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Poincaré duality groups of dimension two

BENO ECKMANN and HEINZ MÜLLER

In this paper we prove that 2-dimensional Poincaré duality groups with positive first Betti number β_1 are surface groups. As a corollary it follows that a connected Poincaré 2-complex with $\beta_1 > 0$ is homotopy equivalent to a closed surface, and so is any finite connected Poincaré 2-complex.

1. Statement of algebraic results

1.1. A Poincaré duality group of dimension n , in short PD^n -group, is a group G acting on \mathbf{Z} such that there are natural duality isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \mathbf{Z} \otimes A) \quad (1)$$

for all integers k and all G -modules A (where G acts diagonally on $\mathbf{Z} \otimes A$); the isomorphisms (1) can be given by the cap-product $e \cap -$ with an element $e \in H_n(G; \mathbf{Z})$ called fundamental class. If (1) holds, the “formal dimension” n (= cohomology dimension of G) and the G -module \mathbf{Z} ($\cong H^n(G; \mathbf{Z}G)$) are determined by G . A PD^n -group G is called orientable or non-orientable according to whether \mathbf{Z} is a trivial G -module or not.

The fundamental group $\pi_1(M^n)$ of a closed connected aspherical n -dimensional manifold is a PD^n -group. In particular, if M^2 is a closed surface of genus ≥ 1 , then $\pi_1(M^2)$ is a PD^2 -group. We will call such a group $\pi_1(M^2)$ a “surface group”; it admits a finite presentation of well-known canonical type. It has been conjectured that these surface groups are the only PD^2 -groups. We will show that this is so except in a very special case which remains open.

1.2. From general arguments [5], [2] it is known that PD^n -groups are of type (FP); this means that there exists a $\mathbf{Z}G$ -projective resolution of the trivial G -module \mathbf{Z} , of finite length and finitely generated over $\mathbf{Z}G$. In particular, a PD^n -group G is finitely generated, and its Betti numbers $\beta_i(G)$ and the Euler characteristic $\chi(G) = \sum_{i=0}^n (-1)^i \beta_i$ are defined. Our main result is

THEOREM 1. *Let G be a PD^2 -group with $\beta_1(G) > 0$. Then G is a surface group.*

The condition $\beta_1(G) > 0$ means, in the orientable case, that $\beta_1(G)$ is an even integer ≥ 2 ; in the non-orientable case, an integer ≥ 1 . Thus $\beta_1(G) > 0$ is equivalent to $\chi(G) \leq 0$ (since $\chi(G) = 2 - \beta_1(G)$ in the orientable, $1 - \beta_1(G)$ in the non-orientable case). If G is non-orientable, it contains an orientable PD^2 -group G_1 as subgroup of index 2. By the multiplicative property of the Euler characteristic (which holds for groups of type (FP), cf. [6]) one has $\chi(G_1) = 2\chi(G)$; hence $\beta_1(G) > 0$ if and only if $\beta_1(G_1) > 0$.

1.3. A group G is said to be of type (FF) if it admits a $\mathbf{Z}G$ -free resolution of finite length and finitely generated over $\mathbf{Z}G$. Obviously surface groups are of type (FF). It is not known whether there exist groups of type (FP) which are not of type (FF).

COROLLARY 1. *A PD^2 -group G of type (FF) is a surface group.*

Proof. We first assume G orientable. Then the method of proof used by J. Cohen [7] is valid for any (FF)-resolution and shows that the assumption $\beta_1(G) = 0$ (i.e. $H_1(G; \mathbf{Z}) = 0$) leads to a contradiction. Hence $\beta_1(G) > 0$, and the assertion follows from Theorem 1.

If G is non-orientable, let G_1 be the orientable subgroup of index 2; it is also of type (FF), and thus $\beta_1(G_1) > 0$. The Euler characteristic argument above then shows that $\beta_1(G) > 0$.

1.4. We thus see that the case $\beta_1(G) = 0$ not covered by Theorem 1 is equivalent to the existence of a PD^2 -group G not of type (FF), but of course of type (FP). We further note that, by Theorem 1, the condition $\beta_1(G) > 0$ not only implies type (FF) but also finite presentability.

