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Quadratic forms over rings of dimension 1

(Dedicated to Professor K. G. Ramanathan on his sixtieth birthday)

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Introduction

The object of this paper is to study quadratic spaces over commutative noetherian domains* of dimension 1 in which 2 is invertible. We prove (Theorem 3.1) that over such a ring, if further, the set of singular prime ideals is finite, then any quadratic space which contains a hyperbolic plane locally at all the prime ideals contains a hyperbolic space of rank 2. The main tool for the proof of this theorem is a result (Theorem 2.1) which seems to be of independent interest, which states that over a semi-local ring of dimension 1 (in which 2 is invertible), if a quadratic space contains locally a hyperbolic plane, then it contains a hyperbolic plane. As an application, we classify (Proposition 4.5) quadratic spaces over $k[t^2, t^3]$, k a field of characteristic $\neq 2$, up to anisotropic spaces and deduce that if k is a quadratically closed field, any quadratic space of rank ≥ 3 over $k[t^2, t^3]$ is extended from k. We show however (Corollary 4.8) that over $\mathbf{R}[t^2, t^3]$ there exist anisotropic quadratic forms of rank ≥ 3 and discriminant 1 which are not extended from $\mathbf{R}$. This follows from the result (Proposition 4.7) that the non free projective module over $\mathbf{R}[X, Y]$ constructed in ([9], Proposition 1) remains non free over $\mathbf{H}[X, Y]/(X^3 - Y^2)$. (This gives incidentally a non free, stably free projective module over a non commutative ring of dimension 1).

Throughout the paper, unless otherwise explicitly stated, R denotes a commutative, noetherian ring with identity in which 2 is invertible. We sometimes denote by q the quadratic space (R^n, q).

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§ 1. Some assorted lemmas

We collect in this section some lemmas which are needed in our later sections. Some of the lemmas are probably well known, but we have included their proofs for the sake of completeness.

* For the falsity of the theorem for nondomains, see, H. Bass "Modules which support nonsingular forms," J. Algebra *13* (1969).

LEMMA 1.1. *Let R denote a commutative ring, which is complete with respect to an ideal I and in which 2 is invertible. If (P, q) is a quadratic space over R such that $(P, q)\otimes_R R/I$ contains a hyperbolic plane, then (P, q) contains a hyperbolic plane.*

Proof. Let $\bar{v}$, $\bar{w} \in P\otimes_R R/I$ be such that $q(\bar{v}) = q(\bar{w}) = 0$ and $q(\bar{v}, \bar{w}) = 1$. Let $v, w \in P$ be lifts of $\bar{v}, \bar{w}$. Then $(Rv+Rw, q \mid Rv+Rw)$ is a quadratic R-space of rank 2 and discriminant $-1+\mu$, $\mu\in I$. The element $1-\mu$ is a square in R since R is I-adic complete. Hence $(Rv+Rw,\ q \mid Rv+Rw)$ is a hyperbolic plane.

LEMMA 1.2. *Let R be a commutative artinian ring. If (R^n, q) is a quadratic space which locally contains a hyperbolic plane, then it contains a hyperbolic plane.*

Proof. Let $\mathfrak{M}_i$, $1\leq i\leq r$ be the maximal ideals of R. We have an isometry

$$q\otimes_R R/\mathrm{rad}\, R \xrightarrow{\sim} \prod_{1\leq i\leq r} q\otimes_R R/\mathfrak{M}_i \tag{$*$}$$

Since, by assumption, $q\otimes R_{\mathfrak{M}_i}$ contains a hyperbolic plane, it follows that each component of the R.H.S. of $(*)$ contains a hyperbolic plane and hence $q\otimes_R R/\mathrm{rad}\, R$ contains a hyperbolic plane. Lemma 1.2 now follows from Lemma 1.1.

