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## A quick proof of the 4-dimensional stable surgery theorem

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In 1971 Cappel and Shaneson published a proof that if  $f:(M^4, \partial) \rightarrow (X, \partial)$  is a smooth surgery problem with trivial obstruction ( $\sigma(f) = o \in L_4^s(\pi_1 X)$ ) then a stable solution for  $f$  exists. That is, for some  $k$  the map  $f \# id:(M \# k(S^2 \times S^2), \partial) \rightarrow (X \# k(S^2 \times S^2), \partial)$  is normally bordant relative to the boundary to a simple homotopy equivalence.

At about the time of the Cappel–Shaneson result the second author discovered a homotopy theoretic proof of a closely related factorization result for surgery maps. The purpose of this note is to give a short geometric proof of this factorization result, and to observe that it implies the stable surgery theorem.

We shall call a surgery map *prepared* if it induces an isomorphism on  $\pi_0$  and  $\pi_1$ , and the intersection form on the kernel  $K_2(M)$  is a direct sum of standard planes. There is no difficulty in constructing a normal bordism of a map with trivial obstruction to a prepared one: First, surgeries on 0 and 1-spheres are used to achieve the homotopy conditions. The surgery obstruction is then defined to be the stable equivalence class of the intersection form on  $K_2(M)$  [Wall]. Vanishing of the obstruction means that after addition of trivial planes, this kernel is isomorphic to a sum of planes. Since surgery on a trivial 1-sphere in  $M$  has the effect of adding a plane to  $K_2(M)$ , repetition of this operation yields a prepared map.

**PROPOSITION 1.** *Any prepared  $f$  factors up to homotopy as a surgery map through a simple homotopy equivalence  $g$ :*

$$\begin{array}{ccc}
 (M^4, \partial) & \xrightarrow{f} & (X, \partial) \\
 \searrow \simeq g & & \nearrow \text{projection} \\
 & (X \# k(S^2 \times S^2), \partial) &
 \end{array}$$

Part of the data of a surgery map is a vector bundle  $\xi$  over  $X$  and a bundle

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map over  $f$ ,  $b : \nu_M \rightarrow \xi$ . By factoring “as a surgery map” we mean that this bundle map also factors through a map to the pull-back;  $c : \nu_M \rightarrow p^* \xi$ , where  $p$  is the projection.

**PROPOSITION 2.** *Proposition 1 implies the stable surgery theorem.*

**Proof of Proposition 2.** Suppose  $f$  is a surgery map with trivial obstruction. As explained above we may assume  $f$  is prepared. We show that  $f \# id_{k(S^2 \times S^2)}$  is normally bordant to the map  $g$  of Proposition 2. Since  $g$  is a simple homotopy equivalence this constitutes a solution of the stabilized surgery problem.

Normal bordism classes (rel  $\partial$ ) correspond to lifts (rel  $\partial$ ) of the classifying map for the normal fibration of  $X$  to  $BO$ ;

$$\begin{array}{ccc} & BO & \\ \nearrow & \downarrow & \\ X & \xrightarrow{\nu_x} & BG. \end{array}$$

(The uniqueness theorem for the normal fibration gives a fiber homotopy equivalence  $\nu_x \simeq \xi$ , which defines a lift.) Both  $g$  and  $f \# id_{k(S^2 \times S^2)}$  have lifts obtained from the lift for  $f$  by composition with the projection:

$$\begin{array}{ccc} & BO & \\ \nearrow_{\nu_x} & \downarrow & \\ X \# k_{(S^2 \times S^2)} & \rightarrow X & \rightarrow BG. \end{array}$$

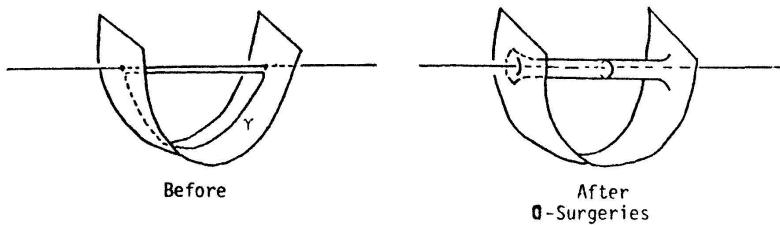
This is the lift corresponding to  $f \# id_{k(S^2 \times S^2)}$  by direct inspection, and it corresponds to  $g$  by the existence of the factorization of the bundle map into  $c : \nu_M \rightarrow p^* \xi$  and the pullback map. Since these maps correspond to the same lift, they are normally cobordant.

The philosophical significance is that the troublesome surgeries on 2-spheres are unnecessary for the stable surgery theorem. Once the surgery map is prepared, its domain is the domain of a stable solution; only a little tinkering is required to find the map  $g$ .

