Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 56 (1981)

Artikel: On one-relator soluble groups.
Autor: Strebel, Ralph
DOI: https://doi.org/10.5169/seals-43234

Comment. Math. Helvetici 56 (1981) 123–131 0010–2571/81/001123–09$01.50+0.20/0

On one-relator soluble groups

RALPH STREBEL

1. Introduction

1.1. Let G be a one-relator group, say

$$G=\langle x_1, x_2, \ldots, x_m; w\rangle,$$

and $\mathfrak{B}$ a soluble variety. Following G. Baumslag we call $G/\mathfrak{B}(G)$, where $\mathfrak{B}(G)$ denotes the $\mathfrak{B}$-verbal subgroup of G, a *one-relator* $\mathfrak{B}$*-group*. G. Baumslag initiated the search for conditions on m, w and $\mathfrak{B}$ that force $G/\mathfrak{B}(G)$ to be infinitely related. He showed this to be the case if $\mathfrak{B}=\mathfrak{A}^2$ (i.e. the variety of all metabelian groups) and $m \geqq 3$, w arbitrary, or $m=2$, w a proper power ([2], p. 67, Theorem F). These results were extended in [4] (p. 259, Theorem B): If $m \geqq 3$ and $\mathfrak{B}$ is a soluble variety containing a subvariety of the form $\mathfrak{A}_p \cdot \mathfrak{A}$, p a prime, then $G/\mathfrak{B}(G)$ is infinitely related. This result is actually best possible; for a soluble variety $\mathfrak{B}$ having no subvariety of the form $\mathfrak{A}_p \cdot \mathfrak{A}$ is nilpotent-by-locally finite (J. Groves [6], p. 392, Theorem A; I'd like to thank P. M. Neumann for drawing my attention to this reference). For $m \geqq 3$ the final answer to G. Baumslag's question therefore reads:

THEOREM A. *Let $G=\langle x_1, \ldots, x_m; w\rangle$ be a one-relator group and $\mathfrak{B}$ a soluble variety. If $m \geqq 3$ then $G/\mathfrak{B}(G)$ is finitely related if and only if $\mathfrak{B}$ is nilpotent-by-locally finite. In particular, a finitely generated $\mathfrak{B}$-free group is finitely related if and only if it is nilpotent-by-finite.*

(For groups $H=\langle x_1, \ldots, x_m; w_1, \ldots, w_n\rangle$ with $m \geqq n+2$ the answer is the same.)

1.2. As the case $m=1$ is clear we are left with $m=2$. This case is surprisingly complex, as was already brought to light by the examples in [3] and by the Baumslag-Boler 2-generator 2-relator group

$$G=\langle x_1, x_2; [x_1, x_2]^{x_1}=[x_1, x_2]\cdot[x_1, x_2]^{x_2}, [[x_1, x_2], [x_1, x_2]^{x_2}]=1\rangle$$
$$=\langle x_1, x_2; x_1^{x_2 x_1}=x_1^{x_2 x_2}, [[x_1, x_2], [x_1, x_2]^{x_2}]=1\rangle,$$

which is a one-relator metabelian group whose derived group is free Abelian of rank $\aleph_0$ ([2], p. 68, Ex. G(ii)).

In [4] the subcase "$m=2$ and w not a commutator word" is dealt with. The authors prove that for $\mathfrak{V}$ a variety with $\mathfrak{A}^2 \leqq \mathfrak{V} \leqq \mathfrak{N}_c\mathfrak{A}$, $c \geqq 1$, $G/\mathfrak{V}(G)$ is finitely related if and only if $G/\mathfrak{A}^2(G) = G/G''$ is, and they give an algorithm which, for a preassigned w, allows to decide whether G/G'' is finitely related or not.