1.5. A further corollary concerns the “Nielsen conjecture” for surface groups.

COROLLARY 2. *Let G be a torsion-free group containing a surface group G_1 as a subgroup of finite index. Then G itself is a surface group.*

Proof. Any torsion-free group G containing a PD^2 -group G_1 as subgroup of finite index is itself a PD^2 -group (cf. [1], [2]). Since $\beta_1(G_1) > 0$, i.e., $\chi(G_1) \leq 0$, the multiplicative property of the Euler characteristic, $\chi(G_1) = |G : G_1| \chi(G)$, yields $\chi(G) \leq 0$. Hence $\beta_1(G) > 0$, and the assertion follows from Theorem 1.

1.6. The relative analogue of a PD^n -group is a PD^n -pair, cf. Bieri-Eckmann [3]. A group pair $(G; S_0, S_1, \dots, S_m)$ consists of a group G and a family of subgroups $\underline{S} = (S_0, S_1, \dots, S_m)$, $m \geq 0$; it is called a PD^n -pair if for some G -action on \mathbf{Z} there are duality isomorphisms between the cohomology of G and the relative homology of $(G; \underline{S})$, analogous to (1) and also given by the cap product

$e \cap -$ with a fundamental class $e \in H_n(G, \mathbb{S}; \mathbb{Z})$. The duality is, of course, of exactly the same form as that of compact manifolds-with-boundary. Examples of PD^2 -pairs are obtained by taking for G the fundamental group of a closed surface with $m+1$ discs removed ($m \geq 0$, and $m \geq 1$ if the surface is the sphere) together with the family of infinite cyclic subgroups generated by the circles bounding the discs. These PD^2 -pairs of groups are called “geometric”.

THEOREM 2. *All PD^2 -pairs of groups are geometric.*

This result is actually a consequence of Corollary 1. Indeed it is shown in [3] that it is implied by the assertion that one-relator PD^2 -groups are surface groups. Since one-relator PD^2 -groups are of type (FF), Corollary 1 tells that this is the case.

However, Theorem 2 will be used in the proof of Theorem 1 and therefore requires a direct proof.

1.7. The proof of Theorem 2 will be given in Section 4, of Theorem 1 in Section 5. In Section 3 we describe the procedure of proof and list some auxiliary results, in particular the “decomposition theorems for group pairs” (H. Müller [10]). Section 2 deals with the topological aspect.

2. Topological application: Poincaré 2-complexes

2.1. A Poincaré n -complex is a CW-complex X dominated by a finite complex and fulfilling Poincaré duality for arbitrary local coefficients, with respect to a dualizing $\pi_1(X)$ -module \mathbb{Z} and a formal dimension n . We will always assume here that it is *connected*.

The study of Poincaré complexes was initiated by Wall in the 60-s. In [15] Wall proved, in particular, that if X is a Poincaré 2-complex with $\pi_1(X)$ finite, then X is homotopy equivalent to S^2 or $\mathbb{R}P^2$; if $\pi_1(X)$ is infinite, then X is aspherical, i.e., it is an Eilenberg-Mac Lane complex $K(G, 1)$ for $G = \pi_1(X)$. In the latter case the investigation is thus reduced to the study of finitely presented PD^2 -groups. Later J. Cohen [7] showed that if X is a *finite* Poincaré 2-complex with $\beta_1(X) = 0$ then the conclusion is the same as for $\pi_1(X)$ finite; and that a Poincaré 2-complex X with $\beta_1(X) = 1$ or 2 is homotopy equivalent to the appropriate closed surface.

2.2. As a consequence of Theorem 1 we obtain

COROLLARY 3. *Let X be a Poincaré 2-complex with $\beta_1(X) > 0$. Then X is homotopy equivalent to a closed surface (of genus ≥ 1).*

Indeed, since $\beta_1(X) > 0$ implies that $\pi_1(X)$ is infinite, $G = \pi_1(X)$ is a PD^2 -group with $\beta_1(G) > 0$ and thus isomorphic to $\pi_1(Y)$, where Y is a closed surface of genus ≥ 1 . The isomorphism provides a homotopy equivalence between $X = K(G, 1)$ and Y .

