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## Spectra of manifolds with small handles

I. CHAVEL<sup>(1)</sup> and E. A. FELDMAN<sup>(1)</sup>

To H. E. RAUCH, in memoriam

In this paper we consider a compact connected  $C^\infty$  Riemannian manifold  $M$  of dimension  $n \geq 2$  and study the effect, on the spectrum of the associated Laplace–Beltrami operator  $\Delta$  acting on functions, of adding a “small” handle to  $M$ .

The handles we consider are defined as follows: Fix two distinct points  $p_1, p_2$  in  $M$  and for  $\varepsilon > 0$  define

$B_\varepsilon \equiv$ : union of the open geodesic disks about  $p_1, p_2$  of radius  $\varepsilon$ ,

$\Omega_\varepsilon \equiv$ :  $M - \overline{B_\varepsilon}$ ,

$\Gamma_\varepsilon \equiv$ : common boundary of  $B_\varepsilon$  and  $\Omega_\varepsilon$ ,

$S_\varepsilon \equiv$ :  $(n - 1)$ -sphere in  $R^n$  of radius  $\varepsilon$ ,

$S \equiv$ :  $S_1$ .

For positive  $\varepsilon$  which is less than  $\frac{1}{4}$  the injectivity radius of  $M$  and less than  $\frac{1}{4}$  the distance from  $p_1$  to  $p_2$ , let  $M_\varepsilon$  be a compact connected  $C^\infty$  Riemannian manifold with  $\overline{\Omega_\varepsilon}$  isometrically imbedded in  $M_\varepsilon$ , and with a diffeomorphism

$$\Psi_\varepsilon : M_\varepsilon - \Omega_{2\varepsilon} \rightarrow [-1, 1] \times S$$

such that

$$C_\varepsilon \equiv M_\varepsilon - \Omega_\varepsilon = \Psi_\varepsilon^{-1} \left[ \left[ -\frac{1}{2}, \frac{1}{2} \right] \times S \right].$$

We refer to such an  $M_\varepsilon$  as *obtained from  $M$  by adding the handle  $C_\varepsilon$  across  $\Gamma_\varepsilon$* .

Denote the respective spectra of  $M, M_\varepsilon$  by

$$\text{spec}(M) \equiv \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\},$$

$$\text{spec}(M_\varepsilon) \equiv \{0 = \sigma_0(\varepsilon) < \sigma_1(\varepsilon) \leq \sigma_2(\varepsilon) \leq \dots\},$$

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where each distinct eigenvalue is repeated according to its multiplicity; and denote the associated theta functions by

$$\Theta(t) \equiv: \sum_{j=0}^{\infty} e^{-\lambda_j t}, \quad \Theta_\varepsilon(t) \equiv: \sum_{j=0}^{\infty} e^{-\sigma_j(\varepsilon)t}.$$

Our interest in this paper is in determining whether the family of Riemannian manifolds  $M_\varepsilon$  can be chosen so that

$$\lim \sigma_j(\varepsilon) = \lambda_j \quad \text{as } \varepsilon \downarrow 0 \tag{1}$$

for all  $j = 1, 2, \dots$ .

Our first comment is that even if (1) is valid for all  $j$ , we do not expect that it be valid uniformly in  $j$ . In fact, when  $M$  is 2-dimensional the Minakshisundaram-Pleijel asymptotic expansion reads as [10, p. 45; 1, pp. 204–222]

$$\Theta(t) \sim \frac{A(M)}{4\pi t} + \frac{\chi(M)}{6} + O(t), \quad \Theta_\varepsilon(t) \sim \frac{A(M_\varepsilon)}{4\pi t} + \frac{\chi(M_\varepsilon)}{6} + O(t), \tag{2}$$

as  $t \downarrow 0$  (where  $A(\cdot)$ ,  $\chi(\cdot)$  denote area and Euler-characteristic, respectively). If (1) were valid uniformly in  $j$  then (2) would imply, by an easy argument, that  $\chi(M_\varepsilon) = \chi(M)$  – an impossibility.

**THEOREM A.** *We always have*

$$\limsup \sigma_j(\varepsilon) \leq \lambda_j \quad \text{as } \varepsilon \downarrow 0 \tag{3}$$

for all  $j = 1, 2, \dots$ . A necessary condition that (1) be valid for all  $j$  is that  $\nu(\varepsilon)$ , the lowest eigenvalue of  $C_\varepsilon$  with Dirichlet data on  $\Gamma_\varepsilon$ , satisfy

$$\lim \nu(\varepsilon) = +\infty \quad \text{as } \varepsilon \downarrow 0. \tag{4}$$

In particular, if for a fixed  $l > 0$ , the (“long-thin”) cylinder  $[-l/2, l/2] \times S_\varepsilon$  is an isometrically imbedded open submanifold of  $C_\varepsilon$  for every  $\varepsilon$ , then  $\nu(\varepsilon) \leq \pi^2/l^2$  and (1) cannot be satisfied for all  $j$ .

To give a sufficient condition we require a definition,

**DEFINITION 1.** For any compact Riemannian manifold  $X$  of dimension

$n \geq 2$ , we define the *isoperimetric constant*  $c_1(X)$  by

$$c_1(X) = \inf_Y \frac{\{\text{vol}_{n-1}(Y)\}^n}{\{\min(\text{vol}_n(X_1), \text{vol}_n(X_2))\}^{n-1}} \quad (5)$$

where  $\text{vol}_k(\cdot)$  denotes  $k$ -dimensional Riemannian measure, and  $Y$  ranges over all compact  $(n-1)$ -dimensional submanifolds of  $X$  which divide  $X$  into 2 open submanifolds  $X_1, X_2$  each having boundary  $Y$ .

**THEOREM B.** *Assume there exists a constant  $c > 0$  such that*

$$c_1(M_\varepsilon) \geq c > 0 \quad (6)$$

for all  $\varepsilon$ . Then (1) is valid for all  $j = 1, 2, \dots$

That (6) is an indication of the “smallness” of  $C_\varepsilon$  is given by

**LEMMA 1.** *The sufficient condition “(6) for all  $\varepsilon$ ” implies*

$$\text{vol}_n(C_\varepsilon) = O(\varepsilon^n), \quad (7)$$

$$\nu(\varepsilon) \geq \text{const}/\varepsilon^2 \quad (8)$$

as  $\varepsilon \downarrow 0$ .

