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Schwarzian derivatives, the Poincaré metric and the kernel function

A. F. BEARDON and F. W. GEHRING [1]

Dedicated to the memory of Professor Z. Nehari

1. Introduction.

Throughout this paper U will denote the open unit disk in the complex plane $\mathbf{C}$ and D a subdomain of $\mathbf{C}$ with U as its universal covering surface. There are many results concerned with the interdependence of the Schwarzian derivative S_f of a function f analytic in D, the hyperbolic or Poincaré metric ρ_D in D and the reduced Bergman kernel K_D of D. In this paper we shall give a complete account of the relationships which exist between these quantitites.

If f is analytic in D, the *Schwarzian derivative* of f is defined by

$$S_f(z)=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

It is known that $S_f=0$ throughout D if and only if f is a Möbius transformation in D. Moreover, the univalence of f is in some sense related to S_f being small.

As U is the universal covering surface of D, there is an analytic projection p of U onto D. The hyperbolic metric

$$\rho_U(z)|dz|=\frac{|dz|}{1-|z|^2}$$

in U projects to the *hyperbolic metric* $\rho_D(z)|dz|$ in D where

$$\rho_D(p(z))|p'(z)|=\rho_U(z). \tag{1.1}$$

Note that this defines ρ_D in D independently of p. It is known that if $D_1\subset D$, then

$$\rho_D(z)\leq\rho_{D_1}(z). \tag{1.2}$$

The reader is referred to [4] for further details.

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If f is analytic in D, the Dirichlet integral of f is

$$I(f)=\iint_D |f'(z)|^2\,dx\,dy.$$

We assume for the moment that D supports non-constant analytic functions with finite Dirichlet integral (that is, D is not in O_{AD}) and set

$$L_2(D)=L_{2,s}(D)=\{f' : f \text{ analytic in } D,\ I(f)<\infty\}.$$

Using the inner product

$$(f, g)=\iint_D f(z)\overline{g(z)}\,dx\,dy$$

on $L_2(D)$ we construct a complete orthonormal basis $\{\varphi_n\}$ and define

$$K_D(z, \bar{w})=K_{D,s}(z, \bar{w})=\sum_{n=1}^{\infty}\varphi_n(z)\overline{\varphi_n(w)}.$$

This is the *reduced kernel* for D and it does not depend on the choice of the φ_n. Again if $D_1\subset D$, then

$$K_D(z, \bar{z})\leq K_{D_1}(z, \bar{z}). \tag{1.3}$$

See [5] and [7] for further details.

We come now to the known results. First, Kraus proved that if f is analytic and univalent in U, then

$$|S_f(z)|\leq 6\rho_U(z)^2.$$

Lehto extended this by showing that if f is analytic and univalent in a simply connected domain D, then

$$|S_f(z)|\leq 12\rho_D(z)^2.$$

Finally Gehring observed that if f is analytic and univalent in an arbitrary domain D, then

$$|S_f(z)|\leq 6\,\mathrm{dist}\,(z, \partial D)^{-2}. \tag{1.4}$$

The constants 6, 12 and 6 here are all best possible; see [9] for references and further details. Note, however, that

$$\rho_D(z)\,\mathrm{dist}\,(z, \partial D) \leq 1. \tag{1.5}$$

We shall prove the following result and thereby complete this aspect of the problem.

THEOREM 1. *Let f be analytic and univalent in any domain D. Then*

$$|S_f(z)| \leq 12\rho_D(z)^2$$

for all z in D.

Certain relationships between S_f and K_D and between K_D and ρ_D are known. We have, for example, the following result due to Bergman and Schiffer. (See p. 215, the remark on p. 234, and p. 239 in [6].)

THEOREM 2. *Let f be analytic and univalent in a domain D which is not in O_{AD}. Then*

$$|S_f(z)| \leq 12\pi K_D(z, \bar{z})$$

for all z in D.

We shall establish the following result in Section 2.

THEOREM 3. *Let D be any domain which is not in O_{AD}. Then*

$$K_D(z, \bar{z}) \leq \frac{1}{\pi}\,\rho_D(z)^2$$

for all z in D.