COROLLARY 4. *A finite Poincaré 2-complex X is homotopy equivalent to a closed surface.*

Proof. If $\pi_1(X)$ is finite, one applies Wall's result mentioned above. If $\pi_1(X) = G$ is infinite, then G is a PD^2 -group of type (FF) , hence isomorphic to a surface group by Corollary 1. Thus $X = K(G, 1)$ is homotopy equivalent to a closed surface.

2.3. Thus all Poincaré 2-complexes X are homotopy equivalent to closed surfaces, except possibly if (a) $\pi_1(X)$ is infinite and $\beta_1(X) = 0$, and (b) X is not homotopy equivalent to a finite complex. Note that each of properties (a) and (b) implies the other. Except for finite presentability of $G = \pi_1(X)$ this exceptional possibility is exactly the same as the case not covered by Theorem 1, cf. 1.4.

3. Splitting of groups and group pairs

3.1. A group G is said to *split over a subgroup H* if it is either (α) an amalgamated free product $G = G_1 *_H G_2$, $G_1 \neq H \neq G_2$ or (β) an HNN-extension $G = G_1 *_H G_1$. Cases where H is finitely generated or even finite will be of special importance.

If G is a PD^2 -group with $\beta_1(G) > 0$ then G admits an infinite cyclic factor group (infinite cyclic groups will be denoted by C in the following, or by $C(g)$ if we want to emphasize a generator g). Since G is of type (FP) , it is “almost finitely presented”. By a theorem of Bieri–Strebel [4], any almost finitely presented group admitting a factor group C splits over a *finitely generated* group L (by a splitting (β)). Thus Theorem 1 is a consequence of

THEOREM 1'. *Let G be a PD^2 -group which splits over a finitely generated subgroup L . Then G is a surface group.*

If one confines attention to *finitely presented* PD^2 -groups only (e.g., in the context of Poincaré 2-complexes or of the Nielsen conjecture), the Bieri–Strebel argument can be replaced by a somewhat simpler one which is just a modification of Moldavanskii's method [9]; cf. Eckmann–Müller [8].

3.2. The proof of Theorem 1' will proceed as follows. By Strebel's theorem [13] the subgroup L , being of infinite index in G , is free. If the rank of L is > 1 , the splitting can be changed so as to become a splitting of G over a subgroup of smaller rank. One is thus reduced to the case where $L = C$ is infinite cyclic. Then the group pairs $(G_1; C)$ and $(G_2; C)$ in case (α) , or $(G_1; C, p^{-1}Cp)$ in case (β) , are PD^2 -pairs; this follows from general results on PD^n -groups and -pairs (Bieri-Eckmann [3]). By our Theorem 2 these PD^2 -pairs are geometric, which easily implies that $G = G_1 *_C G_2$, or $G = G_1 *_{C,p}$ respectively, is a surface group.

3.3. Both the reduction process above and the proof of Theorem 2 are based on "decomposition theorems for group pairs" (H. Müller [10]). For the convenience of the reader we summarize the appropriate definitions and those results which are needed.

In this context, a splitting of G is understood to be over a *finite* subgroup K . A group pair $(G; S_1, S_2, \dots, S_m)$, $m \geq 0$, and a splitting (α) $G = G_1 *_K G_2$ or (β) $G = G_1 *_{K,p}$ are said to be *adapted to each other* if each S_j , $j = 1, \dots, m$ is conjugate to a subgroup of G_1 or G_2 . If for $(G; S_1, S_2, \dots, S_m)$ such a splitting of G exists we simply say that the pair is adapted. If G is finitely generated, the pair $(G; S_1, \dots, S_m)$ is adapted if and only if $\bigcap_{j=1}^m N_j \neq 0$, where N_j is the kernel of the restriction map $\text{res}_j: H^1(G; \mathbf{Z}G) \rightarrow H^1(S_j; \mathbf{Z}G)$. This is just a restatement of Swarup's relative version of Stallings' structure theorem for finitely generated groups with more than one end.