Indeed, one proves (7) by picking  $Y = \Gamma_\varepsilon$ , and  $X_1 = C_\varepsilon$ ,  $X_2 = \Omega_\varepsilon$ .

In order to prove (8) from (6) and (7) let us recall, a definition and Cheeger’s inequality for manifolds with boundary [4; 14].

**DEFINITION.** Let  $M$  be a compact manifold with boundary  $\partial M$ . We define the constant  $h(M)$  by

$$h(M) = \inf_Y \frac{\text{vol}_{n-1}(Y)}{\text{vol}_n(X)}$$

where  $Y$  ranges over all compact  $(n-1)$  dimension submanifolds such that  $\partial M \cap Y = \emptyset$ , which divide  $M$  into  $X$  and  $X'$  where  $\partial \bar{X} \cap \partial M = \emptyset$ .

Cheeger’s argument [4] shows that  $\lambda_1(M) \geq h^2/4$  where  $\lambda_1(M)$  is the first eigenvalue for the Laplacian with Dirichlet boundary data.

Let  $M = C_\varepsilon$ ,  $\partial M = \Gamma_\varepsilon$  and  $X$  and  $Y$  submanifolds of  $M$  as in the above

definition then  $\text{vol}_{n-1}(Y) \geq c^{1/n} \text{vol}_n(X)^{n-1/n}$  and

$$h(C_\varepsilon) \geq \inf_Y c^{1/n} \text{vol}_n(X)^{-1/n} \geq k/\varepsilon$$

follow from (6) and (7). Therefore (8) follows from Cheeger's inequality.

We next remark that whereas the necessary condition for the validity of (1) for all  $j$  is a consequence of the max-min characterization of eigenvalues and thus best interpreted via vibration phenomena, the sufficient condition is obtained by working with the respective fundamental solutions of the heat equation on  $M, M_\varepsilon$ .

Most important is the interpretation of these fundamental solutions via Brownian motion, viz., if

$$p : M \times M \times (0, \infty) \rightarrow R$$

is the fundamental solution of the heat equation on  $M$ , then  $p(x, y, t)$  is the probability density for a Brownian path in  $M$  starting at  $x$  at time 0 to be at  $y$  at time  $t$ . Of course one has a similar statement for

$$p_\varepsilon : M_\varepsilon \times M_\varepsilon \times (0, \infty) \rightarrow R,$$

the fundamental solution of the heat equation on  $M_\varepsilon$ . Similarly, if we let

$$q_\varepsilon : \Omega_\varepsilon \times \Omega_\varepsilon \times (0, \infty) \rightarrow R$$

denote the fundamental solution of the heat equation on  $\Omega_\varepsilon$  with Dirichlet data on  $\Gamma_\varepsilon$  then  $q_\varepsilon(x, y, t)$  is the probability density that a Brownian path starting at  $x \in \Omega_\varepsilon$  at time 0 will be at  $y \in \Omega_\varepsilon$  at time  $t$  *without having hit  $\Gamma_\varepsilon$  between time 0 and time  $t$* . In particular, for  $x, y$  in

$$M_0 \equiv M - \{p_1, p_2\}$$

(we now think of  $q_\varepsilon$  as vanishing on the complement of  $\Omega_\varepsilon \times \Omega_\varepsilon$ )  $q_\varepsilon(x, y, t)$  is a decreasing function in  $\varepsilon$ , and

$$q_\varepsilon \leq p, \quad q_\varepsilon \leq p_\varepsilon \tag{9}$$

on  $M \times M \times (0, \infty), M_\varepsilon \times M_\varepsilon \times (0, \infty)$ , cf. [7; 12] for the application and details in Euclidean space, and [9] for the construction on general Riemannian manifolds.

Our final concern is that we *can* construct manifolds  $M, M_\varepsilon$  for which (6) is satisfied for all  $\varepsilon$ .

**MAIN THEOREM.** *Let  $M$  be a compact 2-dimensional Riemannian manifold,  $K : M \rightarrow \mathbb{R}$  its Gaussian curvature and  $\tilde{M} = \{M - K^{-1}[0]\} \cup \{\text{int } K^{-1}[0]\}$ . Then  $\tilde{M}$  is open and dense in  $M$ . Given any two distinct points  $p_1, p_2$  in  $\tilde{M}$  then  $M_\varepsilon$  may be constructed so that there exists  $c > 0$  for which (6) is valid for all  $\varepsilon$ . Thus  $M_\varepsilon$  may be constructed so that  $A(M_\varepsilon) \rightarrow A(M)$  as  $\varepsilon \downarrow 0$  and so that (1) is valid for all  $j = 1, 2, \dots$*

The theorem suggests that to the question “Can you hear the shape of a drum?” [7] one should answer “For a compact 2-manifold – not really.” For to determine the Euler-characteristic, via (2), by actually listening to its tones (square roots of the eigenvalues) one would have to know *a priori* that what is heard in fact approximates *all* the tones with *uniform* accuracy. Anything less could lead the listener astray in determining the Euler-characteristic.

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This paper is dedicated to the inspiring memory of H. E. Rauch, whom both authors knew and admired as a friend, teacher, and mathematician.

## §1. Proof of Theorem A

Denote the spectrum of  $\Omega_\varepsilon$  with Dirichlet boundary data (distinct eigenvalues are repeated according to multiplicity) by

$$\text{spec}(\Omega_\varepsilon) \equiv: \{0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leq \lambda_3(\varepsilon) \leq \dots\}.$$

Then the max-min characterizations of the eigenvalues [5, Chap. VI] imply that  $\lambda_j(\varepsilon)$  is an increasing function of  $\varepsilon$ , and the validity of the inequalities

$$\lambda_j(\varepsilon) \geq \lambda_{j-1}, \quad \lambda_j(\varepsilon) \geq \sigma_{j-1}(\varepsilon) \tag{10}$$

for  $j = 1, 2, \dots$ . Moreover, in [3] it was shown (cf. [13] for the case of domains in Euclidean space) that

$$\lambda_j(\varepsilon) \rightarrow \lambda_{j-1} \quad \text{as } \varepsilon \downarrow 0 \tag{11}$$

for all  $j = 1, 2, \dots$ . Then (10), (11) imply (3).