If D is not in O_{AD}, then Theorem 1 follows from Theorems 2 and 3. We obtain the general case in Section 3 by means of a limiting argument. We also show there that the constants 12, 12π and $1/\pi$ in Theorems 1, 2 and 3 are all best possible if we restrict D to the class of domains of any fixed connectivity n.

The inequalities in (1.4) and in Theorems 1 and 2 are necessary conditions for the univalence of a function f analytic in D. We conclude this paper by considering in Sections 5 and 6 the domains D for which similar inequalities are sufficient to ensure the univalence of f.

2. Proof of Theorem 3.

The proof is based on the solution of extremal problems and is essentially contained in the work of Ahlfors and Beurling [3]. Explicitly, the proof consists of verifying the following sequence of equalities and inequalities for each fixed point z in D:

$$\begin{aligned}\pi K_D(z, \bar{z}) &= \pi(\inf\{I(g) : |g(z)| = 1\})^{-1}\\ &= \sup\{|f'(z)|^2 : I(f) \leq \pi\}\\ &= (\sup\{|f'(z)| : I(f) \leq \pi\})^2\\ &\leq (\sup\{|f'(z)| : |f| \leq 1\})^2\\ &\leq (\sup\{|f'(z)| : |f| \leq 1, f \text{ multiple valued}\})^2\\ &= \rho D(z)^2.\end{aligned}$$

In this sequence, all functions are supposed to be analytic in D with the meaning of the penultimate line to be clarified shortly. Of course, it is only necessary to prove inequalities at each stage. However, the equalities are of independent interest.

The verification of the above sequence now follows. The first line is well known; see, for example pp. 21–22 in [5] or p. 43 in [7].

To verify the second line, observe first that the map of $\{g : |g'(z)| = 1\}$ into $\{f : I(f) \leq \pi\}$ defined by

$$g \mapsto f = \left(\frac{\pi}{I(g)}\right)^{1/2} g$$

yields

$$\pi(\inf\{I(g) : |g'(z)| = 1\})^{-1} \leq \sup\{|f'(z)|^2 : I(f) \leq \pi\};$$

this is all we need for the proof. Next this inequality shows that

$$\{f : I(f) \leq \pi, f'(z) \neq 0\} \neq \emptyset$$

and the map of $\{f : I(f) \leq \pi,\ f'(z) \neq 0\}$ into $\{g : g(z)| = 1\}$ defined by

$$f \mapsto g = \frac{f}{f'(z)}$$

yields the opposite inequality. Observe now that the third line is trivially true.

The first inequality in the fourth line is contained in [3]; it is the inequality (11) on p. 105 of [3] and involves only the two largest terms. The fifth and sixth lines of the sequence are also suggested in the last paragraph on p. 105 of [3]. We shall include the details here.

By

$$\mathscr{C} = \mathscr{C}(z) = \{f : |f| \leq 1, f \text{ multiple valued}\}$$

we mean the class of functions f which are analytic in a neighborhood of the point z and which can be continued along each path γ in D so that $|f| \leq 1$. Let p be an analytic projection of U onto D chosen so that $p(0) = z$. Then $\mathscr{C}$ consists of all functions of the form $F \circ p^{-1}$ where F is analytic with $|F| \leq 1$ in U and p^{-1} is the local inverse of p defined near z with $p^{-1}(z) = 0$. Since $\mathscr{C}$ contains the class of functions f in line 4, it remains only to verify the equality in line 6.

Choose f in $\mathscr{C}$. Then $f = F \circ p^{-1}$ in a neighborhood of z where F is analytic with $|F| \leq 1$ in U. The Schwarz Lemma gives

$$|f'(z)||p'(0)| = |F'(0)| \leq 1 = \rho_U(0),$$

and as $\rho_D(z)|p'(0)| = \rho_U(0)$, we find that $|f'(z)| \leq \rho_D(z)$.
This proves that

$$\sup\{|f'(z)| : f \in \mathscr{C}\} \leq \rho_D(z). \tag{2.1}$$

Now p^{-1} is in $\mathscr{C}$. Hence if we take $f = p^{-1}$ in the above argument, we find that $|f'(z)| = \rho_D(z)$. Thus equality holds in (2.1) and the proof of Theorem 3 is complete.