In this article the remaining subcase will be treated. To state our answer concisely we introduce some notation. Let F be the free group on x_1 and x_2, let $D_1 : F \to \mathbb{Z}F$ denote the partial derivative with respect to x_1 and $^- : \mathbb{Z}F \to \mathbb{Z}F_{ab}$ the canonical projection. Choose an isomorphism $\theta : F_{ab} \xrightarrow{\sim} \mathbb{Z}^2 \subset \mathbb{R}^2$ onto the standard lattice in the Euclidean plane $\mathbb{R}^2$. If $\lambda = \sum \lambda_q \cdot q$ is an element of the group ring $\mathbb{Z}F_{ab}$ the set

$$\theta_*(\lambda) = \{\theta q \mid \lambda_q \neq 0\}$$

is a finite subset of $\mathbb{Z}^2$ and λ_q will also be called the coefficient of the point θq.

THEOREM B. *Let $G = \langle x_1, x_2; w\rangle$ be a one-relator group with w a commutator word. If $G' \neq G''$ and $l \geqq 3$ then $G/G^{(l)}$ is never finitely related. G/G'', however, may be finitely related: this happens precisely if $\theta_*(\overline{D_1 w/(1-\bar{x}_2)})$ does not lie on a straight line, if the convex polygon $\mathcal{P}$ bounding the convex hull of $\theta_*(\overline{D_1 w/(1-\bar{x}_2)}) \subseteq \mathbb{R}^2$ has no parallel edges and if for every edge s of $\mathcal{P}$ the vertex v_s of $\mathcal{P}$ having greatest distance from the straight line supporting s has coefficient* ± 1.

If, to give an illustration, G is the Baumslag-Boler group

$$G = \langle x_1, x_2;\ x_1^{x_2 x_1} = x_1^{x_2 x_2}\rangle$$

its bounding polygon $\mathcal{P}$ is a triangle all whose vertices have coefficients ± 1 and so G/G'' is finitely related, in accordance with our previous remark.

1.3. Theorem A and Theorem B together reveal that G. Baumslag's question whether $G/G^{(l+1)}$, G a one-relator group, can only be finitely related if $G/G^{(l)}$ is so ([2], p. 71, Problem 6), has a positive answer in all cases, save possibly in the case where $m=2$ and w is not a commutator word. The positive answer, moreover, suggests that the variety $\mathfrak{A}^2$ may play an exceptional role among the varieties $\mathfrak{A}^l$, $l \geqq 2$, in that a finitely generated group of $\mathfrak{A}^l$ which is "close" to being relatively free, containing e.g. an $\mathfrak{A}^l$-free subgroup of rank $\geqq 2$, can at best be finitely related if it is metabelian.

2. Proof of Theorem B: the metabelian case

2.1. Let $G=\langle x_1, x_2; w\rangle$ be a one-relator group with w a commutator word. Then $Q=G_{ab}=G/G'$ is free-Abelian of rank 2 and $A=G'_{ab}$ is a $\mathbb{Z}Q$-module (via conjugation). By [5] (Theorem A and formula (2.3)) the answer to the question whether the quotient group G/G'' is finitely related depends only on the annihilator ideal $I=\mathrm{Ann}_{\mathbb{Z}Q}(A)$ of A. We claim that I is a principal ideal generated by $\overline{D_1 w}/(1-\bar{x}_2)$.

To see this let F be free on x_1 and x_2, let $D_i: F\to\mathbb{Z}F$ denote the partial derivative with respect to x_i, $i=1,2$, and $^-:\mathbb{Z}F\twoheadrightarrow\mathbb{Z}F_{ab}$ the canonical projection. If $G=F/gp_F(w)=\langle x_1, x_2; w\rangle$ is a one-relator group with $w\in F'$, the canonical projection $\pi: F\twoheadrightarrow G$ induces an isomorphism $\pi_*: F_{ab}\xrightarrow{\sim} G_{ab}$ which will be thought of as being the identity.