LEMMA 1.3. *Let $R\subset S$ be commutative semi-local rings and let $\mathfrak{C}$ be an ideal of S which is contained in R. Let A be an Azumaya algebra over R such that $A\otimes_R S$ and $A\otimes_R R/\mathfrak{C}$ are both isomorphic to matrix algebras. Then A is isomorphic to a matrix algebra.*

Proof. If, for any commutative ring, $\mathrm{Az}(-)$ denotes the category of Azumaya algebras over the ring, we have ([2], Theorem 5.3, p 481) the following exact sequence

$$\begin{aligned} K_1\,\mathrm{Az}\,(S)\oplus K_1\,\mathrm{Az}\,(R/\mathfrak{C}) &\to K_1\,\mathrm{Az}\,(S/\mathfrak{C})\to K_0\,\mathrm{Az}\,(R)\\ &\to K_0\,\mathrm{Az}\,(S)\oplus K_0\,\mathrm{Az}\,(R/\mathfrak{C})\to K_0\,\mathrm{Az}\,(S/\mathfrak{C}) \end{aligned}$$

In view of ([3], Proposition 6.8, p 120), we have, for any commutative ring B, $K_1\,\mathrm{Az}\,(B)\xrightarrow{\sim}\mathbf{Q}/\mathbf{Z}\otimes U(B)\oplus\mathbf{Q}/\mathbf{Z}\otimes SK_1(B)$. In particular, if B is semi-local, we have $SK_1(B)=0$ ([2], Cor. 9.2, p 267) and hence $K_1\,\mathrm{Az}\,(B)\xrightarrow{\sim}\mathbf{Q}/\mathbf{Z}\otimes U(B)$. Hence we have an exact sequence

$$\mathbf{Q}/\mathbf{Z}\otimes U(S)\oplus\mathbf{Q}/\mathbf{Z}\otimes U(R/\mathfrak{C})\to\mathbf{Q}/\mathbf{Z}\otimes U(S/\mathfrak{C})\to K_0\,\mathrm{Az}\,(R)\to ..$$

On the other hand, we have ([2], Theorem 5.3, p 481) the exact sequence

$$U(S)\oplus U(R/\mathfrak{C})\to U(S/\mathfrak{C})\to \operatorname{Pic}(R)=(1),$$

so that $U(S)\oplus U(R/\mathfrak{C})\to U(S/\mathfrak{C})$ is a surjection and hence $K_1\operatorname{Az}(S)\oplus K_1\operatorname{Az}(R/\mathfrak{C})\to K_1\operatorname{Az}(S/\mathfrak{C})$ is surjective and $0\to K_0\operatorname{Az}(R)\to K_0\operatorname{Az}(S)\oplus K_0\operatorname{Az}(R/\mathfrak{C})$ is exact. If rank $A=r^2$, then, the element $A-M_r(R)$ of $K_0\operatorname{Az}(R)$ becomes trivial in $K_0\operatorname{Az}(S)\oplus K_0\operatorname{Az}(R/\mathfrak{C})$ and hence is trivial in $K_0\operatorname{Az}(R)$. Hence $A\otimes_R M_s(R)\xrightarrow{\sim} M_{r+s}(R)$. Since R is semi local, we may cancel $M_s(R)$ ([13], Proposition 3.2) to get $A\xrightarrow{\sim} M_r(R)$.

COROLLARY 1.4. *Let R be a commutative semi-local domain of dimension 1. Then, any Azumaya algebra over R which is locally isomorphic to a matrix algebra is isomorphic to a matrix algebra.*

Proof. Let $\bar{R}$ denote the integral closure of R in its quotient field K. Since $\dim R=1$, by Krull–Akizuki Theorem, $\bar{R}$ is noetherian and hence semi-local. The canonical map $\operatorname{Br}(\bar{R})\to\operatorname{Br}(K)$ is injective ([1], 7.2) and since $\bar{R}$ is semi local, $A\otimes_R\bar{R}$ is isomorphic to a matrix algebra. Hence, there exists an integral extension S of R which is an R-module of finite type such that $A\otimes_R S$ is isomorphic to a matrix algebra. If $\mathfrak{C}$ denotes the (non zero) conductor of R in S, $R/\mathfrak{C}$ is artinian and $A\otimes_R R/\mathfrak{C}$ is locally a matrix algebra. Thus, $A\otimes_R R/\mathfrak{C}$ is a matrix algebra modulo rad $R/\mathfrak{C}$ and hence is a matrix algebra. We now apply Lemma 1.3 to complete the proof.

Remark. M. Ojanguren, in his paper entitled 'A non-trivial locally trivial algebra' (J. Algebra 29, 510–512) gives an example of a domain R of dimension 2 and an Azumaya R-algebra which is locally a matrix algebra, but not a matrix algebra. This example can be semi-localised to show that the above corollary does not generalise to higher dimensions.