In the following we assume that $(G; S_1, \dots, S_m)$ is an adapted pair and that G is finitely generated. With respect to the pair $(G; S_1, \dots, S_m)$ a number $n(T)$, called *weight* of T , is associated with every subgroup T of G . The definition uses the restriction map

$$\text{res}: H^1(G; \mathbf{Z}G) \rightarrow H^1(T; \mathbf{Z}G).$$

For simplicity we only consider the case where T is finitely generated. We regard $H^1(T; \mathbf{Z}T)$ as T -submodule of the (right) G -module $H^1(T; \mathbf{Z}G)$ (the embedding is induced by the inclusion $\mathbf{Z}T \rightarrow \mathbf{Z}G$). Since T is finitely generated, we have a decomposition (as abelian group)

$$H^1(T; \mathbf{Z}G) = \bigoplus_{x_i \in G/T} H^1(T; \mathbf{Z}T)x_i$$

(see, e.g., [2] Proposition 5.3).

DEFINITION. The weight $n(T)$ is the minimal number of non-trivial components of $\text{res}(c) \in \bigoplus_{x_i \in G/T} H^1(T; \mathbf{Z}T)x_i$ for all $c \in \bigcap_{j=1}^m N_j$, $c \neq 0$.

3.4. For different values of $n(T)$ various types of a *simultaneous* splitting of G and a graph-decomposition of T are obtained. We describe here only two special cases (Corollaire 2 and Corollaire 5 of [11]). In the statements the splitting $G = G_1 * G_2$ or $G = G_1 *_{e,p}$ written $G * \langle p \rangle$, is always meant to be adapted to the pair $(G; S_1, \dots, S_m)$.

THEOREM A. Assume that T is torsion-free and $n(T) = 1$. Then we have one of the following cases

- 1) $G = G_1 * G_2$, $T = T_1 * T_2$, $T_1 \subset G_1$, $T_2 \subset G_2$;
- 2) $G = G_1 * \langle p \rangle$, $T = T_1 * p T_2 p^{-1}$, $T_1, T_2 \subset G_1$;
- 3) $G = \langle p \rangle$, $T = C(p)$, $S_1 = \dots = S_m = e$ or $m = 0$.

THEOREM B. Assume that G is torsion-free, T infinite cyclic and $n(T) = 2$. Then we have one of the following cases

- 1) $G = G_1 * G_2$, $T = C(g_1 g_2)$, $e \neq g_i \in G_i$, $i = 1, 2$;
- 2) $G = G_1 * \langle p \rangle$, $T = C(p g_1 p^{-1} g_2)$, $e \neq g_1, g_2 \in G_1$;
- 3) $G = \langle p \rangle$, $T = C(p^2)$, $S_1 = \dots = S_m = e$ or $m = 0$.

4. Proof of Theorem 2

4.1. Let $(G; S_0, S_1, \dots, S_m)$, $m \geq 0$, in short $(G; \underline{S})$, be a PD^2 -pair. G acts on \mathbf{Z} , and there is a fundamental class $e \in H_2(G, \underline{S}; \mathbf{Z})$ such that

$$e \cap - : H^k(G; A) \rightarrow H_{2-k}(G, \underline{S}; \mathbf{Z} \otimes A) \quad (2)$$

is an isomorphism for all k and A . The *geometric* PD^2 -pairs (cf. 1.6) are as follows:

Orientable case

- (3) G is freely generated by $t_1, \dots, t_m, x_1, y_1, \dots, x_g, y_g$, $m + g > 0$,
 S_1, \dots, S_m are generated by conjugates to t_1, \dots, t_m and S_0 is generated
 by $t_1 \cdots t_m \cdot \prod_{i=1}^g [x_i, y_i]$.

Non-orientable case

- (4) G is freely generated by $t_1, \dots, t_m, z_0, \dots, z_g$, $m \geq 0$, $g \geq 0$,
 S_1, \dots, S_m are generated by conjugates to t_1, \dots, t_m and S_0 is generated
 by $t_1 \cdots t_m \cdot \prod_{i=0}^g z_i^2$.

