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A quick proof of the 4-dimensional stable surgery theorem

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In 1971 Cappel and Shaneson published a proof that if $f: (M^4, \partial) \rightarrow (X, \partial)$ is a smooth surgery problem with trivial obstruction ($\sigma(f) = 0 \in L_4^s(\pi_1 X)$) then a stable solution for f exists. That is, for some k the map $f \# \text{id}: (M \# k(S^2 \times S^2), \partial) \rightarrow (X \# k(S^2 \times S^2), \partial)$ is normally bordant relative to the boundary to a simple homotopy equivalence.

At about the time of the Cappel–Shaneson result the second author discovered a homotopy theoretic proof of a closely related factorization result for surgery maps. The purpose of this note is to give a short geometric proof of this factorization result, and to observe that it implies the stable surgery theorem.

We shall call a surgery map *prepared* if it induces an isomorphism on π_0 and π_1 , and the intersection form on the kernel $K_2(M)$ is a direct sum of standard planes. There is no difficulty in constructing a normal bordism of a map with trivial obstruction to a prepared one: First, surgeries on 0 and 1-spheres are used to achieve the homotopy conditions. The surgery obstruction is then defined to be the stable equivalence class of the intersection form on $K_2(M)$ [Wall]. Vanishing of the obstruction means that after addition of trivial planes, this kernel is isomorphic to a sum of planes. Since surgery on a trivial 1-sphere in M has the effect of adding a plane to $K_2(M)$, repetition of this operation yields a prepared map.

PROPOSITION 1. *Any prepared f factors up to homotopy as a surgery map through a simple homotopy equivalence g :*

$$\begin{array}{ccc}
 (M^4, \delta) & \xrightarrow{f} & (X, \delta) \\
 \searrow \scriptstyle g \cong & & \nearrow \scriptstyle \text{projection} \\
 & (X \# k(S^2 \times S^2), \delta) &
 \end{array}$$

Part of the data of a surgery map is a vector bundle ξ over X and a bundle

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map over f , $b: \nu_M \rightarrow \xi$. By factoring “as a surgery map” we mean that this bundle map also factors through a map to the pull-back; $c: \nu_M \rightarrow p^*\xi$, where p is the projection.

PROPOSITION 2. *Proposition 1 implies the stable surgery theorem.*

Proof of Proposition 2. Suppose f is a surgery map with trivial obstruction. As explained above we may assume f is prepared. We show that $f \# id_{k(S^2 \times S^2)}$ is normally bordant to the map g of Proposition 2. Since g is a simple homotopy equivalence this constitutes a solution of the stabilized surgery problem.

Normal bordism classes (rel ∂) correspond to lifts (rel ∂) of the classifying map for the normal fibration of X to BO ;

$$\begin{array}{ccc} & & BO \\ & \nearrow \text{dashed} & \downarrow \\ X & \xrightarrow{\nu_x} & BG. \end{array}$$

(The uniqueness theorem for the normal fibration gives a fiber homotopy equivalence $\nu_x \simeq \xi$, which defines a lift.) Both g and $f \# id_{k(S^2 \times S^2)}$ have lifts obtained from the lift for f by composition with the projection:

$$X \# k_{(S^2 \times S^2)} \rightarrow X \begin{array}{ccc} & & BO \\ & \nearrow \text{dashed } \nu_x & \downarrow \\ & \longrightarrow & BG. \end{array}$$

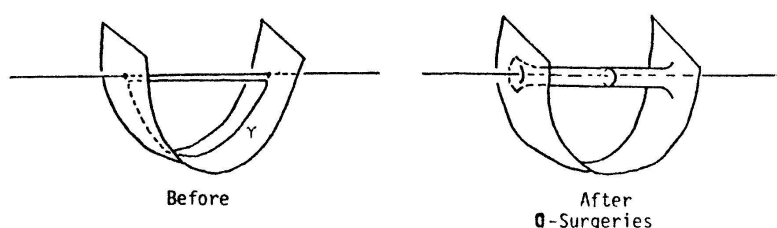
This is the lift corresponding to $f \# id_{k(S^2 \times S^2)}$ by direct inspection, and it corresponds to g by the existence of the factorization of the bundle map into $c: \nu_M \rightarrow p^*\xi$ and the pullback map. Since these maps correspond to the same lift, they are normally cobordant.

The philosophical significance is that the troublesome surgeries on 2-spheres are unnecessary for the stable surgery theorem. Once the surgery map is prepared, its domain is the domain of a stable solution; only a little tinkering is required to find the map g .