An alternative proof is available. The curvature of the metric $(\pi K_D(z, \bar{z}))^{1/2}$ $|dz|$ is at most -4 (see [12] and, for the annulus, [13]). Theorem 3 now follows from, for example, p. 16 in [2].

3. Proof of Theorem 1.

We begin with the following result on the hyperbolic metric (see Theorem 1, [10]).

LEMMA 1. *Let $\{D_n\}$ be a nondecreasing sequence of domains whose union is D. Then*

$$\lim_{n\to\infty} \rho_{D_n}(z) = \rho_D(z)$$

for all z in D.

Now given any domain D it is clearly possible to define a nondecreasing sequence of domains D_n which are not in O_{AD} and whose union is D. Suppose f is analytic and univalent in D. Then for each n, Theorems 2 and 3 give

$$|S_f(z)| \le 12\pi K_{D_n}(z, \bar{z}) \le 12\rho_{D_n}(z)^2,$$

and hence by Lemma 1,

$$|S_f(z)| \le 12 \lim_{n\to\infty} \rho_{D_n}(z)^2 = 12\rho_D(z)^2$$

for all z in D. This completes the proof of Theorem 1.

Remark. The constants 12, 12π and $1/\pi$ in Theorems 1, 2 and 3 are all best possible for domains of each fixed connectivity n.

Proof. Fix n and suppose that $|S_f(z)| \le b_n \rho_D(z)^2$ for all n-tuply connected domains D and all f analytic and univalent in D. Next let L denote the nonpositive half of the real axis and for $0<r<1$ let B denote the closed disk of radius r with center at the origin. Then $D_0 = \mathbf{C} - \mathrm{L}$ and $D_1 = D_0 - B$ are simply connected domains, $f(z) = z^{1/2} - z^{-1/2}$ has an analytic branch in D_0 and

$$\lim_{r\to 0} |S_f(1)| \rho_{D_1}(1)^{-2} = |S_f(1)| \rho_{D_0}(1)^{-2} = 12$$

by Lemma 1. For $0<r<1$ let D denote any n-tuply connected domain bounded by L and by $(n-1)$ disjoint continua in B. Then $D_1 \subset D \subset D_0$ and

$$b_n \ge \limsup_{r\to 0} |S_f(1)|\, \rho_D(1)^{-2} \ge \lim_{r\to 0} |S_f(1)|\, \rho_{D_1}(1)^{-2} = 12$$

by (1.2). Thus Theorem 1 is sharp for all n-tuply connected domains. It then follows that Theorems 2 and 3 are also sharp for domains of connectivity n.

4. Geometric considerations.

We shall need an explicit formula for K_D in the case of an annulus. If

$$D_1 = \{z : \frac{1}{r} < |z| < r\}, \quad 1 < r < \infty,$$

then (see [7], p. 55)

$$\pi K_{D_1}(z, \bar{z}) = |z|^{-2} \sum_{n=1}^{\infty} n \frac{\cosh(2n \log|z|)}{\sinh(2n \log r)}. \tag{4.1}$$

Note that we also have (see [4])

$$\rho_{D_1}(z) = \frac{\pi}{4|z| \log r} \sec\left(\frac{\pi \log|z|}{2 \log r}\right) \tag{4.2}$$

$$= |z|^{-1}\left(\int_0^{\infty} t \frac{\cosh(2t \log|z|)}{\sinh(2t \log r)}\, dt\right)^{1/2}. \tag{4.3}$$

We shall not need the next result, a localisation principle which reduces the problem of estimating ρ_D for a general domain to the case when D is a disk or annulus; nevertheless we feel that it may be of independent interest and we include it for this reason.