4.2. By Theorem 4.2 and 9.3 of [3] we know that a PD^2 -pair $(G; S_0, S_1, \dots, S_m)$ consists of a finitely generated free group G and a family $\underline{S} = (S_0, S_1, \dots, S_m)$ of cyclic subgroups. Moreover, the fundamental class $e \in H_2(G; \underline{S}; \mathbf{Z})$ determines fundamental classes e_i for the PD^1 -groups S_0, \dots, S_m , namely the components of $\partial e \in H_1(\underline{S}; \mathbf{Z}) = \bigoplus_{i=0}^m H_1(S_i; \mathbf{Z})$, where ∂ is the connecting homomorphism in the exact homology sequence of G modulo \underline{S} . By [3], Theorem 2.1 one has the following commutative diagram

$$\begin{array}{ccccc}
 0 \rightarrow H^1(G; \mathbf{Z}G) & \xrightarrow{\{\text{res}_i\}} & \bigoplus_{i=0}^m H^1(S_i; \mathbf{Z}G) & \xrightarrow{\delta} & H^2(G, \underline{S}; \mathbf{Z}G) \rightarrow 0 \\
 & & \cong \downarrow \{e_i \cap -\} & & \cong \downarrow (e \cap -) \\
 & & \bigoplus_{i=0}^m H_0(S_i; \mathbf{Z} \otimes \mathbf{Z}G) & \xrightarrow{\text{cor}} & H_0(G; \mathbf{Z} \otimes \mathbf{Z}G) \\
 & & \cong \downarrow j & & \cong \downarrow \\
 & & \bigoplus_{i=0}^m (\mathbf{Z} \otimes_{S_i} \mathbf{Z}G) & \xrightarrow{p} & \mathbf{Z}
 \end{array} \tag{5}$$

where the top row is exact and $p(1 \otimes_{S_i} y) = 1 \cdot y$ for $y \in G$.

4.3. We now prove, by induction on the rank $rk(G)$, that $(G; \underline{S})$ has a presentation (3) or (4) and thus is geometric.

If $rk(G) = 1$ then $\bigoplus_{i=0}^m (\mathbf{Z} \otimes_{S_i} \mathbf{Z}G)$ is free Abelian of rank 2, by (5). This is possible only if either $m = 1$ and $S_0 = S_1 = G$; or if $m = 0$ and $S_0 = C(a^2)$ where $G = \langle a \rangle$. Thus we either have a presentation (3) with $m = 1$, $g = 0$, or a presentation (4) with $m = 0$, $g = 0$.

If $rk(G) \geq 2$ we put $T = S_0$ and determine the weight $n(T)$ with respect to the pair $(G; S_1, \dots, S_m)$, which is adapted by (5). We consider elements $\text{res}_0(c)$, $0 \neq c \in \bigcap_{j=1}^m N_j$ (i.e., elements $(d, 0, \dots, 0) \in \text{im}\{\text{res}_i\}$, $d \neq 0$) and count the number of components of d in $H^1(T; \mathbf{Z}G) = \bigoplus_{x_v \in G/T} H^1(T; \mathbf{Z}T)_{x_v}$. From (5) we see that $\text{im}\{\text{res}_i\} = \ker \delta = \ker pj\{e_i \cap -\}$, and $pj\{e_i \cap -\}$ restricted to any $H^1(T; \mathbf{Z}T)_{x_v}$ is bijective. Thus the minimal number of components of elements $d \neq 0$ is two, i.e., the weight of $T = S_0$ is 2. By Theorem B we therefore have one of the two following cases:

1) $G = G_1 * G_2$; $S_0 = C(g_1 g_2)$, $e \neq g_i \in G_i$, $i = 1, 2$, and the subgroups S_1, \dots, S_k are conjugate to subgroups of G_1 , while S_{k+1}, \dots, S_m are conjugate to subgroups of G_2 , for some k , $0 \leq k \leq m$.

2) $G = G_1 * \langle p \rangle$; $S_0 = C(p g_1 p^{-1} g_2)$, $e \neq g_1, g_2 \in G_1$, and S_1, \dots, S_m are conjugate to subgroups of G_1 .

Since hypothesis and assertion are invariant under conjugation we may assume that S_1, \dots, S_m are actually subgroups of G_1 or G_2 respectively.