LEMMA 1. *If $G=\langle x_1, x_2; w\rangle$ is a one-relator group and w a commutator word then G'_{ab} is a cyclic $\mathbb{Z}G_{ab}$-module with $\overline{D_1 w}/(1-\bar{x}_2)$ as its defining annihilator.*

Proof. There is an exact sequence of left $\mathbb{Z}G$-modules

$$\mathbb{Z}G\xrightarrow{\partial_2}\mathbb{Z}G\oplus\mathbb{Z}G\xrightarrow{\partial_1}\mathbb{Z}G\xrightarrow{\varepsilon}\mathbb{Z}\to 0 \tag{1}$$

where ε is the augmentation, ∂_1 sends $(1,0)$ to $1-\pi(x_1)$ and $(0,1)$ to $1-\pi(x_2)$, while $\partial_2(1)=((\pi\circ D_1)w, (\pi\circ D_2)w)$. If this beginning of a $\mathbb{Z}G$-free resolution is used to compute $H_1(G',\mathbb{Z})=G'_{ab}$ one gets

$$G'_{ab}\cong \ker(\mathbb{Z}\otimes_{G'}\partial_1)/\mathrm{im}(\mathbb{Z}\otimes_{G'}\partial_2).$$

In the special case $w_0=x_1x_2x_1^{-1}x_2^{-1}$ we have $G'=G''$ and so $\ker(\mathbb{Z}\otimes_{G'}\partial_1)$ is generated by

$$(\overline{D_1 w_0}, \overline{D_2 w_0})=(1-\bar{x}_2, -1+\bar{x}_1).$$

In the general case, $\mathrm{im}(\mathbb{Z}\otimes_{G'}\partial_2)$ is the cyclic submodule generated by $(\overline{D_1 w}, \overline{D_2 w})$. Since we know a priori that $\mathrm{im}(\mathbb{Z}\otimes_{G'}\partial_2)\subseteq\ker(\mathbb{Z}\otimes_{G'}\partial_1)$ there must exist $\mu\in\mathbb{Z}G_{ab}$ such that $\overline{D_1 w}=\mu(1-\bar{x}_2)$ and $\overline{D_2 w}=\mu(-1+\bar{x}_1)$. It follows that $G'_{ab}\cong\mathbb{Z}G_{ab}/(\mathbb{Z}G_{ab}\cdot\mu)$. Note that μ is unique, $\mathbb{Z}G_{ab}$ being a unique factorization domain. □

Remark. If $w=([x_1, x_2]^{c_1})^{f_1}\cdot\dots\cdot([x_1, x_2]^{c_k})^{f_k}$ where $c_1,\dots,c_k$ are integers and $f_1,\dots,f_k$ elements of F then $\overline{D_1 w}/(1-\bar{x}_2)=c_1\bar{f}_1+\cdots+c_k\bar{f}_k$. This shows that every element of $\mathbb{Z}F_{ab}$ is the defining annihilator of G'_{ab} for a suitable w.

2.2. We describe next the necessary and sufficient condition stated in [5] for G/G'' to be finitely related. Set $A = G'_{ab}$ and $Q = G_{ab}$ for short. Let $C(A) = 1 + \mathrm{Ann}_{\mathbb{Z}Q}(A)$ denote the set of centralizers of A. By Lemma 1 we have

$$C(A) = \{1 + \mu(\overline{D_1 w/(1 - \bar{x}_2)}) \mid \mu \in \mathbb{Z}Q\}.$$

Let $S^1 \subseteq \mathbb{R}^2$ denote the unit circle in the Euclidean plane $\mathbb{R}^2$, and choose an isomorphism $\theta : Q = G_{ab} \xrightarrow{\sim} \mathbb{Z}^2 \subseteq \mathbb{R}^2$ onto the standard lattice of $\mathbb{R}^2$. Define the subset $\Sigma_A \subseteq S^1$ by

$$\Sigma_A = \bigcup_{\lambda \in C(A)} \{u \mid \text{for all } q \in \operatorname{supp}(\lambda) \text{ one has } \langle u, \theta q \rangle > 0\}. \tag{2}$$

Then the metabelian group G/G'' is finitely related if, and only if, $\Sigma_A \cup -\Sigma_A = S^1$, where $-\Sigma_A = \{-u \in S^1 \mid u \in \Sigma_A\}$ is the set of antipodes of Σ_A ([5], Theorem A, formula (2.3) and Remark *2.3*). So it remains only to determine when $\Sigma_A \cup -\Sigma_A = S^1$.