LEMMA 2. *Each point z of D is contained in a subdomain D_1 which is either an annulus or a disk and for which*

$$\tfrac{1}{12}\rho_{D_1}(z) < \rho_D(z) \leq \rho_{D_1}(z).$$

Proof. Fix z in D and let B denote the largest open disk contained in D with center z; clearly B has radius $d = \operatorname{dist}(z, \partial D)$. Next choose a point w in ∂D with $|w - z| = d$ and let r be the largest positive number for which the annulus

$$A = \left\{\zeta : \frac{1}{r} < \frac{|\zeta - w|}{d} < r\right\}$$

lies in D. When no such A exists we take $r = 1$. In our present notation, the sharper form of Theorem 1 in [4] (preceding (2.4) in [4]) yields the following counterpart to inequality (1.5):

$$1 \leq 2(5 + \log r)\, d\rho_D(z). \tag{4.4}$$

Now suppose that $\log r \leq \pi/4$. As $\rho_B(z) = 1/d$, we find that

$$1 \leq \frac{\rho_B(z)}{\rho_D(z)} \leq 2\left(5 + \frac{\pi}{4}\right) < 12$$

from (1.2) and (4.4). Suppose next that $\log r > \pi/4$. Then we can apply (4.2) to obtain

$$\rho_A(z) = \frac{\pi}{4(\log r)d},$$

and we find that

$$1 \le \frac{\rho_A(z)}{\rho_D(z)} \le \frac{\pi}{2}\left(\frac{5}{\log r} + 1\right) < 12$$

from (1.2) and (4.4). The conclusion in Lemma 2 now follows by taking $D_1 = B$ when $\log r \le \pi/4$ and $D_1 = A$ when $\log r > \pi/4$.

As an illustration, we use Lemma 2 and (1.3) and obtain

$$\frac{K_D(z, \bar{z})}{144\rho_D(z)^2} \leqslant \frac{K_{D_1}(z, \bar{z})}{\rho_{D_1}(z)^2}.$$

If D_1 is a disk, the right hand side is $1/\pi$ ([5], p. 34): if D_1 is an annulus, (4.1) and (4.3) imply that the right hand side is at most $2/\pi$ (trivially) or $1/\pi$ (by contour integration).

5. Criteria for univalence.

Suppose that f is analytic in a domain D which is not in O_{AD}. Then Theorem 2, Theorem 1 and (1.4) yield respectively the following necessary conditions for f to be univalent in D:

$$|S_f(z)| \le 12\pi K_D(z, \bar{z}),$$
$$|S_f(z)| \le 12\rho_D(z)^2,$$
$$|S_f(z)| \le 6 \operatorname{dist}(z, \partial D)^{-2}.$$

Moreover the constants 12π, 12 and 6 here are all best possible. It is thus of interest to ask under what circumstances do there exist constants a, b and c such that one or more of the inequalities

$$\left.\begin{aligned} &|S_f(z)| \le aK_D(z, \bar{z}), \\ &|S_f(z)| \le b\rho_D(z)^2, \\ &|S_f(z)| \le c \operatorname{dist}(z, \partial D)^{-2} \end{aligned}\right\} \tag{5.1}$$

is a sufficient condition for f to be univalent in D.

A complete answer to this question can be given when D is simply connected. In this case, $K_D(z, \bar{z}) = \frac{1}{\pi}\rho_D(z)^2$ by p. 34 in [5] while

$$\tfrac{1}{4}\operatorname{dist}(z, \partial D)^{-1} \le \rho_D(z) \le \operatorname{dist}(z, \partial D)^{-1}$$

by Koebe's theorem and (1.5). Hence

$$\frac{1}{16\pi}\operatorname{dist}(z, \partial D)^{-2} \le K_D(z, \bar{z}) = \frac{1}{\pi}\rho_D(z)^2 \le \frac{1}{\pi}\operatorname{dist}(z, \partial D)^{-2}, \tag{5.2}$$

and up to multiplicative constants the conditions in (5.1) are equivalent. Thus the following result characterizes the simply connected domains for which any of the conditions in (5.1) implies univalence. (See Theorem 2 in [1] and Theorem 5 in [9].)

