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## On multiple points of smooth immersions

FELICE RONGA

### §1. Introduction

Let  $f: V^n \rightarrow W^{n+r}$  be a smooth immersion, where  $V^n$  and  $W^{n+r}$  are smooth manifolds of dimension  $n$  and  $n+r$  respectively; we denote by  $V^{(k)}$  the  $k$ -fold product of  $V$ ,  $\Delta_V(k) = \{(x_1, \dots, x_k) \in V^{(k)} \mid \exists i \neq j \text{ with } x_i = x_j\}$ ,  $\delta_W(k) = \{(y, \dots, y) \in W^{(k)}\}$ . We shall say that  $f$  is regular if  $f^k: V^{(k)} \rightarrow W^{(k)}$  is transversal to  $\delta_W(k)$  outside  $\Delta_V(k)$ . This means that if  $f(x_1) = \dots = f(x_k) = y$ ,  $x_i \neq x_j$ , the vector spaces  $\text{Im}(df_{x_1}), \dots, \text{Im}(df_{x_k})$  are in general position in  $TW_y$ .

The following theorem has been proved by Ralph J. Herbert in his thesis [3]:

**1.1 THEOREM.** *Let  $f: V^n \rightarrow W^{n+r}$  be a regular proper immersion and set  $N_k = \{y \in W \mid \#(f^{-1}(y)) = k\}$ ,  $M_k = f^{-1}(N_k)$ . Then  $\bar{M}_k$  and  $\bar{N}_k$  carry fundamental classes over the integers modulo two; denoting by  $m_k$  and  $n_k$  their Poincaré duals in  $V$  and  $W$  respectively and by  $e = e(N_f)$  the Euler class of the normal bundle  $N_f$  of  $f$ , we have:*

$$m_k = f^*(n_{k-1}) - e \cdot m_{k-1} \tag{*}$$

*If  $r$  is even and  $V$  and  $W$  oriented,  $\bar{M}_k$  and  $\bar{N}_k$  carry fundamental classes over the integers, and the above formula is valid in integral cohomology.*

The fundamental classes are meant as in [2], §2.2.

#### Remarks.

(i) If  $r$  is even and  $N_f$  only is oriented, we still have integral dual classes, for which (\*) stays valid.

(ii) In proving (\*) we will exhibit minimal desingularisations of  $\bar{M}_k$  and  $\bar{N}_k$  which provide fundamental classes in bordism theory (oriented bordism if  $r$  is even and  $N_f$  oriented, complex bordism if  $N_f$  has a stable complex structure,

unoriented bordism otherwise). In the corresponding cobordism theories (\*) still holds

(iii) From (\*) we deduce:

$$m_k = \sum_{j=0, \dots, k-1} (-1)^j e^j f^*(n_{k-1-j})$$

In particular if  $W = \mathbf{R}^{n+r}$ ,  $m_k = (-1)^{k-1} e^{k-1}$ . This recovers the formula for triple points of immersed surfaces in  $\mathbf{R}^3$  given in [1].

(iv) When  $r$  is even and  $N_f$  oriented, the orientations we shall give for the dual classes to  $\bar{M}_k$  and  $\bar{N}_k$  are such that  $f_!(m_k) = k \cdot n_k$ , where  $f_! : H^*(V) \rightarrow H^*(W)$  is the Gysin homomorphism associated to  $f$ . Defining  $\varphi_h : H^*(V) \rightarrow H^*(V)$  by  $\varphi_h(a) = f^* f_!(a) - h(e \cdot a)$ , we deduce from (\*):

$$(k-1)! m_k = \varphi_{k-1} \cdot \varphi_{k-2} \cdot \dots \cdot \varphi_1(1)$$

Herbert's theorem corrects a formula given in [4]. The purpose of this note is to give a simple proof of (\*). My contribution is the idea of proving (\*) using Proposition 2.2 below, which is a generalization of a proposition of D. Quillen ([5], prop. 3.3).

Particular cases of (\*) were known before Herbert's thesis. In [7], p. 131, H. Whitney shows that  $m_2 = f^* f_!(1) - e$ ; Herbert's method for proving (\*) appears to be a generalisation of Whitney's method, which also inspired our approach. By different methods, the case of triple points of surfaces in  $\mathbf{R}^3$  is treated in [1] and [6] deals with the number of triple points of an immersion  $V^{4n} \rightarrow \mathbf{R}^{6n}$ .