2.3. Every $u \in S^1$ leads to an $\mathbb{R}$-grading of $\mathbb{Z}Q$ by setting $\mathbb{Z}Q = \bigoplus\{R\rho \mid \rho \in \mathbb{R}\}$, where R_ρ is the $\mathbb{Z}$-submodule of $\mathbb{Z}Q$ generated by all elements $q \in Q$ with $\langle u, \theta q \rangle = \rho$. Let $\lambda = 1 + \mu \cdot \mu_0$, $\mu_0 = \overline{D_1 w/(1 - \bar{x}_2)}$, be a centralizer of A. It contributes to Σ_A as given in (2) precisely if the homogeneous components of negative degree of $\mu \cdot \mu_0$ are all zero, while the homogeneous component of degree zero equals -1. This is only possible when the homogeneous component of μ_0 having lowest degree is a unit of $\mathbb{Z}Q$, i.e., of the form $\pm q$, $q \in Q$. If, conversely, the homogeneous component of μ_0 having lowest degree is a unit of $\mathbb{Z}Q$, say e, then $1 + (-e^{-1})\mu_0$ will be a centralizer contributing to Σ_A. If, for $u \in S^1$ and $\lambda \in \mathbb{Z}Q$, we denote by λ_u the homogeneous component of λ with lowest degree in the $\mathbb{R}$-grading associated with u, we have therefore

$$\Sigma_A = \{u \in S^1 \mid (\overline{D_1 w/(1 - \bar{x}_2)})_u \text{ is a unit of } \mathbb{Z}Q\}. \tag{3}$$

Assume now that $\Sigma_A \cup -\Sigma_A = S^1$ and set $\mu_0 = \overline{D_1 w/(1 - \bar{x}_2)}$ for short. Then for every $u \in S^1$ either $(\mu_0)_u$ or $(\mu_0)_{-u}$ is a unit of $\mathbb{Z}Q$. Let us see what this implies for the convex polygon $\mathcal{P}$ bounding the convex hull of the finite set

$$\theta_*(\mu_0) = \{\theta q \mid q \in \operatorname{supp}(\mu_0)\}.$$

Let u be a unit vector orthogonal to the straight line l supporting an edge s of $\mathcal{P}$ and such that the remaining edges of $\mathcal{P}$ lie in the positive halfplane determined by l and u. Then the support of $(\mu_0)_u$ has at least two elements and so $(\mu_0)_u$ cannot be a unit, whence $(\mu_0)_{-u}$ must be a unit. It follows that no other edge can be parallel to s and that the vertex v_s of greatest distance from l must have coefficient ± 1 (and cannot lie on l). Thus the conditions on $\mathcal{P}$ stated in Theorem B are necessary. Suppose, conversely, they are satisfied. Then $\Sigma_A \cup -\Sigma_A$ is a *closed, non-void* set. On the other hand it is immediate from (2) that Σ_A, and hence also $\Sigma_A \cup -\Sigma_A$, are *open*. Since S^1 is connected this gives $\Sigma_A \cup -\Sigma_A = S^1$. □

3. Proof of Theorem B: the meta metabelian case

3.1. We begin with some auxiliary results. Let $K \lhd L \lhd G$ be normal subgroups of a group G and suppose $K \leqq [L, L]$. For every trivial $\mathbb{Z}G$-module A the pair of extensions

$$\begin{array}{ccccc} K & \lhd & L & \twoheadrightarrow & L/K \\ \downarrow{=} & & \downarrow & & \downarrow \\ K & \lhd & G & \twoheadrightarrow & G/K \end{array} \tag{4}$$

where all maps are the canonical ones, gives rise to a commutative diagram

$$\begin{array}{ccccccc} H_2(L, A) & \to & H_2(L/K, A) & \to & K_{ab} \otimes_L A & \to & 0 \\ \downarrow & & \downarrow & & \downarrow 1_* & & \\ H_2(G, A) & \to & H_2(G/K, A) & \to & K_{ab} \otimes_G A & \to & 0. \end{array} \tag{5}$$