THEOREM 4. *Let D be a simply connected domain. If ∂D is a K-quasiconformal circle, then there exists a positive constant b depending only on K such that the inequality*

$$|S_f(z)| \le b\rho_D(z)^2 \tag{5.3}$$

implies univalence for each f analytic in D. Conversely if there exists a positive constant b such that (5.3) *implies univalence for each f analytic in D, then ∂D is a K-quasiconformal circle where K depends only on b.*

Here a set E in the extended plane $\bar{\mathbf{C}}$ is said to be a K-quasiconformal circle if there exists a K-quasiconformal mapping of $\bar{\mathbf{C}}$ onto itself which maps the unit circle onto E.

The situation is more complicated when D is multiply connected. In this case (5.2) is replaced by the chain of inequalities

$$K_D(z, \bar{z}) \le \frac{1}{\pi}\rho_D(z)^2 \le \frac{1}{\pi}\operatorname{dist}(z, \partial D)^{-2};$$

here we assume that D is not in O_{AD}. Moreover, there is no counterpart for the first inequality in (5.2). For example, if D is the unit disk U punctured at the origin, then

$$K_D(z, \bar{z})\operatorname{dist}(z, \partial D)^2 \le \frac{1}{\pi}|z|^2(1-|z|^2)^{-2} \to 0$$

as $z \to 0$. Thus the conditions in (5.1) are not equivalent with the first implying the second, the second implying the third and the third not necessarily implying the first.

B. Osgood has recently established the following univalence criterion for multiply connected domains in [11].

THEOREM 5. *Suppose that the components of ∂D consist of a finite number of points and K-quasiconformal circles. Then there exists a positive constant c such that the inequality*

$$|S_f(z)| \leq c \operatorname{dist}(z, \partial D)^{-2} \tag{5.4}$$

implies univalence for each f analytic in D.

Theorem 5 is an analogue for multiply connected domains of the first half of Theorem 4. Moreover since (5.4) is the least restrictive condition in (5.1), we see as a consequence that the other two conditions in (5.1) also imply univalence for f analytic in D whenever D is bounded by a finite number of points and K-quasiconformal circles.

Theorem 5 contains two hypotheses about the domain D: first, that each component of ∂D is a point or a K-quasiconformal circle and second, that ∂D has only finitely many components. We conclude this paper with an analogue for the second half of Theorem 4 which shows that each of the above hypotheses is essential. (Cf. Theorem 6 in [9].)

THEOREM 6. *Let D be a domain which is not in O_{AD} and suppose there exists a positive constant a such that the inequality*

$$|S_f(z)| \leq aK_D(z, \bar{z}) \tag{5.5}$$

implies univalence for each f analytic in D. Then each component of ∂D is a point or a K-quasiconformal circle where K depends only on a. In addition, D does not separate the boundary components of any circular ring domain with conformal modulus less than a positive constant m which depends only on a.

Here D is said to *separate* the boundary components of a ring domain R if $D \subset R$ and if distinct components of ∂R cannot be joined in $\bar{\mathbf{C}} - D$.

We postpone the proof of Theorem 6 until Section 6 in order to give first some consequences of this result.

Suppose that D is not in O_{AD}. Since (5.5) is the most restrictive condition in (5.1), we see that the conclusions of Theorem 6 hold if either of the other two conditions in (5.1) imply univalence for f analytic in D. These conclusions hold trivially if D is in O_{AD}, since then each component of ∂D must be a point. (See p. 112 in [3].)

If we combine the above remarks with those following the statement of Theorem 5, we obtain the following characterization of the finitely connected domains for which any of the conditions in (5.1) implies univalence.

COROLLARY 1. *Let D be a finitely connected domain. Then there exists a positive constant a, b or c such that the corresponding inequality in* (5.1) *implies univalence for each f analytic in D if and only if each component of ∂D is a point or a K-quasiconformal circle.*

Corollary 1 shows that the first hypothesis in Theorem 5 (that each component of ∂D be a point or a K-quasiconformal circle) is necessary. We show next by means of an example that the second hypothesis (that the number of components of ∂D be finite) is also necessary.

