)_{p} = \text{Im } z(-\sin \beta_{p}, \cos \beta_{p})_{p}.$$

The contribution to the integral is then $\int 2\varphi \sin \beta_p \cos \beta_p ds$. At p we have $\operatorname{Im} z d\theta = -dx = \operatorname{Im} z \sin \beta_p ds$ and finally the integral

$$\int 2\varphi \cos \beta_p \, d\theta.$$

Now the contribution of p to the integral (1.1) tends to $\cos \beta_p$ as φ approximates $\frac{1}{2}$ the Dirac delta. A check of the calculations shows the integral is indeed unchanged if s is replaced by -s. Finally each intersection of α and γ contributes such a term.

4. Remarks

Our calculations suggest that the appropriate Beltrami differential representing a Fenchel-Nielsen twist is a distribution. This is consistent with the point of view of the Fenchel-Nielsen twist found in the theory of amalgamation of Fuchsian groups.

Finally, we give an alternative formula for the cosine. Let the axis of two hyperbolic transformations A and B intersect at a point p. Let δ be the angle at p formed by the segments along the axes of A and B to their respective attractive fixed points. Then

$$\cos \delta = \frac{-\operatorname{sgn} (\operatorname{tr} A \operatorname{tr} B)(\operatorname{tr} A \operatorname{tr} B - 2 \operatorname{tr} A B)}{(\operatorname{tr}^2 A - 4)^{1/2} (\operatorname{tr}^2 B - 4)^{1/2}}$$

where tr denotes the trace of a matrix and sgn denotes the sign of a real number.

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