Its rows are exact (here the assumption that $K \leqq [L, L]$ and A be a trivial G-module are used). Each $g \in G$ defines by conjugation an action on each of the groups occurring in (4) and, as all the canonical maps in (4) are compatible with this action, it follows that (5) is a diagram of (right) $\mathbb{Z}G$-modules and $\mathbb{Z}G$-homomorphisms. Note that $H_2(G, A)$ and $H_2(G/K, A)$ are trivial G-modules, inner automorphisms inducing the identity in homology, and that the G-action on $K_{ab} \otimes_L A$ resp. $K_{ab} \otimes_G A$ is the obvious one induced by conjugation (and so trivial in the second case). By applying the right exact functor $(?)_G = ? \otimes_G \mathbb{Z}$ to the diagram (5) one therefore obtains the commutative diagram with exact rows

$$\begin{array}{ccccccc} H_2(L, A)_G & \to & H_2(L/K, A)_G & \to & K_{ab} \otimes_G A & \to & 0 \\ \downarrow & & \downarrow & & \downarrow{=} & & \\ H_2(G, A) & \to & H_2(G/K, A) & \to & K_{ab} \otimes_G A & \to & 0. \end{array} \tag{6}$$

In particular, one gets

COROLLARY 2. *Let $L \lhd G$ be a normal subgroup of G and $A = \mathbb{Q}$. If $H_2(L, \mathbb{Q})_G$ and $H_2(G, \mathbb{Q})$ are finite dimensional then $H_2(G/[L, L], \mathbb{Q})$ is infinite dimensional if and only if $H_2(L_{ab}, \mathbb{Q})_G$ is so.*

LEMMA 3. *Let $L \lhd G$ be a normal subgroup. If $H_2(G, \mathbb{Q})$ and $H_2(L, \mathbb{Q})_G$ are finite and $H_2(G/L, \mathbb{Q})$ is infinite dimensional, then $H_2(G/[L, L], \mathbb{Q})$ is also infinite dimensional.*

Proof. The 5-term sequence associated with the extension $L \lhd G \twoheadrightarrow G/L$ shows $L_{ab} \otimes_G \mathbb{Q} \cong L/[L, G] \otimes_{\mathbb{Z}} \mathbb{Q}$ to be infinite dimensional. Hence $L/[L, G]$ has infinite torsion-free rank and the same is true of its exterior square

$$L/[L, G] \wedge L/[L, G] \cong H_2(L/[L, G], \mathbb{Z}).$$

The canonical projection $L/[L, L] \twoheadrightarrow L/[L, G]$ induces a *surjective* $\mathbb{Z}G$-module homomorphism $H_2(L_{ab}, \mathbb{Q}) \twoheadrightarrow H_2(L/[L, G], \mathbb{Q})$ and the target module is a G-trivial, infinite dimensional vector space over $\mathbb{Q}$. Consequently $H_2(L_{ab}, \mathbb{Q})_G$ is infinite dimensional and Corollary 2 permits to conclude. □

3.2. **E**-*groups.* Following [8] a group X will be called an **E**-*group* if its Abelianization X_{ab} is *torsion-free* and if there exists a $\mathbb{Z}X$-projective resolution $\cdots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow \mathbb{Z}$ such that $\partial_2 \otimes_X \mathbb{Z} : P_2 \otimes_X \mathbb{Z} \to P_1 \otimes_X \mathbb{Z}$ is *injective.* A basic result about **E**-groups states that the commutator subgroup of an **E**-group is again an **E**-group; in particular, $H_2(X^{(l)}, \mathbb{Z}) = 0$ for every $l \geqq 0$ (see [8], p. 303, Theorem A).