With these preliminaries, establishing the necessary condition is done as follows: Let the union of the spectra of  $C_\varepsilon, \Omega_\varepsilon$  with Dirichlet data on  $\Gamma_\varepsilon$  be

denoted by

$$\text{spec}(C_\varepsilon) \cup \text{spec}(\Omega_\varepsilon) \equiv: \{0 < \mu_0(\varepsilon) \leq \mu_1(\varepsilon) \leq \dots\}$$

where the eigenvalues have been re-listed in non-decreasing order and repeated according to multiplicity. Then a max-min argument [5, p. 408] implies

$$\sigma_j(\varepsilon) \leq \mu_j(\varepsilon) \tag{12}$$

for all  $j = 0, 1, 2, \dots$ . Assume

$$\alpha \equiv: \liminf \nu(\varepsilon) \quad \text{as } \varepsilon \downarrow 0$$

is finite, and let  $\lambda_k$  be the first eigenvalue of  $M$  which is strictly greater than  $\alpha$  (in particular,  $\lambda_{k-1} < \lambda_k$ ). Then for any  $\varepsilon$  for which we have

$$\nu(\varepsilon) < \lambda_k$$

we also have  $\nu(\varepsilon) < \lambda_k \leq \lambda_{k+1}(\varepsilon)$ , i.e.,

$$\nu(\varepsilon) \in \{\mu_0(\varepsilon), \dots, \mu_k(\varepsilon)\},$$

which implies

$$\sigma_k(\varepsilon) \leq \mu_k(\varepsilon) \leq \max\{\nu(\varepsilon), \lambda_k(\varepsilon)\}.$$

Thus  $\alpha < \lambda_k$  implies by (11) that

$$\liminf \sigma_k(\varepsilon) \leq \liminf \max\{\nu(\varepsilon), \lambda_k(\varepsilon)\} = \max\{\alpha, \lambda_{k-1}\} < \lambda_k$$

as  $\varepsilon \downarrow 0$ . It is therefore impossible that  $\delta_k(\varepsilon) \rightarrow \lambda_k$  as  $\varepsilon \downarrow 0$ .

**COROLLARY 1.** *If*

$$\liminf \nu(\varepsilon) = \alpha < +\infty \quad \text{as } \varepsilon \downarrow 0,$$

*and  $\lambda_k$  is the first eigenvalue of  $M$  greater than  $\alpha$ , then*

$$\liminf \sigma_k(\varepsilon) < \lambda_k \quad \text{as } \varepsilon \downarrow 0.$$

*Remark 1.* We note that (4) is also a necessary condition that  $\Theta_\varepsilon(t) \rightarrow \Theta(t)$ ,

for any given  $t > 0$ ,  $\varepsilon \downarrow 0$ . Indeed, (12) implies that

$$\Theta_\varepsilon(t) \geq e^{-\nu(\varepsilon)t} + \sum_{j=1}^{\infty} e^{-\lambda_j(\varepsilon)t}.$$

But in [3] it was proved that the series on the right-hand side of the above inequality tends to  $\Theta(t)$ , uniformly on compact subsets of  $(0, \infty)$ , as  $\varepsilon \downarrow 0$ . That (4) is a consequence of  $\Theta_\varepsilon(t) \rightarrow \Theta(t)$  is immediate.

## §2. Proof of Theorem B

LEMMA 2. *Let  $dM$ ,  $dM_\varepsilon$  denote the respective volume elements of  $M$ ,  $M_\varepsilon$  (of course they agree on  $\Omega_\varepsilon$ ), and let  $f$  be any bounded measurable function compactly supported on  $M_0$ . Then*

$$\lim \int_{M_\varepsilon} p_\varepsilon(x, w, t) f(w) dM_\varepsilon = \int_M p(x, y, t) f(y) dM(y),$$

uniformly in  $(x, t) \in$  compact subsets of  $M_0 \times (0, \infty)$ , as  $\varepsilon \downarrow 0$ . In particular we have

$$\lim p_\varepsilon(x, y, t) = p(x, y, t) \quad \text{as } \varepsilon \downarrow 0 \quad (13)$$

on  $M_0 \times M_0 \times (0, \infty)$ .

*Proof.* In [3] it was shown (cf. [13] for the case of domain in Euclidean space) that

$$\lim q_\varepsilon(x, y, t) = p(x, y, t) \quad \text{as } \varepsilon \downarrow 0$$

uniformly on compact subsets of  $M_0 \times M_0 \times (0, \infty)$ . Let  $K$  be a compact subset of  $M_0$  and pick  $\varepsilon$  sufficiently small so that  $\Omega_\varepsilon$  contains  $K$  and the support of  $f$ . Then for  $x \in K$ ,  $t \in [a, b] \subseteq (0, \infty)$  we have

$$\begin{aligned} & \left| \int_{M_\varepsilon} \{p_\varepsilon(x, w, t) - q_\varepsilon(x, w, t)\} f(w) dM_\varepsilon(w) \right| \\ & \leq \max |f| \int_{M_\varepsilon} \{p_\varepsilon(x, w, t) - q_\varepsilon(x, w, t)\} dM_\varepsilon(w) \\ & = \max |f| \left\{ 1 - \int_{\Omega_\varepsilon} q_\varepsilon(x, y, t) dM(y) \right\} = \max |f| \int_M \{p(x, y, t) - q_\varepsilon(x, y, t)\} dM(y), \end{aligned}$$

since

$$\int_{M_\varepsilon} p(x, w, t) dM_\varepsilon(w) = 1 = \int_M p(x, y, t) dM(y).$$

Thus

$$\begin{aligned} & \left| \int_{M_\varepsilon} p_\varepsilon(x, w, t) f(w) dM_\varepsilon(w) - \int_M p(x, y, t) f(y) dM(y) \right| \\ & \leq 2 \max |f| \int_M \{p(x, y, t) - q_\varepsilon(x, y, t)\} dM(y) \end{aligned}$$

which goes to 0, uniformly in  $(x, t) \in K \times [a, b]$ , as  $\varepsilon \downarrow 0$ . Thus the lemma is proven.

To prove Theorem B we first reduce the problem to showing that for  $t$  bounded away from 0,  $p_\varepsilon(z, w, t)$  is uniformly bounded above *independent of  $\varepsilon$* .