LEMMA 1.5. *Let R be a commutative ring of dimension* 1 *for which* Pic R *is n divisible for some integer $n\geqslant 1$. Then, any Azumaya algebra of rank n^2 over R which is trivial in* Br (R) *is isomorphic to $M_n(R)$. In fact, more explicitly, if Q is a projective R-module of rank n, then* $\operatorname{End}_R Q\xrightarrow{\sim} M_n(R)$.

Proof. Let $A=\operatorname{End}_R Q$, Q being a projective R-module of rank n. Since R is of dimension 1, we can write $Q=P\oplus R^{n-1}$ with $P\in\operatorname{Pic} R$. By our assumption, there exists $P_1\in\operatorname{Pic} R$ such that $\otimes^n P_1\xrightarrow{\sim} P$. We have

$$R^n\otimes_R P_1\xrightarrow{\sim}(\otimes^n P_1)\oplus R^{n-1}\xrightarrow{\sim} P\oplus R^{n-1},$$

so that

$$M_n(R) = \mathrm{End}_R R^n \xrightarrow{\sim} \mathrm{End}_R R^n \otimes_R \mathrm{End}_R P_1$$
$$\xrightarrow{\sim} \mathrm{End}_R (R^n \otimes P_1) \xrightarrow{\sim} \mathrm{End}_R (P \oplus R^{n-1})$$

§ 2. Quadratic forms over semi-local rings of dimension 1

THEOREM 2.1. *Let R be a noetherian semi-local domain of dimension 1 in which 2 is invertible. If a quadratic form over R contains locally a hyperbolic plane, then it contains a hyperbolic plane.*

Proof. Let $\bar{R}$ denote the integral closure of R in its quotient field K. By Krull–Akizuki theorem, $\bar{R}$ is noetherian and hence Dedekind. Since R is semi-local, $\bar{R}$ is semi-local and hence is a principal ideal domain.

Let q be a quadratic form over R which contains locally a hyperbolic plane. Then, $q \otimes_R K$ contains a hyperbolic plane and it follows that $q \otimes_R \bar{R}$ contains a hyperbolic plane, since $\bar{R}$ is a principal ideal domain.

Let $(\mu_1, \ldots, \mu_n) \in \bar{R}^n$ be a unimodular isotropy of q in $\bar{R}^n$ and let $\Sigma \mu_i \nu_i = 1$, $\nu_i \in \bar{R}$. If $S = R[\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n]$, S is finitely generated as an R-module, hence noetherian semi-local and $q \otimes_R S$ contains a hyperbolic plane. Let $q \otimes_R S \xrightarrow{\sim} q_1 \perp h$, where h denotes a hyperbolic plane over S.

Let $\mathfrak{C}$ denote the conductor of R in S. Since S is finitely generated as an R-module, $\mathfrak{C} \neq 0$ and we have the following cartesian square

$$\begin{array}{ccc} R & \rightarrow & S \\ \downarrow & & \downarrow \\ R/\mathfrak{C} & \rightarrow & S/\mathfrak{C} \end{array}$$

Since $R/\mathfrak{C}$ is Artinian, Lemma 1.2 implies that $q \otimes_R R/\mathfrak{C} \xrightarrow{\sim} q_2 \perp h$ where h is a hyperbolic plane over $R/\mathfrak{C}$. Since moreover $q \otimes_R S \xrightarrow{\sim} q_1 \perp h$, we have $q_1 \otimes_R S/\mathfrak{C} \xrightarrow{\sim} q_2 \otimes_R S/\mathfrak{C}$. In view of ([2], Theorem 5.1, p 479) there exists a quadratic form q' over R such that $q' \otimes_R S \xrightarrow{\sim} q_1$, $q' \otimes_R R/\mathfrak{C} \xrightarrow{\sim} q_2$. Let $\mathrm{disc}\, q = -\lambda\, \mathrm{disc}\, q'$ with $\lambda \in U(R)$. Since q and q' differ by a hyperbolic plane over S and $R/\mathfrak{C}$, λ is a square in both S and $R/\mathfrak{C}$. Let $q' = \langle \nu_1, \ldots, \nu_n \rangle$ be a diagonalization of q' over R. Then $\langle \lambda \nu_1, \nu_2, \ldots, \nu_n \rangle$ becomes isometric to q_1 over S and q_2 over $R/\mathfrak{C}$. Replacing q' by $\langle \lambda \nu_1, \nu_2, \ldots, \nu_n \rangle$ we may assume to start with that $\mathrm{disc}\, q = -\mathrm{disc}\, q' = \mathrm{disc}\, q' \cdot \mathrm{disc}\, h$.