**Proof of Proposition 1.** Assume  $f$  is prepared.  $K_2(M)$  has a preferred basis represented by framed immersed spheres  $a_1, \dots, a_k, b_1, \dots, b_k$  with algebraic intersections  $\lambda(a_i, a_j) = \lambda(b_i, b_j) = 0$ ,  $\mu(a_i) = \mu(b_i) = 0$  and  $\lambda(a_i, b_i) = \delta_{ij} \in Z[\pi_1 X]$ .

The framing of each sphere's normal bundle is determined by null homotopies for these spheres in  $X$  together with the bundle map  $b: \nu_M \rightarrow \xi$  covering  $f$ .

In dimension four this data may not be sufficient to produce disjointly embedded wedges of spheres. However, we can find framed disjointly embedded wedges of oriented surfaces  $A_1 \vee B_1, \dots, A_k \vee B_k$  representing the preferred basis, which are nullhomotopic in  $X$ . Suppose we have an algebraically cancelling pair of intersection points. Choose an arc between these points on one surface, and modify the other surface by an ambient  $o$ -surgery: replace discs by the normal sphere bundle restricted to the arc. Algebraically cancelling means first



the intersection points have opposite sign (so the result of the  $o$ -surgery is oriented) and second the loop formed by arcs on the two surfaces ( $\gamma$  in the picture) is nullhomotopic. The nullhomotopy may be used to construct a homotopy of the surged surface into the original one. Therefore nullhomotopy in  $X$  is also preserved by this operation.

Again these surfaces are framed by the nullhomotopy in  $X$  and the bundle map. The framing determines maps on the closed regular neighborhoods  $h_i: (\nu(A_i \vee B_i), \partial) \rightarrow (S^2 \times S^2 - \text{int } D^4, \partial)$ ,  $1 \leq k$ .

Assume, as in [Wall, Chapter 2] that  $X$  has a top 4-cell. Let  $D_1, \dots, D_k$  be disjoint 4-discs in the top cell. Then there is a map  $f'$  homotopic (rel  $\partial$ ) to  $f$  such that  $(f')^{-1}(D_i, \partial) = (\nu(A_i \vee B_i), \partial)$ : First find a map  $f''$  (using the nullhomotopies) such that  $f''(\nu(A_i \vee B_i), \partial) = (D_i, \partial)$  and (by transversality) the rest of the inverse image of  $D_i$  consists of discs mapping difformorphically to  $D_i$ . Since  $f''$  is degree 1, the extra discs may be cancelled by a further homotopy. The result is  $f'$ .

The factorization  $g$  is constructed by cutting and pasting:  $g = f' | (M - \coprod \nu(A_i \vee B_i)) \cup \coprod h_i$ . This does not change the isomorphism on  $\pi_1$ , and the following homology calculation (with  $\mathbb{Z}[\pi_1 X]$  coefficients) shows that  $g$  is a simple homotopy equivalence.

Let  $\nu$  denote  $\coprod_{i=1}^k \nu(A_i \vee B_i)$ ,  $M^- = M - \text{int } \nu$ . From the Mayer–Vietoris sequences of kernel modules of

$$K_2^f(\partial \nu) \rightarrow K_2^f(\nu) \oplus K_2^f(M^-) \rightarrow K_2^f(M) \rightarrow 0$$

we see that

$$K_2^f(\partial \mathbf{n}) \xrightarrow{\text{inc}*} K_2^f(M^-)$$

is onto, the middle arrow having been constructed to be a simple isomorphism when restricted to the first summand. Now consider the same sequence replacing  $f$  by  $g$ .  $K_2^f(\partial \mathbf{n}) = K_2^g(\partial \mathbf{n})$  and  $K_2^f(M^-) = K_2^g(M^-)$  so the map

$$K_2^g(\partial \mathbf{n}) \xrightarrow{\text{inc}*} K_2^g(M^-)$$

remains an epimorphism. By construction  $K_2^g(\mathbf{n}) \cong 0$ . Consequently  $K_2^g(M) \cong 0$ . A standard argument using Poincaré duality shows that  $g$  induces an isomorphism on  $H_*(\mathbf{ ; Z}[\pi_1 X])$  for all  $*$  and by Whitehead's theorem must be a homotopy equivalence. The simplicity of  $K_2^f(\mathbf{n}) \rightarrow K_2^f(M)$  implies that  $g$  is in fact a simple homotopy equivalence.

Finally the nullhomotopy of the  $A_i \vee B_i$  in  $X$ , and the bundle map  $b : \nu_M \rightarrow \xi$  define a framing of the restriction of  $\nu_M$  to the neighborhood  $\mathbf{n}_i$ . This can be interpreted as a factorization of  $b$  through the pullback  $p^* \xi$ , since this pullback is trivial on the summands  $\# S^2 \times S^2$ .

## REFERENCES

SYLVAIN CAPPELL and JULIUS SHANESON, *On four dimensional surgery and applications*, Comment. Math. Helv. 46 (1971), 500–528.  
 C. T. C. WALL, *Surgery of Compact Manifolds*, Academic Press, 1970.

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