Assume that this has in fact been accomplished. Then one has by the Sturm–Liouville expansion (cf. below) of  $p_\varepsilon, p$  that for any fixed  $t > 0$ ,

$$\begin{aligned} \Theta_\varepsilon(t) - \Theta(t) &= \int_{M_\varepsilon} p_\varepsilon(z, z, t) dM_\varepsilon(z) - \int_M p(x, x, t) dM(x) \\ &= \int_{C_\varepsilon} p_\varepsilon(z, z, t) dM_\varepsilon(z) - \int_{B_\varepsilon} p(x, x, t) dM(x) \\ &\quad + \int_{\Omega_\varepsilon} \{p_\varepsilon(x, x, t) - p(x, x, t)\} dM(x) \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

Indeed, the first two integrands are bounded and the volumes of  $C_\varepsilon, B_\varepsilon$  tend to 0. The convergence of the third integral follows from (13) and Lebesgue's dominated convergence theorem. Thus  $p_\varepsilon$  uniformly bounded independent of  $\varepsilon$  implies for  $t > 0$ ,

$$\lim \Theta_\varepsilon(t) = \Theta(t) \quad \text{as } \varepsilon \downarrow 0. \tag{14}$$

Finally assume there exists  $k \geq 1$  such that  $\liminf \sigma_k(\varepsilon) < \lambda_k$  as  $\varepsilon \rightarrow 0$ . Let  $\varepsilon_l$  be a sequence going to 0, with  $\sigma_k(\varepsilon_l) \rightarrow \sigma_k < \lambda_k$  as  $l \rightarrow \infty$ . Then by (14), (3) and Fatou's lemma we have

$$\begin{aligned} \Theta(t) &= \lim \Theta_{\varepsilon_l}(t) \geq \sum_{j=0}^{\infty} \liminf \exp(-\sigma_j(\varepsilon_l)t) = \sum_{j=0}^{\infty} \exp(-\limsup \sigma_j(\varepsilon_l)t) \\ &= e^{-\sigma_k t} + \sum_{j \neq k} \exp(-\limsup \sigma_j(\varepsilon_l)t) > \Theta(t) \end{aligned}$$

which implies a contradiction.

So to prove Theorem B we must bound  $p_\varepsilon$  above independently of  $\varepsilon$ . To do so we require some estimates of P. Li [8].

**DEFINITION 2.** Given a compact Riemannian manifold  $X$  of dimension  $n \geq 2$  we define the *Sobolev constant* of  $X$ ,  $c_0(X)$ , by

$$c_0(X) \equiv: \inf_f \left[ \left\{ \int_M |\nabla f|^2 \right\}^n / \inf_{\beta \in \mathbb{R}} \left\{ \int_M |f - \beta|^{n/(n-1)} \right\}^{n-1} \right]$$

where  $f$  ranges over the Sobolev space of functions with  $L^1$ -derivatives.

**LEMMA (P. Li) 3.** Let  $v = \text{vol}_n(X)$ ,  $c_0 = c_0(X)$ . Then there exist constants depending only on  $n$  such that

for any eigenfunction  $f$  with eigenvalue  $\tau \neq 0$  we have

$$\|f\|_\infty^2 \leq \text{const} \begin{cases} \|f\|_2^2 (\tau^{n/2}/c_0) \exp \{ \text{const} (c_0/v)^{2/n}/\tau \}, & n \geq 3 \\ \|f\|_2^2 (\tau^2 v/c_0^2) \exp \{ \text{const} c_0/\tau v \}, & n = 2 \end{cases} \quad (15)$$

for the  $k^{\text{th}}$  eigenvalue  $\tau_k$  of  $X$  we have

$$k \leq \text{const} \begin{cases} \{ \tau_k (v/c_0)^{2/n} \}^{n-1} & n \geq 3 \\ \{ \tau_k v/c_0 \}^2 & n = 2 \end{cases} \quad (16)$$

for all  $k = 1, 2, \dots$

Before turning to the proof of Theorem B we remark (as in [8]) that the argument of [2, Section 3], when applied to compact  $X$  without boundary, yields

$$c_1(X) \leq c_0(X) \leq 2c_1(X). \quad (17)$$

Also, by considering arbitrarily small geodesic disks, one has under *all* circumstances

$$c_1(X) \leq n^{n-1} \text{vol}_{n-1}(S). \quad (18)$$

We now prove Theorem B; recall that we must establish an upper bound on  $p_\varepsilon$  which is independent of  $\varepsilon$ . Assume (6). Then Lemma 1 implies

$$\lim \text{vol}_n(M_\varepsilon) = \text{vol}_n(M) \quad \text{as } \varepsilon \downarrow 0. \quad (19)$$

Also,  $\sigma_1(\varepsilon)$  is bounded away from zero, by either using Cheeger's inequality [4]

with (19) or by using (17), (19), and (16) for  $k=1$ . Thus we have that  $\{\sigma_1(\varepsilon), \text{vol}_n(M_\varepsilon), c_0(M_\varepsilon)\}$  are all restricted to a compact subset of  $(0, \infty)$ . Then there exist constants *independent of  $\varepsilon$*  for which Li's estimates now read as

$$\|f\|_\infty^2 \leq \text{const} \|f\|_2^2 \begin{cases} \tau^{n/2} & n \geq 3 \\ \tau^2 & n = 2 \end{cases} \quad (15')$$

$$\tau_k \geq \text{const} \begin{cases} k^{1/(n-1)} & n \geq 3 \\ k^{1/2} & n = 2. \end{cases} \quad (16')$$

Now fix  $t > 0$  and let  $\{\Phi_j(\varepsilon)\}$  be an orthonormal basis of  $L^2(M_\varepsilon)$  consisting of eigenfunctions corresponding respectively to  $\{\sigma_j(\varepsilon)\}$ . Then the eigenfunction expansion of  $p_\varepsilon$  is given by, and satisfies,

$$\begin{aligned} p_\varepsilon(z, w, t) &= \sum_{j=0}^{\infty} e^{-\sigma_j(\varepsilon)t} \Phi_j(\varepsilon)(z) \Phi_j(\varepsilon)(w) \\ &\leq \sum_{j=0}^{\infty} e^{-\sigma_j(\varepsilon)t} \|\Phi_j(\varepsilon)\|_\infty^2. \end{aligned}$$