## §2. Proofs

We adopt the following notations: a smooth map  $\alpha : A \rightarrow X$  means a  $C^\infty$  map between  $C^\infty$  manifolds.  $TA$  denotes the tangent bundle of  $A$ ,  $N_\alpha = \alpha^*(TX) - TA$  the virtual normal bundle of  $\alpha$ ; if  $\alpha$  is an immersion,  $N_\alpha$  denotes the genuine normal bundle of  $\alpha$ , namely  $\alpha^*(TX)/d\alpha(TA)$ , where  $d\alpha : TA \rightarrow \alpha^*TX$  denotes the derivative of  $\alpha$ .

Let  $f : V^n \rightarrow W^{n+r}$  be a smooth regular proper immersion. We set:

$$- N_k(f) = \{y \in W \mid \#(f^{-1}(y)) = k\}, \quad M_k(f) = f^{-1}(N_k)$$

$$- \hat{M}_k(f) = \{(x_1, \dots, x_k) \in V^{(k)} - \Delta_V(k) \mid f(x_i) = f(x_j)\}$$

The group of permutations of  $k$  objects  $S_k$  acts fixed-point free on  $\hat{M}_k(f)$  in the obvious way.

$$- \tilde{N}_k(f) = \hat{M}_k/S_k, \quad \tilde{M}_k(f) = \hat{M}_k/S_{k-1},$$

where  $S_{k-1}$  acts on the last  $k - 1$  coordinates.

We write  $[x_1, \dots, x_k]$ , resp.  $(x_1, [x_2, \dots, x_k])$  for the class of  $(x_1, \dots, x_k) \in \hat{M}_k$  in  $\tilde{N}_k$ , resp.  $\tilde{M}_k$ . We define  $f_k : \tilde{M}_k \rightarrow V$ ,  $f_k(x_1, [x_2, \dots, x_k]) = x_1$  and  $g_k : \tilde{N}_k \rightarrow W$ ,  $g_k([x_1, \dots, x_k]) = f(x_1) (= f(x_2) = \dots = f(x_k))$ . We set  $\tilde{M}_k^0 = f_k^{-1}(M_k)$ ,  $\tilde{N}_k^0 = g_k^{-1}(N_k)$ . Recall that  $N_f^{(k)}$  denotes the  $k$ -fold product of  $N_f$ .

2.1 LEMMA.

(i)  $f_k$  and  $g_k$  are proper immersions with normal bundles  $N_{g_k} = (N_f^{(k)} | \hat{M}_k)/S_k$  and  $N_{f_k} = (0 \times N_f^{(k-1)} | \hat{M}_k)/S_{k-1}$ .

(ii)  $\tilde{M}_k^0$  and  $\tilde{N}_k^0$  are open dense in  $\tilde{M}_k$  and  $\tilde{N}_k$  respectively,  $f_k | \tilde{M}_k^0 : \tilde{M}_k^0 \rightarrow M_k$  and  $g_k | \tilde{N}_k^0 : \tilde{N}_k^0 \rightarrow N_k$  are diffeomorphisms.

(iii)  $f_k(\tilde{M}_k) = \bar{M}_k = \bigcup_{h \geq k} M_h$ ,  $g_k(\tilde{N}_k) = \bar{N}_k = \bigcup_{h \geq k} N_h$ .

*Proof.* Since  $\hat{M}_k = (f^k)^{-1}(\delta_W(k)) - \Delta_V(k)$ , we deduce from the transversality of  $f^k$  to  $\delta_W(k)$  outside  $\Delta_V(k)$  that  $T(\hat{M}_k)_{(x_1, \dots, x_k)} = \{(v_1, \dots, v_k) \in T(V)_{(x_1, \dots, x_k)}^{(k)} | df_{x_i}(v_i) = df_{x_j}(v_j)\}$ . So,  $v_1 = 0$  implies  $v_2 = \dots = v_k = 0$ . Hence  $f_k$  and  $g_k$  are immersions; it is easily seen that their normal bundles are as stated.