Denoting, for any commutative ring B, $K_iO(B)$, $i = 0, 1$, the K_i-groups of the category Q of quadratic spaces over B, we have the exact sequence ([2], Theorem 5.3, p 481)

$$K_1O(S/\mathfrak{C}) \xrightarrow{\phi} K_0O(R) \xrightarrow{\eta} K_0O(S) \oplus K_0O(R/\mathfrak{C}) \to K_0O(S/\mathfrak{C})$$

The forms q and $q' \perp h$ map into the same element in $K_0O(S) \oplus K_0O(R/\mathfrak{C})$ under η and hence the class of $q - q' \perp h$ in $K_0O(R)$ is in the image of ϕ.

Since $S/\mathfrak{C}$ is semi local, the canonical map $O_4(S/\mathfrak{C}) \to K_1O(S/\mathfrak{C})$ is surjective ([14], Theorem 3.1, p 317). Let $\alpha \in O_4(S/\mathfrak{C})$ be such that the class of $q - q' \perp h$ in $K_0O(R)$ is the image of the class $[\alpha]$ of α under ϕ. By definition, $\phi[\alpha] = q_1 - h \perp h$, where q_1 is a quadratic R-space of rank 4 which becomes hyperbolic over S and $R/\mathfrak{C}$. We therefore have $q - q' \perp h = q_1 - h \perp h$ in $K_0O(R)$ so that

$$q \perp h \perp h \perp H \xrightarrow{\sim} q_1 \perp q' \perp h \perp H$$

for some hyperbolic space H over R. Since R is semi-local, we have ([12], Theorem 8.1), $q \perp h \xrightarrow{\sim} q_1 \perp q'$. Since disc $q = -$disc q', it follows that disc $q_1 = 1$. Since q_1 is a rank 4 quadratic space of discriminant 1, it follows from ([8], Theorem 4.6) that such a space is given by the reduced norm of $\mathrm{Hom}_A(P, Q)$, where A is an Azumaya algebra of rank 4 over R and P and Q projective A-modules of rank 1. Since by our choice q_1 is hyperbolic over S and $R/\mathfrak{C}$, it follows that $A \otimes_R S$ and $A \otimes_R R/\mathfrak{C}$ are both Braner equivalent to ([8], Theorem 4.6) and hence isomorphic ([13], Proposition 3.2) to the 2×2 matrix algebras over S and $R/\mathfrak{C}$ respectively. Lemma 1.3 now shows that $A \xrightarrow{\sim} M_2(R)$, P and Q are therefore free and hence q_1 is hyperbolic i.e. $q_1 \xrightarrow{\sim} h \perp h$. We therefore have $q \perp h \xrightarrow{\sim} q' \perp h \perp h$ and by ([12], Theorem 8.1), $q \xrightarrow{\sim} q' \perp h$, i.e. q contains a hyperbolic plane.

§ 3. Quadratic forms over rings of dimension 1

THEOREM 3.1. *Let R be a noetherian domain of dimension 1 in which 2 is invertible. Suppose that the singular set* Sing (R) *of* Spec (R) *is finite and non empty. If (P, q) is a quadratic R-space such that $(P, q) \otimes_R R_{\mathfrak{p}}$ contains a hyperbolic plane for $\mathfrak{p} \in \mathrm{Sing}(R)$, then (P, q) contains a hyperbolic space of rank* 2.