COROLLARY 2. *Let $\{B_n\}$ be any sequence of disjoint closed disks which lie in the unit disk U and let*

$$D = U - \bigcup_{n=1}^{\infty} B_n.$$

If

$$\sup_n \text{hyp area}\,(B_n) = \infty, \tag{5.6}$$

then there exists no positive constant a, b or c for which the corresponding condition in (5.1) *implies univalence for each f analytic in D.*

Here the hyperbolic area in (5.6) is taken with respect to U, that is

$$\text{hyp area}\,(B_n) = \iint_{B_n} (1-|z|^2)^{-2}\,dx\,dy.$$

Proof. The domain D is clearly not in O_{AD}. For each n let R_n denote the circular ring domain bounded by ∂U and ∂B_n. Then D separates the boundary components of ∂R_n while an elementary calculation shows that the conformal modulus of R_n is given by

$$\text{mod}\, R_n = \frac{1}{2}\log\left(1 + \frac{\pi}{\text{hyp area}\,(B_n)}\right).$$

(Cf. p. 8 in [8].) From (5.6) we find that $\inf_n \operatorname{mod} R_n = 0$, and the desired conclusion now follows from Theorem 6.

Now let D be as in Corollary 2. Then the components of ∂D are all circles. If the B_n are chosen so that (5.6) holds, we obtain an example which shows that the second hypothesis in Theorem 5 (that D be finitely connected) is essential.

On the other hand B. Osgood has shown in [11] that for certain choices of the disks B_n one obtains infinitely connected circle domains D in which each of the conditions in (5.1) implies univalence.

6. Proof of Theorem 6.

By hypothesis D is a domain which is not in O_{AD}. We begin with a pair of lemmas, the first of which yields a lower bound for K_D.

LEMMA 3. *If w_1 and w_2 are finite points which lie in the same component of $\bar{\mathbf{C}} - D$, then*

$$K_D(z, \bar{z}) \geq \frac{1}{16\pi}\left(\frac{|w_1 - w_2|}{|z - w_1||z - w_2|}\right)^2$$

for all z in D.

Proof. Let C denote the component of $\bar{\mathbf{C}} - D$ containing w_1 and w_2, and let C_1 and D_1 denote the images of C and D under $f(z) = (z - w_1)/(z - w_2)$. Then $D_2 = \bar{\mathbf{C}} - C_1$ is a simply connected subdomain of $\mathbf{C}$ which contains the domain D_1 but not the point $z = 0$. Hence (1.3) and the first inequality in (5.2) give

$$K_{D_1}(w, \bar{w}) \geq K_{D_2}(w, \bar{w}) \geq \frac{1}{16\pi} \operatorname{dist}(w, \partial D_2)^{-2} \geq \frac{1}{16\pi} |w|^{-2}$$

for all w in D_1. Then as

$$K_D(z, \bar{z}) = K_{D_1}(f(z), \overline{f(z)})|f'(z)|^2$$

we obtain

$$K_D(z, \bar{z}) \geq \frac{1}{16\pi}\left|\frac{f'(z)}{f(z)}\right|^2 = \frac{1}{16\pi}\left(\frac{|w_1 - w_2|}{|z - w_1||z - w_2|}\right)^2$$

for all z in D.

LEMMA 4. *If D separates the boundary components of a circular ring domain R with modulus less than $\log 2$, then there exists a function f analytic but not univalent in D with*

$$|S_f(z)| \leq (12\pi \bmod R) K_D(z, \bar{z})$$

for all z in D.