Let us check that  $\hat{M}_k$  is closed in  $V^{(k)}$ : if not, there are sequences  $\{x_1^h\}$ ,  $\{x_2^h\} \subset V$ ,  $f(x_1^h) = f(x_2^h)$ ,  $x_1^h \neq x_2^h$ , with  $\lim_{h \rightarrow \infty} (x_1^h) = \lim_{h \rightarrow \infty} (x_2^h) = x$ . We write  $f$  in local coordinates as a map  $f : \mathbf{R}^n \rightarrow \mathbf{R}^{n+r}$ ; we can assume that  $x_1^h - x_2^h / \|x_1^h - x_2^h\|$  tends to  $v \in \mathbf{R}^n$ ,  $\|v\| = 1$ . But then  $df_x(v) = 0$  and  $f$  is no longer an immersion. Hence  $\tilde{M}_k$  and  $\tilde{N}_k$  are closed in  $V^{(k)}/S_{k-1}$  and  $V^{(k)}/S_k$  respectively and since  $f$  is proper we deduce that  $f_k$  and  $g_k$  are proper. This proves (i). The assertions (ii) and (iii) follow from the fact that  $f_k$  and  $g_k$  are proper and, using the implicit function theorem, by writing  $f$  locally as a linear map.

We digress now to sub-cartesian diagrams; they generalize the notion of clean intersection of Quillen ([5], §3), which concerns the case when  $\alpha$  and  $\beta$  below are embeddings.

DEFINITION. The diagram of smooth proper immersions:

$$\begin{array}{ccc} Z & \xrightarrow{f_B} & B \\ f_A \downarrow & & \downarrow \beta \\ A & \xrightarrow{\alpha} & X \end{array}$$

is said to be sub-cartesian if:

(i)  $f_A \times f_B : Z \rightarrow A \times B$  is an embedding onto  $A \times_X B = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ .

(ii) the following sequence is exact:

$$0 \rightarrow TZ \xrightarrow{d(f_A \times f_B)} f_A^*TA \times f_B^*TB \xrightarrow{(d\alpha, -d\beta)} f_A^*\alpha^*TX$$

where  $(d\alpha, -d\beta)$  is meant to send  $(v, w) \in (f_A^*TA \times f_B^*TB)_z$  to  $d\alpha(v) - d\beta(w)$ . The vector bundle  $E = f_A^*\alpha^*TX / \text{Im}(d(f_A \times f_B))$  over  $Z$  is called the excess vector bundle.

*Remarks.*

(i) The above diagram is cartesian if and only if  $E$  is the zero bundle.

(ii) We have not assumed  $Z$  to have constant dimension, hence  $E$  won't have constant rank in general.

(iii) The above condition (ii) is equivalent to say that if for  $a \in A$  and  $b \in B$  we choose open neighbourhoods  $A'$  and  $B'$  respectively such that  $\alpha \mid A'$  and  $\beta \mid B'$  are embeddings, then  $\alpha(A') \cap \beta(B')$  is a sub-manifold of  $X$  and  $T(\alpha(A') \cap \beta(B')) = T(\alpha(A')) \cap T(\beta(B'))$ . This is to say that  $\alpha(A')$  and  $\beta(B')$  intersect cleanly in  $X$  in the terminology of [5].

**2.2 PROPOSITION.** For  $c \in H^*(B)$  we have:

$$\alpha^*\beta_!(c) = f_{A!}(e(E) \cdot f_B^*(c))$$

where  $e$  denotes the Euler class,  $\beta_!$  and  $f_{A!}$  are the Gysin homomorphisms associated to those maps. The cohomology is taken over the integers whenever  $N_\beta$  and  $E$  are oriented, the integers modulo two otherwise. (The proposition and its proof remain valid in any generalized cohomology theory in which  $N_\beta$  and  $E$  have orientations.)

*Proof.* We replace  $Z$  by its image in  $A \times B$ , still denoted by  $Z$ . We provide  $TX$  with a metric and identify  $E$  with the orthogonal to  $\text{Im}(d(f_A \times f_B \mid_Z))$  in  $f_A^*\alpha^*TX$ . Let  $e : TX \rightarrow X$  be the exponential mapping associated to the metric; for  $x \in X$  there is an open neighbourhood  $U_x$  of  $0 \in TX_x$  such that  $e_x = e \mid U_x$  is a diffeomorphism onto an open neighbourhood of  $x$  in  $X$ . Let  $\Omega$  be a closed tubular neighbourhood of  $Z$  in  $A \times B$ ; it is a manifold with boundary  $\partial\Omega$ . If  $\Omega$  is small enough, for  $(a, b) \in \Omega$  we have  $b \in e_{\alpha(a)}(U_{\alpha(a)})$ . Let  $v : Z \rightarrow E$  be a section transversal to the zero section and denote by  $\bar{E}$  and  $\bar{v}$  extensions of  $E$  and  $v$  to  $\Omega$ , with

$\bar{E}$  still a sub-bundle of  $TX' = p_A^* \alpha^*(TX) | \Omega$ , where  $p_A : \Omega \rightarrow A$  denotes the obvious projection. Define the section  $w : \Omega \rightarrow TX'$  by  $w(a, b) = e_{\alpha(a)}^{-1}(\beta(b))$ , and the section  $\bar{w} : \Omega \rightarrow TX'$  by  $\bar{w} = w + \bar{v}$ . If  $\Omega$  is small enough,  $w(a, b) \notin \bar{E}_{(a,b)}$  for  $(a, b) \notin Z$  and hence, setting  $Z'' = \{(a, b) \in \Omega \mid \bar{w}(a, b) = 0\}$ , we have  $z'' = \{(a, b) \in Z \mid v(a, b) = 0\}$ .