Proof. Let $S = R - \bigcup_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$. Then $S^{-1}R$ is semi-local and by our assumption $(P, q) \otimes_R S^{-1}R$ contains locally a hyperbolic plane. Hence by Theorem 2.1, $(P, q) \otimes_R S^{-1}R$ contains a hyperbolic plane. Let $v \in P$ be a unimodular isotropy in $(P, q) \otimes_R S^{-1}R$. Let K denote the quotient field of R. Then, for any $\mathfrak{p} \in \mathrm{Spec}(R) - \mathrm{Sing}(R)$, $R_{\mathfrak{p}}$ being a discrete valuation ring, since $P_{\mathfrak{p}}/Kv \cap P_{\mathfrak{p}}$ is torsion free, it is projective. On the other hand, if $\mathfrak{p} \in \mathrm{Sing}(R)$, $(Kv \cap P)_{\mathfrak{p}} = ((S^{-1}R) \cdot v)_{\mathfrak{p}}$ is a direct summand of $P_{\mathfrak{p}}$, since v is unimodular in $P \otimes_R S^{-1}R$. Thus $(P/Kv \cap P)_{\mathfrak{p}}$ is projective for any $\mathfrak{p} \in \mathrm{Spec}(R)$ and hence $P/Kv \cap P$ is projective. Thus, $Kv \cap P$ is a totally isotropic direct summand of P of rank 1 which can be completed to a hyperbolic space of rank 2 ([4], 4.10.1).

COROLLARY 3.2. *Let R be an affine domain of dimension* 1 *over a field k of characteristic* $\neq 2$. *Then, every quadratic R-space which locally contains a hyperbolic plane contains a hyperbolic space. In particular, the canonical map* $W(R) \to \prod_{\mathfrak{p} \in \mathrm{Spec}(R)} W(R_{\mathfrak{p}})$ *is injective,* $W(-)$ *denoting the Wittring.*

§ 4. Quadratic spaces over $k[t^2, t^3]$

Let k be any field of characteristic $\neq 2$. The conductor of the subring $k[t^2, t^3]$ of the polynomial ring $k[t]$ in $k[t]$ is the ideal (t^2, t^3) and we have the Cartesian square

$$\begin{array}{ccc} k[t^2, t^3] & \longrightarrow & k[t] \\ \downarrow & & \downarrow \\ k = k[t^2, t^3]/(t^2, t^3) & \longrightarrow & k[t]/(t^2) \end{array}$$

We begin by recording some results which we require. In what follows we shall write $R = k[t^2, t^3]$.

LEMMA 4.1. $\mathrm{Pic}(R) \xrightarrow{\sim} k$.

Proof. We have, in view of ([2], Theorem 5.3, p 481), an exact sequence

$$1 \to U(R) \to U(k) \times U(k[t]) \xrightarrow{\eta} U(k[t]/(t^2) \to \mathrm{Pic}(R) \to \mathrm{Pic}(k[t]) \times \mathrm{Pic}(k) = 1$$

We therefore have the exact sequence

$$1 \to k^* \to k^* \times k^* \xrightarrow{\eta} U(k[t]/(t^2)) \to \mathrm{Pic}(R) \to 1$$

An element $\lambda + \mu t \in k[t]/(t^2)$ is a unit if and only if $\lambda \in k^*$ and then the map $\lambda + \mu t \to (\lambda, \lambda^{-1}\mu)$ gives an isomorphism $U(k[t]/(t^2)) \to k^* \times k$. Hence the cokernel of η is isomorphic to k and $\mathrm{Pic}(R) \xrightarrow{\sim} k$.

LEMMA 4.2. *The inclusion* $k \to k[t^2, t^3]$ *induces an isomorphism of the* 2-*torsion subgrup of* $\mathrm{Br}(k)$ *onto the* 2-*torsion subgroup of* $\mathrm{Br}(k[t^2, t^3])$.

Proof. We have ([6], Theorem 2.2) a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1=\operatorname{Pic}(k[t]/(t^2)) & \to & \operatorname{Br}(R) & \to & \operatorname{Br}(k[t])\oplus\operatorname{Br}(k) & \to & \operatorname{Br}(k[t]/(t^2)) & \to & 1\\
 & & \uparrow & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\psi} & & \\
1 & \to & \operatorname{Br}(k) & \to & \operatorname{Br}(k)\oplus\operatorname{Br}(k) & \to & \operatorname{Br}(k) & \to & 1
\end{array}$$

where the vertical maps are induced by inclusions. The map ψ is an isomorphism in view of ([13], Cor. 2.5). On the other hand in view of ([1], 7.6) φ induces an isomorphism of the 2-torsion subgroup of $\operatorname{Br}(k)\oplus\operatorname{Br}(k)$ on to the 2-torsion subgroup of $\operatorname{Br}(k[t])\oplus\operatorname{Br}(k)$. The lemma is now immediate.