Proof. Suppose first that R is an annulus of the form

$$R = \left\{ z : \frac{1}{r} < |z| < r \right\}, \quad 1 < r < \infty. \tag{6.1}$$

Since D separates the components of ∂R, we can find points x_1 and x_2 in D with $-r < x_1 < -1/r$ and $1/r < x_2 < r$. Let $x_0 = \frac{1}{2}(x_1 + x_2)$ and set $f(z) = (z - x_0)^2$. Then f is analytic in D with $f(x_1) = f(x_2)$ and $S_f(z) = -\frac{3}{2}(z - x_0)^{-2}$. Next as $D \subset R$ we have

$$K_D(z, \bar{z}) \geq K_R(z, \bar{z}) \geq (\pi |z|^2 \sinh(2 \log r))^{-1}$$

by (1.3) and (4.1). Then since $\log r^2 = \bmod R < \log 2$, we find that

$$|x_0| < \frac{1}{2}\left(r - \frac{1}{r}\right) < \frac{1}{2r}, \quad \sinh(2 \log r) < 2 \bmod R,$$

and we obtain

$$\frac{|S_f(z)|}{K_D(z, \bar{z})} \leq \frac{3\pi}{2}\left(\frac{|z|}{|z - x_0|}\right)^2 \sinh(2 \log r) \leq 12\pi \bmod R$$

for z in D.

For the general case we can find a Möbius transformation g which maps R onto an annulus R_1 of the form in (6.1). If f_1 is the function defined above corresponding to $D_1 = g(D)$, then $f = f_1 \circ g$ is analytic but not univalent in D and

$$\begin{aligned}
|S_f(z)| &= |S_{f_1}(g(z))| \, |g'(z)|^2 \\
&\leq (12\pi \bmod R_1) K_{D_1}(g(z), \overline{g(z)}) |g'(z)|^2 \\
&= (12\pi \bmod R) K_D(z, \bar{z}).
\end{aligned}$$

This completes the proof of Lemma 4.

We turn now to the proof of Theorem 6. By hypothesis there exists a positive constant a such that the inequality

$$|S_f(z)| \le aK_D(z, \bar{z}) \tag{6.2}$$

implies univalence for each f analytic in D. Let

$$b = \max\left(\frac{20\pi}{a} + 1, 3\right)$$

and for each point z in $\mathbf{C}$ and each $r > 0$ let $B(z, r)$ denote the open disk with center z and radius r. We show first that D is b-*locally connected*, that is for all z and r, points in $D \cap \overline{B(z, r)}$ can be joined in $D \cap \overline{B(z, br)}$ and points in $D - B(z, r)$ can be joined in $D - B(z, r/b)$.

Assume otherwise. Then by Lemmas 1 and 2 in [9] there exist finite points w_1 and w_2 which lie in the same component of $\bar{\mathbf{C}} - D$ such that the function

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies the inequality

$$|h(z_1) - h(z_2) - 2\pi i| \le \frac{4}{b - 1}$$

for some pair of points z_1 and z_2 in D. Since $b \ge 3$

$$|h(z_1) - h(z_2)| \ge 2\pi - \frac{4}{b - 1} > 4.$$

Now set

$$f(z) = \exp(ch(z)), \; c = \frac{2\pi i}{h(z_1) - h(z_2)}.$$

Then f is analytic in D with

$$|S_f(z)| = \frac{|1 - c^2|}{2}\left(\frac{|w_1 - w_2|}{|z - w_1||z - w_2|}\right)^2 \le 8\pi|1 - c^2| K_D(z, \bar{z})$$

by Lemma 3. Our choice of b gives $8\pi|1 - c^2| \le a$ and thus f satisfies (6.2). But $f(z_1) = f(z_2)$ and we have a contradiction.

We conclude that the domain D is indeed b-locally connected and hence, by

Lemma 5 of [9], that each component of ∂D is either a point or a K-quasiconformal circle where K depends only on b. This establishes the first conclusion in Theorem 6.

For the second conclusion let R be a circular ring domain whose boundary components are separated by D and suppose that $\operatorname{mod} R < m$, where $m = \min(\log 2, a/12\pi) > 0$. Then by Lemma 4 there exists an f analytic but not univalent in D with

$$|S_f(z)| \leq (12\pi \operatorname{mod} R) K_D(z, \bar{z}) \leq a K_D(z, \bar{z}).$$

Again we have a contradiction. Hence $\operatorname{mod} R \geq m$ and the proof of Theorem 6 is complete.

7. Final remark.

The authors have just learned that J. Burbea has recently established results similar to Theorems 1 and 3.

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