As an illustration, let $G = F/gp_F(w) = \langle x_1\, x_2; w \rangle$ be a one-relator group with $w \in F'$. Then $H_2(G, \mathbb{Z}) \neq 0$ and so G itself is not an **E**-group. But the situation improves if we pass to G'. To see this view the beginning (1) of a $\mathbb{Z}G$-free resolution as a sequence of right $\mathbb{Z}G$-modules. Then $\partial_2 \otimes_{G'} \mathbb{Z}$, being isomorphic to $\partial_2 \otimes_G \mathbb{Z}G_{ab}$, is injective if, and only if, $\overline{D_1 w} \neq 0$, and this, in turn, happens precisely when the canonical projection $F/F'' \twoheadrightarrow G/G''$ is *not* injective, i.e. if $w \notin F''$. If therefore $w \in F' \setminus F''$ and G'_{ab} is torsion-free, G' will be an **E**-group. Since $H_2(G, \mathbb{Q})$ is at most 1-dimensional iterated application of Lemma 3 will then establish

PROPOSITION 4. *Let $G = F/gp_F(w) = \langle x_1, x_2; w \rangle$ be a one-relator group with $w \in F' \setminus F''$. If G'_{ab} is torsion-free and if there is a natural number $l_0 > 1$ for which $H_2(G/G^{(l_0)}, \mathbb{Q})$ is infinite dimensional then $H_2(G/G^{(l)}, \mathbb{Q})$ will remain infinite dimensional for all $l \geqq l_0$.*

Remark. In [1] H. Abels exhibits a finitely related, soluble group L having an infinitely related quotient group $\bar{L}$, thereby answering a question P. Hall raised in

1954. By varying Abels' example one can concoct a finitely related soluble group L_1 having a homomorphic image $\bar{L}_1$ whose $H_2(\bar{L}_1, \mathbb{Q})$ is infinite dimensional. This shows that in Proposition 4 the (implicit) assumption that G' be an **E**-group is not redundant.

3.3. We are now ready to prove the remainder of Theorem B. Let $G = F/gp_F(w) = \langle x_1, x_2; w\rangle$ be a one-relator group and w a commutator word. If G/G'' is infinitely related then so is every soluble quotient of G which maps onto G/G'' ([5], Theorem B). Assume therefore G/G'' to be finitely related. Then $\Sigma_A \cup -\Sigma_A = S^1$, where $A = G'_{ab}$. (This follows from Theorem A of [5] and has been discussed in Subsection *2.2.*) Since S^1 is connected and Σ_A as well as $-\Sigma_A$ is open there exists $u \in S^1$ with $\{u, -u\} \subseteq \Sigma_A$, and because the unit vectors with rational slope are dense in S^1, u can be chosen to have rational slope. Let t be a primitive element of the free Abelian group G_{ab} which maps under the identification $\theta: G_{ab} \xrightarrow{\sim} \mathbb{Z}^2 \subseteq \mathbb{R}^2$ onto a vector orthogonal to u, and enlarge $\{t\}$ to a basis $\{s, t\}$. By (3) we know that $\{u, -u\} \subseteq \Sigma_A$ holds if and only if the homogeneous components of both lowest and highest degree of $\overline{D_1 w}/(1-\bar{x}_2)$ (in the $\mathbb{R}$-grading associated with u) are units. With respect to the basis $\{s, t\}$ the defining annihilator $\overline{D_1 w}/(1-\bar{x}_2)$ must therefore have the form

$$\overline{D_1 w}/(1-\bar{x}_2) = c_n s^n + c_{n+1} s^{n+1} + \cdots + c_{n+k} s^{n+k},$$

where each c_j, $j = n, \ldots, n+k$, is an element of the group ring $\mathbb{Z}gp(t)$, and c_n and c_{n+k} are units. Consequently $A = G'_{ab}$, viewed as a $\mathbb{Z}gp(t)$-module, is a free $\mathbb{Z}gp(t)$-module of rank k and so, in particular, free Abelian of rank $\aleph_0$, unless $G'_{ab} = 0$. As there is nothing to be proved in the latter case let us assume $G'_{ab} \neq 0$. Then G'_{ab} is torsion-free and $\overline{D_1 w} \neq 0$ and thus G' is an **E**-group. Comparing Theorem B with Proposition 4 and taking into account that $G/G^{(l)}$ cannot be finitely related if $H_2(G/G^{(l)}, \mathbb{Q})$ is infinite dimensional, we see that Theorem B will be established if we can show that $H_2(G/G''', \mathbb{Q})$ is infinite dimensional. We shall achieve this goal by a variation on the proof of Lemma 3.