LEMMA 4.3. $K_1O(k[t]/(t^2)) \xrightarrow{\sim} \mathbf{Z}/2\mathbf{Z}\times k^*/k^{*2}$

Proof. The inclusion $k \xrightarrow{i} k[t]/(t^2)$ and the supplementation $\varepsilon: k[t]/(t^2)\to k$ defined by $\varepsilon(a+bt)=a$ induce maps $K_1O(k)\xrightarrow{i} K_1O(k[t]/(t^2))\xrightarrow{\varepsilon} K_1O(k)$ with $\varepsilon i=$ identity so that $K_1O(k)$ is a direct summand of $K_1O(k[t]/(t^2))$. Since $k[t]/(t^2)$ is local, the projection $O_2(k[t]/(t^2))\to K_1O(k[t]/(t^2))$ is surjective ([4], Th. 3.5). We have

$$O_2(k[t]/(t^2))=\left\{\begin{pmatrix}\lambda & \mu\\ \nu & \delta\end{pmatrix}\in GL_2(k[t]/(t^2)) \;\middle|\; \lambda\delta+\mu\nu=1,\quad \lambda\mu=\nu\delta=0\right\}$$
$$=\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\ \begin{pmatrix}0 & \lambda\\ \lambda^{-1} & 0\end{pmatrix}\;\middle|\; \lambda\in U(k[t]/(t^2))\right\}.$$

Hence any element of $K_1O(k[t]/(t^2))$ is the class of $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ or $\begin{pmatrix}0 & \lambda\\ \lambda^{-1} & 0\end{pmatrix}$, $\lambda\in U(k[t]/(t^2))$. Since

$$\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^{-2}\end{pmatrix}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1},$$

it follows that $\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^{-2}\end{pmatrix}$ represents the trivial element of $K_1O(k[t]/(t^2))$ and $K_1O(k[t]/(t^2))$ is generated by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$, λ running through representatives of square classes of $U(k[t]/(t^2))$. Since $U(k[t]/(t^2))\,|\,(U(k[t]/(t^2)))^2 \xrightarrow{\sim} k^*/k^{*2}$ it follows that $K_1O(k)\to K_1O(k[t]/(t^2))$ is surjective and hence an isomorphism.

LEMMA 4.4. *The inclusion* $k\to R$ *induces an isomorphism* $K_0O(k)\xrightarrow{\sim} K_0O(R)$.

Proof. In view of ([2], Th. 5.3, p 481), we have an exact sequence

$$K_1O(k[t])\oplus K_1O(k)\xrightarrow{\eta} K_1O(k[t]/(t^2))$$
$$\to K_0O(R)\to K_0O(k[t])\oplus K_0O(k)\to K_0O(k[t]/(t^2))\to 0$$

Since $U(k[t]/(t^2))/U(k[t]/(t^2))^2 \xrightarrow{\sim} k^*/k^{*2}$, in view of Lemma 4.3, the map $K_1O(k)\to K_1O(k[t]/(t^2))$ induced by the inclusion $k\to k[t]/(t^2)$ is surjective and hence η is surjective. We therefore have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} O\to K_0O(R) & \to & K_0O(k[t])\oplus K_0O(k) & \to & K_0O(k[t]/(t^2)) & \to 0 \\ \theta\uparrow & & \phi\uparrow & & \psi\uparrow & \\ O\to K_0O(k) & \to & K_0O(k)\oplus K_0O(k) & \to & K_0O(k) & \to 0 \end{array}$$

the vertical maps being induced by the inclusions. From ([5], Theorem 1.1) it follows that ϕ is an isomorphism. Also, since $k[t]/(t^2)$ is local and $U(k[t]/(t^2))/U(k[t]/(t^2))^2 \xrightarrow{\sim} k^*/k^{*2}$ ψ is an isomorphism. Hence θ is an isomorphism.