We proceed with estimate for the case  $n=2$  as this is the situation in which we will construct our explicit examples (the argument for  $n > 2$  is similar). From (15') we have

$$p_\varepsilon(z, w, t) \leq \text{const} \left\{ 1 + \sum_{j=1}^{\infty} \sigma_j^2(\varepsilon) e^{-\sigma_j(\varepsilon)t} \right\}$$

with the constant independent of  $\varepsilon$ . Now (16') implies the existence of a positive integer  $J$ , independent of  $\varepsilon$ , such that for all  $j \geq J$  we have

$$\sigma_j^5(\varepsilon) e^{-\sigma_j(\varepsilon)t} \leq 1.$$

Then (16') implies that

$$\sum_{j=1}^{\infty} \sigma_j^2(\varepsilon) e^{-\sigma_j(\varepsilon)t} \leq \sum_{j < J} \sigma_j^2(\varepsilon) + \text{const} \sum_{j \geq J} j^{-3/2}$$

which is bounded above, independently of  $\varepsilon$ , by (3).

This concludes the proof of Theorem B.

### §3. The construction of $M_\epsilon$ for the main theorem

Let  $M$  be 2-dimensional and  $p \in \tilde{M}$ , i.e., either  $K(p) \neq 0$  or  $K$  vanishes identically on some neighborhood of  $p$ . To  $p$  we associate a number  $\alpha(p)$  with the following list of properties:

(i)  $\alpha$  will be less than the convexity radius of  $M$  (in particular, it is less than  $\frac{1}{2}$  the injectivity radius of  $M$ ). If  $K$ , the Gauss curvature of  $M$ , has maximum equal to  $\kappa$  then  $\alpha$  will be chosen so that it is also less than  $\pi/2\sqrt{\kappa}$ .

(ii) Set

$B(p; r) \equiv$ : metric disk about  $p$  of radius  $r$ .

Then we require that  $K$  either vanishes identically on  $B(p; \alpha)$  or never vanishes on  $B(p; \alpha)$ .

Should  $K$  never vanish on  $B(p; \alpha)$  then  $\alpha$  will be sufficiently small so that

$$\inf |K(q)| > \left(\frac{2}{3}\right) \sup |K(q)| \quad (20)$$

where  $q$  ranges over  $B(p; \alpha)$ .

(iii) Let  $dA$  denote the Riemannian element of area and, as in the introduction,  $A(\cdot)$  denote the area. Then we require that

$$A(B(p; \alpha)) < A(M)/8 \quad (21)$$

$$\iint_{B(p; \alpha)} K dA < \pi/2. \quad (22)$$

In particular for

$$K^+ \equiv: \max \{K, 0\}$$

we have

$$\iint_{B(p; \alpha)} K^+ dA < \pi/2. \quad (23)$$

*Remark 2.* In a moment we shall change the Riemannian metric in a compact subset of  $B(p; \alpha)$ , when  $K(p) \neq 0$ , such that the new Gauss curvature does not change sign. The Gauss–Bonnet formula implies that the left-hand side of (22) does *not* change, hence (23) remains valid in the new Riemannian metric.

If  $K(p) \neq 0$ , then for every  $\varepsilon \in (0, \alpha/2)$  we introduce a new Riemannian metric on  $B(p; \varepsilon)$ . The details will be given for  $K(p) > 0$ , as the case  $K < 0$  is similar. Let

$$k_1 \equiv: \inf K, \quad k_2 \equiv: \sup K$$

on  $B(p; \alpha)$ . Then  $k_1/k_2 > \frac{2}{3}$  implies that for all  $r$  satisfying  $0 < r < \alpha$  we have

$$\cos \sqrt{k_1} r < 2 \left\{ \frac{\sin \sqrt{k_2} r}{\sqrt{k_2} r} - \frac{1}{2} \right\}.$$

Introduce geodesic polar coordinates  $(r, \theta)$  about  $p$  and write the given Riemannian metric as

$$ds^2 = dr^2 + \eta^2(r, \theta) d\theta^2.$$

Then  $\eta$  satisfies Jacobi's equation

$$\frac{\partial^2}{\partial r^2} \eta + K\eta = 0$$

with initial data

$$\eta(0, \theta) = 0, \quad (\partial\eta/\partial r)(0, \theta) = 1.$$

Standard Sturmian arguments imply

$$\begin{aligned} \frac{\partial\eta}{\partial r}(r, \theta) &\leq \cos \sqrt{k_1} r < 2 \left\{ \frac{\sin \sqrt{k_2} r}{\sqrt{k_2} r} - \frac{1}{2} \right\} \\ &\leq 2 \left\{ \frac{\eta(r, \theta)}{r} - \frac{1}{2} \right\} = \frac{\eta(r, \theta) - r/2}{r/2} \end{aligned}$$

i.e.,

$$\frac{\partial\eta}{\partial r} \leq \frac{\eta - r/2}{r/2}. \quad (24)$$

Geometrically, (24) implies that for each fixed  $\theta$ , the tangent line to the curve  $y = \eta(x, \theta)$  (in the  $(x, y)$ -plane) at  $x = r$  intersects the line  $y = x$  for some  $x(r, \theta)$  satisfying  $r/2 < x(r, \theta) < r$ .

Given  $\varepsilon$  satisfying  $0 < \varepsilon < \alpha/2$ , set  $x_1 \equiv: x(3\varepsilon/4, \theta)$  and replace  $y = \eta(x, \theta)$  for

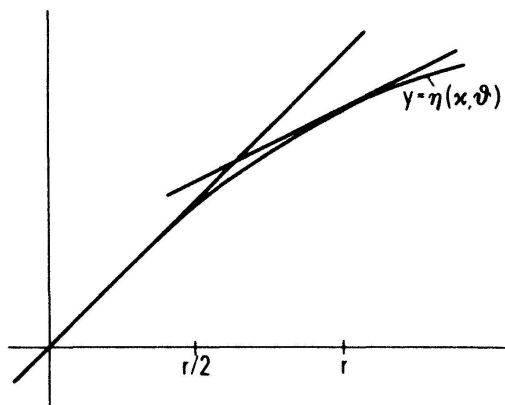


Figure 1

$0 \leq x \leq \varepsilon$  by  $y = \tilde{\eta}(x, \theta)$  where

$$\tilde{\eta}(x, \theta) = x \quad \text{for } 0 \leq x \leq x_1,$$

$\tilde{\eta}(x, \theta)$  is given by the tangent line to  $y = \eta(x, \theta)$  at  $x = 3\varepsilon/4$  for  $x_1 \leq x \leq 3\varepsilon/4$ ,  
 $\tilde{\eta}(x, \theta) = \eta(x, \theta)$  for  $3\varepsilon/4 \leq x \leq \alpha$ .