It follows from the exact sequence (ii) of the definition of a sub-cartesian diagram that  $\bar{w}$  is transversal to the zero section in  $TX'$ . Hence the map  $F : \Omega \rightarrow X \times X$ ,  $F(a, b) = (e_{\alpha(a)}(\bar{w}(a, b)), \beta(b))$  is transversal to  $\Delta_X$  and  $F^{-1}(\Delta_X) = Z''$  if  $v$  has been chosen near enough the zero section in  $E$ .

Let  $\alpha' : A \rightarrow X$  be near  $\alpha$  such that  $\alpha' \times \beta : A \times B \rightarrow X \times X$  is transversal to  $\Delta_X$  and set  $Z' = (\alpha' \times \beta)^{-1}(\Delta_X)$ . The following diagram is cartesian:

$$\begin{array}{ccc}
 Z' & \xrightarrow{f_B} & B \\
 f'_A \downarrow & & \downarrow \beta \\
 A & \xrightarrow{\alpha'} & X
 \end{array}$$

where  $f'_A$  and  $f'_B$  are the obvious projections; hence  $\alpha'^* \beta_!(c) = f'_{A!} f'_B{}^*(c)$ . If  $\alpha'$  is near enough to  $\alpha$ ,  $F' = (\alpha' \times \beta) | \Omega$  and  $F$  are homotopic through maps transversal to  $\Delta_X$  and sending  $\partial\Omega$  into  $X \times X - \Delta_X$ . Hence there is an isotopy of  $\Omega$  leaving  $\partial\Omega$  fixed and sending  $Z'$  onto  $Z''$ .

Consider the inclusions  $i : Z \subset \Omega$ ,  $i' : Z' \subset \Omega$ ,  $i'' : Z'' \subset \Omega$ ,  $j : Z'' \subset Z$ , the projection  $p_A : \Omega \rightarrow A$  and the associated Gysin homomorphisms  $i_! : H^*(Z) \rightarrow H^*(\Omega, \partial\Omega)$ , similarly for  $i'_!$  and  $i''_!$ , and  $p_{A!} : H^*(\Omega, \partial\Omega) \rightarrow H^*(A)$ . Since  $Z'$  and  $Z''$  are isotopic in  $\Omega$  rel.  $\partial\Omega$ ,  $i'_! f_B{}^* = i''_!(f_B j)^*$ . Also, since  $Z''$  is the set of zeroes of  $v : Z \rightarrow E$  which is transversal to the zero section,  $j_!(1) = e(E)$ . Hence, using that  $f'_A = p_{A!} i'_!$ ,  $i'' = ij$ ,  $f_A = p_{A!} i_!$  and  $j_!(j^*(x)) = j_!(1) \cdot x : \alpha^* \beta_!(c) = \alpha'^* \beta_!(c) = f'_{A!} f'_B{}^*(c) = p_{A!} i'_! f_B{}^*(c) = p_{A!} i'_! j^* f_B{}^*(c) = p_{A!} i_! j_! j^* f_B{}^*(c) = (p_{A!} i_!)(j_!(1) \cdot f_B{}^*(c)) = f_{A!}(e(E) \cdot f_B{}^*(c))$ .