PROPOSITION 4.5. *Let k be any field of characteristic $\neq 2$. Then, any quadratic space over $R=k[t^2,t^3]$ is isometric to $q_1\perp H(P)$ or $q_1\perp H(R^n)$, where q_1 is anisotropic and P a projective R-module of rank $\leqslant 1$.*

Proof. Since by Lemma 4.4, $K_0O(k)\xrightarrow{\sim} K_0O(R)$, any quadratic R-space q is stably extended from a quadratic space q_0 over k. Since $q\otimes_R k[t]$ is extended from q_0 and q is isotropic, it follows that q_0 is isotropic and hence contains a hyperbolic plane. Hence $q\otimes R_{\mathfrak{p}}$ contains a hyperbolic plane for each prime ideal $\mathfrak{p}$ of R. By Theorem 3.1, it follows that q contains a hyperbolic space $H(P)$, where P is a projective R-module of rank 1. Hence we may write any quadratic space q over R as $q_1\perp H(Q)$, where q_1 is anisotropic and Q a projective R-module. If rank $Q>1$, then $Q=P\oplus R^n$ with P a projective R-module of rank 1. We claim that $H(Q)\xrightarrow{\sim} H(R^{n+1})$. In fact, it suffices to show that $H(P\oplus R)\xrightarrow{\sim} H(R^2)$.

In fact, more generally, if R is any commutative noetherian ring of dimension 1 for which Pic R is 2 divisible, then, for any $P\in \mathrm{Pic}(R)$, $H(P\oplus R)\xrightarrow{\sim} H(R^2)$. For, if $P=Q\otimes_R Q$, with $Q\in \mathrm{Pic}\,R$, $P\oplus R\xrightarrow{\sim} Q\oplus Q$ and $H(Q\oplus Q)=((Q_1\oplus Q_2)\oplus(Q_1^*\oplus Q_2^*),\ h)$ where $Q_1\xrightarrow{\sim} Q$, $Q_2\xrightarrow{\sim} Q$. We have $Q_1\oplus Q_2^*\xrightarrow{\sim}(Q_1\otimes_R Q_2^*)\oplus R\xrightarrow{\sim} R^2$ is a totally isotropic direct summand of $H(Q\oplus Q)$. This completes the proof of Proposition 4.5.

COROLLARY 4.6. *Let k be a quadratically closed field of characteristic $\neq 2$. Then, any quadratic space of rank $\geqslant 3$ over $R=k[t^2,t^3]$ is extended from k.*

Proof. The underlying module of any discriminant module over R is free since $\mathrm{Pic}(R) \xrightarrow{\sim} k$ has no elements of order 2. Hence the underlying module of any quadratic space over R is free. Every quadratic space q over R of rank $\geqslant 2$ is isotropic, since it is stably extended from k by Lemma 4.4 and k is quadratically closed. Hence if rank $q \geqslant 4$, it follows from Proposition 4.5 that it is isometric to $\langle 1 \rangle \perp H(R^n)$ or $H(R^n)$ and is hence extended from k. Let now q be a quadratic space of rank 3 over R. Then, since $\mathrm{disc}\, q = 1$, (k being quadratically closed), it follows from ([8], Th. 4.9) that q is isometric to the orthogonal complement of 1 in (A, Nrd) where A is an Azumaya algebra of rank 4 over R. From Lemma 4.2 it follows that A is Brauer equivalent to $M_2(R)$ and Lemma 1.5 shows that $A \xrightarrow{\sim} M_2(R)$. Hence q is extended.

We conclude by showing that there exist non extended (anisotropic) quadratic spaces of all ranks $\geqslant 3$ over $\mathbf{R}[t^2, t^3]$, where $\mathbf{R}$ denotes the field of real numbers.

The exact sequence of $\mathbf{H}[X, Y]$-modules

$$0 \to P \to \mathbf{H}[X, Y]^2 \xrightarrow{\eta} \mathbf{H}[X, Y] \to 0 \tag{$*$}$$

defined by $\eta(1,0) = X + i$, $\eta(0, 1) = Y + j$ gives a projective $\mathbf{H}[X, Y]$-module P of rank 1 which is not free ([9], Proposition 1). The reduced norm on $\mathrm{End}_{\mathbf{H}[X,Y]} P$ or P give anisotropic quadratic spaces of rank 3 and 4 respectively and of discriminant 1 over $\mathbf{R}[X, Y]$ which are not extended from $\mathbf{R}$([8], Th. 4.6 and 4.9).