Now smooth  $\tilde{\eta}$  to  $\eta_\varepsilon$  which is a  $C^\infty$  function in  $(r, \theta)$  and which satisfies

$$\begin{aligned} \eta_\varepsilon(r, \theta) &= r & 0 \leq r \leq 5\varepsilon/16 \\ \eta_\varepsilon(r, \theta) &= \eta(r, \theta) & \varepsilon \leq x \leq \alpha \end{aligned} \tag{25}$$

$$\frac{\partial \eta_\varepsilon}{\partial r} > 0, \quad \frac{\partial^2 \eta_\varepsilon}{\partial r^2} \leq 0 \quad 0 \leq x \leq \alpha.$$

Finally replace

$$ds^2 = dr^2 + \eta^2(r, \theta) d\theta^2$$

on  $B(p; \alpha)$ , by

$$ds_\varepsilon^2 = dr^2 + \eta_\varepsilon^2(r, \theta) d\theta^2.$$

Then the new metric is flat on the geodesic disk of radius  $5\varepsilon/16$ , has non-negative Gauss curvature on  $B(p; \alpha)$ , and agrees with the original metric on  $B(p; \alpha) \cap \overline{\Omega}_\varepsilon$ . One sees easily that the smoothing may be chosen so that (21) can be replaced by

$$A(B(p; \alpha)) < A(M)/6. \tag{21'}$$

A similar argument can be carried out when  $K(p_1) < 0$ .

We are now ready to define  $M_\epsilon$ . Given  $p_1, p_2 \in \tilde{M}$  fix  $\alpha$  to be less than  $\alpha(p_1)$ ,  $\alpha(p_2)$  and less than  $\frac{1}{2}$  the distance from  $p_1$  to  $p_2$ . For each  $i = 1, 2$ , if  $K(p_i) = 0$  leave well enough alone. If  $K(p_i) \neq 0$  then for every  $\epsilon < \alpha/2$  introduce the change of metric in  $B(p_i; \epsilon)$  just described. Call the new Riemannian manifold  $M_\epsilon^*$ .

Now attach the tube

$$T \equiv: [-\epsilon/4, \epsilon/4] \times S_{\epsilon/4}$$

to  $M_\epsilon^*$  “at right angles” to the geodesic circles about  $p_1, p_2$  of radius  $\epsilon/4$ , identifying  $\{-\epsilon/4\} \times S_{\epsilon/4}$  with the geodesic circle about  $p_1$ , and  $\{\epsilon/4\} \times S_{\epsilon/4}$  with that about  $p_2$ , of radius  $\epsilon/4$ . Put differently, in the polar coordinate system about each  $p_i$ , replace

$$\eta_\epsilon(r, \theta) = r \quad 0 \leq r \leq 5\epsilon/16$$

with

$$\zeta_\epsilon(r, \theta) = \begin{cases} \epsilon/4 & 0 \leq r \leq \epsilon/4 \\ r & \epsilon/4 \leq r \leq 5\epsilon/16 \end{cases}$$

and identify the two circles  $\{0\} \times S_{\epsilon/4}$ .

The resulting manifold with “creased” Riemannian metric will be our  $M_\epsilon$  and we shall estimate  $C_1(M_\epsilon)$  for this “creased” metric from below. Once we, in fact, verify the existence of  $c$  for which (6) is valid for all  $\epsilon$ , it is easy to smooth the “crease,” with non-positive curvature near the crease, such that (6) is valid with  $c$  replaced by  $c/2$  for all  $\epsilon$ .

#### §4. Estimating the isoperimetric constant

LEMMA (F. Fiala [6, p. 336; 11, p. 12]) 3. *Let  $M$  be a complete Riemannian surface with Riemannian measure  $dA$ , Gauss curvature  $K$ ,  $K^+ = \max\{K, 0\}$ , and  $D$  a simply connected domain in  $M$  of area  $A$  with smooth boundary of length  $L$ . Then*

$$\iint_D K^+ dA < 2\pi$$

implies

$$L^2 - 4\pi A + 2A \iint_D K^+ dA \geq 0.$$

COROLLARY 1. *In the above,*

$$\iint_D K^+ dA < \pi$$

*implies*

$$L^2 \geq 2\pi A.$$

LEMMA (S. T. Yau [4, p. 489]) 4. *For a compact  $n$ -dimensional Riemannian manifold  $M$ , to evaluate  $c_1(M)$  it suffices to let  $Y$  range over those compact  $(n-1)$ -manifolds which separate  $M$  into connected open submanifolds  $X_1, X_2$ .*

*Remark 3.* In [14] Yau proves this fact for Cheeger's isoperimetric constant. However his induction argument and the Minkowski inequality (Triangle inequality if  $\dim M = 2$ ) immediately yields the above lemma.

In what follows  $L_\varepsilon(\cdot)$ ,  $dA_\varepsilon$ ,  $A_\varepsilon(\cdot)$ ,  $K_\varepsilon$  will denote length, area element, area, and Gauss curvature of  $M_\varepsilon$ . We also write  $K_\varepsilon^+ = \max\{K_\varepsilon, 0\}$ .

To start with our estimate of  $c_1(M_\varepsilon)$  from below, let  $\gamma$  be a compact 1-manifold imbedded in  $\Omega_\varepsilon$ , separating  $M_\varepsilon$  into  $M_1, M_2$ , and assume  $C_\varepsilon \subseteq M_1$ . Set  $M_1^* = (M_1 - C_\varepsilon) \cup B_\varepsilon$ , where  $B_\varepsilon$  is, as originally defined, the union of the open geodesic disks of radius  $\varepsilon$  in  $M$  about  $p_1, p_2$ . One easily checks that  $A_\varepsilon(C_\varepsilon)/A(B_\varepsilon)$  is bounded away from  $0, +\infty$  independently of  $\varepsilon$ . Thus

$$\begin{aligned} L_\varepsilon^2(\gamma)/\min(A_\varepsilon(M_1), A_\varepsilon(M_2)) &\geq \text{const } L^2(\gamma)/\min(A(M_1^*), A(M_2)) \\ &\geq \text{const } c_1(M) > 0 \end{aligned}$$

which is independent of  $\varepsilon$ .