*Proof of 1.1.* Consider the diagram:

$$\begin{array}{ccc}
 \tilde{M}_k \cup \tilde{M}_{k-1} & \xrightarrow{p} & \tilde{N}_{k-1} \\
 f_k \cup f_{k-1} \downarrow & & \downarrow g_{k-1} \\
 V & \xrightarrow{f} & W
 \end{array}$$

where  $p(x_1, [x_2, \dots, x_k]) = [x_2, \dots, x_k]$ ,  $p(x_1, [x_2, \dots, x_{k-1}]) = [x_1, \dots, x_{k-1}]$ . It

follows from the transversality of  $f^k : V^{(k)} - \Delta_V(k) \rightarrow W^{(k)}$  to  $\delta_W(k)$  that the above diagram is sub-cartesian, the excess bundle being zero on  $\tilde{M}_k$  and  $f_{k-1}^*(N_f)$  on  $\tilde{M}_{k-1}$ . From 2.1 we deduce that  $f_{k*}([\tilde{M}_k])$  and  $g_{k*}([\tilde{N}_k])$ , where  $[ ]$  denotes the fundamental class, are fundamental classes for  $\bar{M}_k$  and  $\bar{N}_k$  respectively, for which  $m_k = f_{k!}(1)$ ,  $n_k = g_{k!}(1)$ . Applying 2.2 to the above diagram with  $c = 1$  we get:

$$f^*(n_{k-1}) = f^*(g_{k-1!}(1)) = f_{k!}(1) + f_{k-1!}(f_{k-1}^*(e(N_f))) = m_k + e \cdot m_{k-1}.$$

If  $r$  is even and  $N_f$  oriented, the induced orientation on  $N_f^{(k)} | \hat{M}_k^0$  is invariant by the action of  $S_k$  and 2.1 (i) shows that  $N_{f_k \cup f_{k-1}}$  and  $N_{g_{k-1}}$  are oriented. The above calculations hold in integral cohomology. If  $W$  is not orientable,  $m_k$  and  $n_k$  can be interpreted as follows. Let  $\theta_W$  denote the sheaf of orientations of  $W$ ; then  $f^*(\theta_W) = \theta_V$  since  $N_f$  is oriented, and also  $f_k^*(\theta_V) = \theta_{\tilde{M}_k}$ ,  $g_k^*(\theta_W) = \theta_{\tilde{N}_k}$ . Letting  $[ ]$  denote the fundamental class with twisted coefficients, we have that  $f_{k*}([M_k])$  and  $g_{k*}([N_k])$  are fundamental classes for  $\bar{M}_k$  and  $\bar{N}_k$  respectively with twisted coefficients, whose Poincaré duals are  $m_k = f_{k!}(1)$  and  $n_k = g_{k!}(1)$ .

In the terminology of [7], the above considerations amount roughly to say that the homological intersection of  $f(V)$  and  $\bar{N}_{k-1}$  in  $W$  consists of the “far intersection” (that is  $\bar{M}_k$ ) plus the “near intersection” (that is the set of zeroes of a section of the non-zero part of the excess bundle).

### §3. Divisibility conditions

**3.1 PROPOSITION.** *If the compact oriented manifold  $V^{4pr}$  immerses in  $\mathbf{R}^{4pr+2r}$ ,  $\bar{P}_r(V)^p$  is divisible by  $2p + 1$ , where  $\bar{P}_r(V)$  denotes the  $r$ -th Pontriagin class of the stable normal bundle of  $V$ .*

*Proof.* Let  $f : V^{4pr} \rightarrow \mathbf{R}^{4pr+2r}$  be an immersion; after perturbing it slightly we can assume it to be regular. Then  $M_{2p+1}$  consists of isolated points whose number equals  $m_{2p+1}$  evaluated on  $[V]$ ; since  $e(N_f)^2 = \bar{P}_r$ , by 1.1  $m_{2p+1} = (-1)^{2p+1} \cdot \bar{P}_r(V)$ . If  $x_1, \dots, x_{2p+1} \in V$  are distinct and  $f(x_1) = \dots = f(x_{2p+1}) = y$ , the orientation we have given to  $N(f_{2p+1})$  shows that they are all counted with the same sign, say  $\epsilon_y$ . Hence  $(-1)^{2p+1} \cdot \bar{P}_r(V)$  evaluated on  $[V]$  equals  $(\sum_{y \in N_{2p+1}} \epsilon_y) \cdot (2p + 1)$ .

For example, if  $V^{4n}$  immerses in  $\mathbf{R}^{4n+2}$ ,  $P_1^n$  is divisible by  $2n + 1$ . (The case  $n = 1$  was considered by J. H. White in [6]). If  $V^{12}$  immerses in  $\mathbf{R}^{18}$ ,  $\bar{P}_3 = P_1^3 - 2P_1P_2 + P_3$  is divisible by 3. If  $V^{16}$  immerses in  $\mathbf{R}^{20}$ ,  $(P_1^2 - P_2)^2$  is divisible by 5.

In fact 3.1 is probably a consequence of the integrality of the  $L$ -genus, taking inaccount that  $\bar{P}_i = 0$  for  $i > r$ .

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