PROPOSITION 4.7. *The module* $\bar{P} = P \otimes_{\mathbf{H}[X,Y]} \mathbf{H}[X, Y]/(X^3 - Y^2)$ *remains non-free over* $\mathbf{H}[X, Y]/(X^3 - Y^2)$.

Proof. The projection of P on the first factor of $\mathbf{H}[X, Y]^2$ is a left ideal of $\mathbf{H}[X, Y]$ isomorphic to P. It is generated by ([11], p 143) $1 + iX + jY - kXY$ and $1 + Y^2$. If we identify $\mathbf{H}[X, Y]/(X^3 - Y^2)$ with the subring $\mathbf{H}[t^2, t^3]$ of $\mathbf{H}[t]$ by $X \to t^2$, $Y \to t^3$, we see that $\bar{P}$ is isomorphic to the left ideal $\mathfrak{a}$ of $H[t^2, t^3]$ generated by $1 + it^2 + jt^3 - kt^5$ and $1 + t^6$. We prove that this ideal is not principal which shows that $\bar{P}$ is not free. Suppose that $\mathfrak{a}$ is principal. Then it must be generated by an element whose degree is 2 or 3 since $\mathbf{H}[t^2, t^3]$ contains no linear polynomials.

Case 1. Let $\mathfrak{a}$ be generated by $1 + a_1t^2 + a_2t^3$ with $a_2 \neq 0$. Then $1 + it^2 + jt^3 - kt^5 = (1 + b_1t^2)(1 + a_1t^2 + a_2t^3)$ gives $a_1 + b_1 = i$, $a_2 = j$, $b_1a_1 = 0$, $b_1a_2 = -k$ which together imply $a_1 = 0$, $b_1 = i$ so that we find $k = ij = -k$, a contradiction.

Case 2. Let $\mathfrak{a}$ be generated by $1 + a_1t^2$, $a_1 \neq 0$. Then $1 + it^2 + jt^3 - kt^5 = (1 + b_1t^2 + b_2t^3)(1 + a_1t^2)$ gives $a_1 = i$. The equation

$$1 + t^6 = (1 + c_1t^2 + c_2t^3 + c_3t^4)(1 + it^2)$$

leads to $c_1=-i$, $c_2=0$, $c_1i+c_3=0$, $c_3i=1$, so that $c_3=-1=-i$, a contradiction once again. This proves the Proposition.

COROLLARY 4.8. *There exist (anisotropic) non extended quadratic forms of rank* $\geqslant 3$ *and discriminant* 1 *over* $\mathbf{R}[t^2, t^3]$.

Proof. The reduced norm on $\mathrm{End}_{\mathbf{H}[t^2,t^3]}\,\bar{P}$ gives a quadratic form of rank 4 and discriminant 1 over $\mathbf{R}[t^2, t^3]$. Since $\bar{P}$ is not free in view of Proposition 4.7, the orthogonal complement of 1 in $\mathrm{End}_{\mathbf{H}[t^2,t^3]}\,\bar{P}$ gives a non extended quadratic form of rank 3 ([8], Th. 4.9). Similarly the norm form on P gives a quadratic form of rank 4 and discriminant 1 over $\mathbf{R}[t^2, t^3]$ which is not extended. Non extended forms of rank $\geqslant 5$ can be constructed out of these as in ([8], Proposition 7.3).

Remark 1. Explicit non extended quadratic forms over $\mathbf{R}[t^2, t^3]$ of ranks 4 and 3 can be written down by substituting $X=t^2$, $Y=t^3$ in the non extended matrices over $\mathbf{R}[X, Y]$ constructed in [10] and [7]. By the method of [10], it can also be proved that there exist an infinity of mutually inequivalent quadratic spaces of rank 4 and discriminant 1 over $\mathbf{R}[t^2, t^3]$.

Remark 2. The ring $\mathbf{H}[t^2, t^3]$ is an example of a (non commutative) ring of dimension 1 over which we have the non-free but stably free projective module $\bar{P}$. In fact $\bar{P}\oplus\mathbf{H}[t^2, t^3]\xrightarrow{\sim}\mathbf{H}[t^2, t^3]^2$.

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