Proof of Proposition 1. Assume f is prepared. $K_2(M)$ has a preferred basis represented by framed immersed spheres $a_1, \dots, a_k, b_1, \dots, b_k$ with algebraic intersections $\lambda(a_j, a_j) = \lambda(b_j, b_j) = 0$, $\mu(a_i) = \mu(b_i) = 0$ and $\lambda(a_i, b_i) = \delta_{ij} \in \mathbb{Z}[\pi_1 X]$.

The framing of each sphere's normal bundle is determined by null homotopies for these spheres in X together with the bundle map $b: \nu_M \rightarrow \xi$ covering f .

In dimension four this data may not be sufficient to produce disjointly embedded wedges of spheres. However, we can find framed disjointly embedded wedges of oriented surfaces $A_1 \vee B_1, \dots, A_k \vee B_k$ representing the preferred basis, which are nullhomotopic in X . Suppose we have an algebraically cancelling pair of intersection points. Choose an arc between these points on one surface, and modify the other surface by an ambient o -surgery: replace discs by the normal sphere bundle restricted to the arc. Algebraically cancelling means first



the intersection points have opposite sign (so the result of the o -surgery is oriented) and second the loop formed by arcs on the two surfaces (γ in the picture) is nullhomotopic. The nullhomotopy may be used to construct a homotopy of the surged surface into the original one. Therefore nullhomotopy in X is also preserved by this operation.

Again these surfaces are framed by the nullhomotopy in X and the bundle map. The framing determines maps on the closed regular neighborhoods $h_i: (\mathfrak{n}(A_i \vee B_i), \partial) \rightarrow (S^2 \times S^2 - \text{int } D^4, \partial)$, $1 \leq k$.

Assume, as in [Wall, Chapter 2] that X has a top 4-cell. Let D_1, \dots, D_k be disjoint 4-discs in the top cell. Then there is a map f' homotopic (rel ∂) to f such that $(f')^{-1}(D_i, \partial) = (\mathfrak{n}(A_i \vee B_i), \partial)$: First find a map f'' (using the nullhomotopies) such that $f''(\mathfrak{n}(A_i \vee B_i), \partial) = (D_i, \partial)$ and (by transversality) the rest of the inverse image of D_i consists of discs mapping diffeomorphically to D_i . Since f'' is degree 1, the extra discs may be cancelled by a further homotopy. The result is f' .

The factorization g is constructed by cutting and pasting: $g = f' | (M - \amalg \mathfrak{n}(A_i \vee B_i)) \cup \amalg h_i$. This does not change the isomorphism on π_1 , and the following homology calculation (with $Z[\pi_1 X]$ coefficients) shows that g is a simple homotopy equivalence.

Let \mathfrak{n} denote $\amalg_{i=1}^k \mathfrak{n}(A_i \vee B_i)$, $M^- = M - \text{int } \mathfrak{n}$. From the Mayer-Vietoris sequences of kernel modules of

$$K_2^f(\partial \mathfrak{n}) \rightarrow K_2^f(\mathfrak{n}) \oplus K_2^f(M^-) \rightarrow K_2^f(M) \rightarrow 0$$

we see that

$$K_2^f(\partial \mathfrak{n}) \xrightarrow{\text{inc}_*} K_2^f(M^-)$$

is onto, the middle arrow having been constructed to be a simple isomorphism when restricted to the first summand. Now consider the same sequence replacing f by g . $K_2^f(\partial \mathfrak{n}) = K_2^g(\partial \mathfrak{n})$ and $K_2^f(M^-) = K_2^g(M^-)$ so the map

$$K_2^g(\partial \mathfrak{n}) \xrightarrow{\text{inc}_*} K_2^g(M^-)$$

remains an epimorphism. By construction $K_2^g(\mathfrak{n}) \cong 0$. Consequently $K_2^g(M) \cong 0$. A standard argument using Poincaré duality shows that g induces an isomorphism on $H_*(; Z[\pi_1 X])$ for all $*$ and by Whitehead's theorem must be a homotopy equivalence. The simplicity of $K_2^f(\mathfrak{n}) \rightarrow K_2^f(M)$ implies that g is in fact a simple homotopy equivalence.

Finally the nullhomotopy of the $A_i \vee B_i$ in X , and the bundle map $b : \nu_M \rightarrow \xi$ define a framing of the restriction of ν_M to the neighborhood \mathfrak{n}_i . This can be interpreted as a factorization of b through the pullback $p^*\xi$, since this pullback is trivial on the summands $\#S^2 \times S^2$.

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