Also if  $\gamma$  is any compact imbedded 1-manifold of length  $\geq \alpha$  separating  $M_\varepsilon$  into  $M_1, M_2$ , then

$$L_\varepsilon^2(\gamma)/\min(A_\varepsilon(M_1), A_\varepsilon(M_2)) \geq \alpha^2/A_\varepsilon(M_\varepsilon) \geq \text{const} > 0$$

by (19).

Thus our task is to estimate

$$L_\varepsilon^2(\gamma)/\min(A_\varepsilon(M_1), A_\varepsilon(M_2))$$

where  $\gamma$  ranges over compact imbedded 1-manifolds separating  $M_\epsilon$  into connected  $M_1, M_2$  and such that

$$L_\epsilon(\gamma) < \alpha, \quad \gamma \cap \text{int}(C_\epsilon) \neq \emptyset.$$

Since  $\epsilon < \alpha/2$  we immediately have

$$\gamma \subseteq M_\epsilon - \overline{\Omega_\alpha}.$$

We will always have one of the domains, say  $M_1$ , contained in  $M_\epsilon - \overline{\Omega_\alpha}$  and (19), (21') allow us to assume  $A_\epsilon(M_1) < A_\epsilon(M_\epsilon)/2$ ; so we are always estimating

$$L_\epsilon^2(\gamma)/A_\epsilon(M_1)$$

from below.

Since  $M_\epsilon - \overline{\Omega_\alpha}$  is topologically a cylinder, we have that  $\gamma$  is an imbedded circle bounding a disk  $M_1 \subseteq M_\epsilon - \overline{\Omega_\alpha}$  or  $\gamma$  is a pair of imbedded circles bounding a cylinder  $M_1 \subseteq M_\epsilon - \overline{\Omega_\alpha}$ . In the first case, then by smoothing the "crease" with non-positive curvature we would have (from Remark 2)

$$\int_{M_\epsilon - \overline{\Omega_\alpha}} K_\epsilon^+ dA_\epsilon < \pi$$

and therefore

$$L_\epsilon^2(\gamma) \geq 2\pi A_\epsilon(M_1)$$

in the approximating metric, which implies the inequality for our "creased" metric. So we need only consider the second case.

First let  $d(\cdot)$  denote distance in  $M_\epsilon^*$  and define for  $i = 1, 2$

$$\Gamma_i(t) \equiv: \{q \in M_\epsilon^* : d(q, p_i) = t\}$$

$$W_i \equiv: \{q \in M_\epsilon^* : \epsilon/4 \leq d(q, p_i) < \alpha\}.$$

We are now considering  $\gamma = \gamma_1 \cup \gamma_2$  where  $\gamma_1, \gamma_2$  are imbedded circles bounding a cylinder  $M_1$  in  $M_\epsilon - \overline{\Omega_\alpha}$ . We think of  $\gamma_i$  as "closest" to  $\Gamma_i(\alpha)$ . First we shall assume that  $\gamma_i$  does not cross  $\Gamma_i(\epsilon/4)$  for either  $i = 1, 2$ , and then reduce the general case (viz., where at least one  $\gamma_i$  crosses  $\Gamma_i(\epsilon/4)$ ) to this one. The assumption that neither  $\gamma_i$  crosses  $\Gamma_i(\epsilon/4)$  involves three essential possibilities: (i)  $\gamma_1, \gamma_2 \subseteq T$ , (ii)  $\gamma_1 \subseteq W_1, \gamma_2 \subseteq W_1 \cup T$ , (iii)  $\gamma_1 \subseteq W_1, \gamma_2 \subseteq W_2$ .

(i) If  $\gamma_1, \gamma_2 \subseteq T$  then  $M_1 \subseteq T$ , and  $L_\varepsilon(\gamma_i) \geq \pi\varepsilon/2$ ,  $A_\varepsilon(M_1) \leq \pi\varepsilon^2/4$ , which implies

$$L_\varepsilon^2(\gamma) \geq 4\pi A_\varepsilon(M_1).$$

(ii) If  $\gamma_1 \subseteq W_1$ ,  $\gamma_2 \subseteq W_1 \cup T$  then  $M_1 \subseteq T \cup (M_1 \cap W_1)$ , which implies

$$A_\varepsilon(M_1) \leq \pi\varepsilon^2/4 + A_\varepsilon(M_1 \cap W_1).$$

Now think of  $\gamma_1$  as bounding a disk  $D \subseteq B(p_1; \alpha) \subseteq M_\varepsilon^*$ . Then

$$A_\varepsilon(D) = \pi(\varepsilon/4)^2 + A_\varepsilon(M_1 \cap W_1) \geq A_\varepsilon(M_1)/4,$$

and by Remark 2 and Fiala's inequality we therefore have

$$L_\varepsilon^2(\gamma) \geq L_\varepsilon^2(\gamma_1) \geq 2\pi A_\varepsilon(D) \geq (\pi/2)A_\varepsilon(M_1).$$

This argument, of course, covers the possibility:  $\gamma_2 \subseteq W_2$ ,  $\gamma_1 \subseteq W_2 \cup T$ .

(iii) If  $\gamma_1 \subseteq W_1$ ,  $\gamma_2 \subseteq W_2$  then our argument is similar to the one just given. Let  $D_i$  correspond to  $\gamma_i$  as  $D$  corresponded to  $\gamma_1$  in (ii). Then we have

$$\begin{aligned} L_\varepsilon^2(\gamma) \sum_i L_\varepsilon^2(\gamma_i) &\geq \sum_i 2\pi A_\varepsilon(D_i) \\ &= 2\pi \sum_i \{\pi\varepsilon^2/16 + A_\varepsilon(M_1 \cap W_i)\} \\ &> \sum_i \{\pi\varepsilon^2/8 + A_\varepsilon(M_1 \cap W_i)\} = \pi A_\varepsilon(M_1). \end{aligned}$$

In summary, when neither  $\gamma_i$  crosses  $\Gamma_i(\varepsilon/4)$  we have

$$L_\varepsilon^2(\gamma) \geq (\pi/2)A_\varepsilon(M_1). \quad (26)$$

For the general case, we shall assume  $\gamma_1$  crosses  $\Gamma_1(\varepsilon/4)$  transversally with an even number of intersections, and show that we can replace  $\gamma_1$  with  $\tilde{\gamma}_1$  having fewer intersections (therefore, ultimately no intersections), shorter length, and enclosing with  $\gamma_2$  larger area. Thus (26) will remain valid in the general case.