Since G'_{ab} is a free $\mathbb{Z}gp(t)$-module of positive rank its exterior square $G'_{ab} \wedge G'_{ab}$ is a free $\mathbb{Z}gp(t)$-module of rank $\aleph_0$. As $H_2(G', \mathbb{Q}) = 0$, the 5-term sequence associated with $G'' \lhd G' \twoheadrightarrow G'_{ab}$ produces an isomorphism

$$H_2(G'_{ab}, \mathbb{Q}) \xrightarrow{\sim} G''/[G'', G'].$$

Hence $B = G''/[G'', G']$ is a free $\mathbb{Z}gp(t)$-module of infinite rank, and

$$\bar{B} = G''/[G'', G'] \otimes_{gp(t)} \mathbb{Z}$$

is free Abelian of rank $\aleph_0$. On the other hand, G/G'' is finitely related whence B is a finitely generated $\mathbb{Z}G_{ab}$-module and $\bar{B}$ is a finitely generated $\mathbb{Z}gp(s)$-module. As $\mathbb{Q}gp(s)$ is a principal ideal domain, $\bar{B}\otimes_{\mathbb{Z}}\mathbb{Q}$ maps onto $\mathbb{Q}gp(s)$. The exterior square of $\mathbb{Q}gp(s)$ is a free $\mathbb{Q}gp(s)$-module of infinite rank, whence $V= H_2(\mathbb{Q}gp(s),\mathbb{Q})\otimes_{gp(s)}\mathbb{Q}$ is a G-trivial, infinite dimensional vector space over $\mathbb{Q}$. Collecting the various maps one gets a chain of $\mathbb{Q}G$-module epimorphisms

$$H_2(G''_{ab},\mathbb{Q}) \twoheadrightarrow H_2(G''/[G'',G'],\mathbb{Q}) \xrightarrow{\sim} H_2(G''/[G'',G']\otimes_{\mathbb{Z}}\mathbb{Q},\mathbb{Q}) \twoheadrightarrow$$
$$\twoheadrightarrow H_2(\mathbb{Q}gp(s),\mathbb{Q}) \twoheadrightarrow V,$$

which shows that $H_2(G''_{ab},\mathbb{Q})_G$ is infinite dimensional. The proof is completed by an appeal to Corollary 2. □

3.4. Remarks. (a) The topological argument used in *3.3.* to prove that G''_{ab} is a finitely generated $\mathbb{Z}gp(t)$-module for a suitable $t\in G_{ab}$ generalizes to arbitrary finitely related metabelian groups, proving that *every finitely related metabelian group whose Abelianization G_{ab} has torsion-free rank >1 contains a finitely generated normal subgroup $N\lhd G$ with infinite cyclic quotient group.* Moreover, if $M\lhd G$ is a finitely generated normal subgroup with infinite cyclic quotient group then all normal subgroups M_1 (with infinite cyclic quotient group) which are sufficiently close to M, will be finitely generated. (Proofs of these assertions will appear in a joint paper with Robert Bieri.) The second property holds in fact for every finitely generated group G with G_{ab} of torsion-free rank >1 (W. D. Neumann [7]).

(b) Although the subcase where $m=2$ and w is not a commutator word looks, on the face of it, more tractable than the other subcase, I have not succeeded in proving that $G/G^{(l)}$ is infinitely related if $l\geqq 3$ and $G''\neq G'''$. One might be tempted to proceed similarly: If G/G'' is infinitely related every soluble quotient of G which can be mapped onto G/G'' is infinitely related; in the contrary case G' can be shown to be an **E**-group and by the analogue of Proposition 4 it will then do to establish that $H_2(G''_{ab},\mathbb{Q})_G$ is infinite dimensional. Note, however, that in the present situation $G'_{ab}\otimes_{\mathbb{Z}}\mathbb{Q}$ is finite dimensional so that the strategy used in *3.3.* fails.

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Mathematisches Institut der
Universität Heidelberg
Im Neuenheimer Feld 288
D-6900 Heidelberg

Received November 29, 1979