So we now have  $\gamma_1$  crossing  $\Gamma_1(\varepsilon/4)$  transversally. We shall think of  $M_\varepsilon - \overline{\Omega_\alpha}$  as the cylinder  $R^2 - \{0\}$ , with  $\gamma_1$  and  $\Gamma_1(\varepsilon/4)$  winding about 0 once, let

$D \equiv$ : component of  $R^2 - \{\gamma_1\}$  not containing  $M_1$ ,

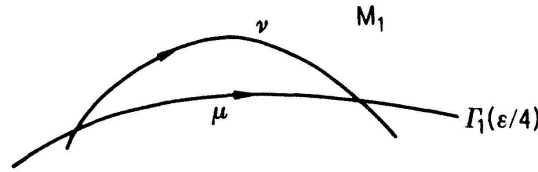


Figure 2

assume for convenience that  $D$  contains  $0$ , and let

$$\{D_1, \dots, D_k\} \equiv: \text{components of } D - \Gamma_1(\epsilon/4).$$

We say that a component  $D_l$  is *simple* if its boundary consists of the union of 2 smooth arcs  $\nu$  and  $\mu$ , with  $\nu$  part of  $\gamma_1$  and  $\mu$  part of  $\Gamma_1(\epsilon/4)$ .

Given a simple  $D_l$  we have that either  $D_l \subseteq W_1$  or  $D_l \subseteq W_2 \cup T$ , with  $\nu \subseteq W_1$  or  $\nu \subseteq W_2 \cup T$  respectively. However, in either case

$$L(\nu) > L(\mu)$$

– note that we are using here the hypothesis that  $\alpha$  was picked to be less than the convexity radius of  $M$ . Then replace  $\gamma_1$  by  $\tilde{\gamma}_1$  by first replacing  $\nu$  by  $\mu$ , then sliding  $\mu$  a drop to the side of  $\Gamma_1(\epsilon/4)$  opposite to  $D_l$ , and, finally, smoothing the corners. Define  $\tilde{\gamma} = \tilde{\gamma}_1 \cup \gamma_2$ . Then  $L_\epsilon(\tilde{\gamma}) \leq L_\epsilon(\gamma)$ , and  $\tilde{M}_1$ , the domain in  $M_\epsilon - \overline{\Omega}_\alpha$  bounded by  $\tilde{\gamma}$ , contains  $M_1 \cup D_l$  which implies  $A_\epsilon(\tilde{M}_1) \geq A_\epsilon(M_1)$ . Thus

$$L_\epsilon^2(\gamma)/A_\epsilon(M_1) \geq L_\epsilon^2(\tilde{\gamma})/A_\epsilon(\tilde{M}_1).$$

The curve  $\tilde{\gamma}$  has the same properties as  $\gamma$ , so we may repeat the argument just given until we are left with a curve not intersecting  $\Gamma_1(\epsilon/4)$ . In this last case we already have the estimate (26). Thus the last thing for us to verify is that given  $\gamma$ , a simple  $D_l$  exists.

**LEMMA 5.** *Let  $\Gamma, \Lambda$  be two simply closed smooth curves in  $\mathbb{R}^2 - \{0\}$  which wind about the origin once and which meet each other transversally. Let  $G$  be the component of  $\mathbb{R}^2 - \Lambda$  containing  $0$ , and let  $\{G_1, \dots, G_r\}$  be the components of  $G - \Gamma$ . Then there exists a component  $G_s$  whose boundary consists of two smooth arcs; one a part of  $\Gamma$  and one a part of  $\Lambda$ .*

*Proof.* We can assume  $\Gamma$  is the unit circle which  $\Omega$  divides into a finite number of arcs alternately belonging to  $G$  and  $\mathbb{R}^2 - (G \cup \Lambda)$ .

Let  $G_0$  be the component of  $G - \Gamma$  containing  $0$ , and consider the segments

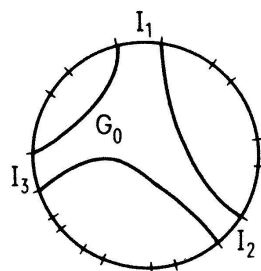


Figure 3

$I_1, \dots, I_{l_1}$  of  $\overline{G_0} \cap \Gamma$ . (Of course if  $l_1 = 1$  then  $G_0$  is simple. So assume otherwise.) To each  $I_{j_1}$ ,  $j_1 = 1, \dots, l_1$  associate the component  $G_{j_1}$  of  $G - \Gamma$  distinct from  $G_0$  having  $I_{j_1}$  as part of its boundary.  $G_1, \dots, G_{l_1}$  are all distinct for otherwise  $\Lambda$  would not be simple by the Jordan curve theorem.

If there exists  $G_{j_1}$  having only  $I_{j_1}$  for the intersection of  $\partial G_{j_1}$  with  $\Gamma$  then we are done, for then  $G_{j_1}$  is then simple. So assume the opposite. Then to each  $j_1 = 1, \dots, l_1$  associate the segments  $I_{j_1,1}, \dots, J_{j_1,l_2}$  of  $\overline{G_{j_1}} \cap \Gamma - I_{j_1}$  and to each segment  $I_{j_1,j_2}$  associate the component of  $G - \Gamma$  distinct from  $G_{j_1}$ ,  $G_{j_1,j_2}$ , having  $I_{j_1,j_2}$  as part of its boundary. Again, the collection  $\{G_{j_1,j_2}\}$  are all distinct.

By continuing this process, if necessary, we exhaust all the arcs of  $G - \Gamma$  and the process stops. But then every component of the last step is simple.

This concludes the proof of Lemma 5 and, with it, the proof of the main theorem.

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