Case 1). We can write G as $G = (G_1 * C(g_2)) *_{C(g_2)} G_2$. The subgroups $S_0 = C(g_1 g_2)$ and S_1, \dots, S_k are in $G_1 * C(g_2)$, and the S_{k+1}, \dots, S_m in G_2 . If $G_2 \neq C(g_2)$, Theorem 8.1 of [3] tells that $(G_2; C(g_2), S_{k+1}, \dots, S_m)$ is a PD^2 -pair. We claim that this is also true if $G_2 = C(g_2)$; namely, that pair is then $(C(g_2); C(g_2), C(g_2))$.

To prove this we note that quite generally, in Case 1), diagram (5) implies that $\text{res}: H^1(G; \mathbf{Z}G) \rightarrow \bigoplus_{i=k+1}^m H^1(S_i; \mathbf{Z}G)$ is surjective, and so is $\text{res}: H^1(G_2; \mathbf{Z}G_2) \rightarrow \bigoplus_{i=k+1}^m (S_i; \mathbf{Z}G_2)$. If $G_2 = C(g_2)$, then $H^1(G_2; \mathbf{Z}G_2) = \mathbf{Z}$, so this is possible only if $k = m$, or $k = m - 1$ and $S_m = G_2 = C(g_2)$. Assume $k = m$; then all subgroups S_1, \dots, S_m are in G_1 , hence $H^1(G, \underline{S}; \mathbf{Z}) \neq 0$, since $G = G_1 * C(g_2) = G_1 * C(g_1 g_2) = G_1 * S_0$. However, for a PD^2 -pair $H^1(G, \underline{S}; \mathbf{Z}G) = 0$, so $k = m$ is not possible and we are left with $k = m - 1$ and $(G_2; C(g_2), S_{k+1}, \dots, S_m) = (C(g_2); C(g_2), C(g_2))$, which is a PD^2 -pair.

Thus $(G_2; C(g_2), S_{k+1}, \dots, S_m)$ is a PD^2 -pair, and so is $(G_1; C(g_1), S_1, \dots, S_k)$. By induction hypothesis they have presentations of the type (3) or (4). It follows immediately that $(G; \underline{S})$ has a presentation (3) or (4): This is obvious if both above pairs have a presentation (3), or both a presentation (4). Otherwise one gets a presentation (4), i.e. non-orientable, by using transformations of the form

$$a^2[b, c] = \bar{a}^2 \bar{b}^2 \bar{c}^2; \quad \bar{a} = a^2 b c a^{-1}, \quad \bar{b} = a c^{-1} b^{-1} a^{-1} c a^{-1}, \quad \bar{c} = a c^{-1} \quad (6)$$

Case 2). Write G as $G = (G_1 * C(a)) *_{C(ag_2^{-1}), p} G_2$ with $p^{-1}(ag_2^{-1})p = g_1$. The subgroups $S_0 = C(a)$ and S_1, \dots, S_m are in $G_1 * C(a)$. By [3], Theorem 8.3, $(G_1 * C(a); C(a), S_1, \dots, S_m, C(ag_2^{-1}), C(g_1))$ is a PD^2 -pair. By the method used in Case 1) it follows that $(G_1; S_1, \dots, S_m, C(g_1), C(g_2))$ is a PD^2 -pair; the induction hypothesis tells that it has a presentation of the type (3) or (4). We may assume that this presentation is as follows.

G_1 is freely generated by t_0, t_1, \dots, t_m and some x_i, y_i (orientable case (3)) or some z_i (non-orientable case (4)); and S_i is conjugate to $C(t_i)$, $i = 1, \dots, m$, $C(g_1)$ to $C(t_0)$, i.e., g_1 is conjugate to t_0 or t_0^{-1} ; and $g_2 = t_0 \cdots t_m r$ where $r = \prod [x_j, y_j]$ or $\prod z_j^2$ respectively. S_0 is generated by $p g_1 p^{-1} t_0 \cdots t_m r$. By changing p if necessary we may assume $g_1 = t_0^{\pm 1}$. Using transformations of the form

$$p t p^{-1} t = \bar{p}^2 \bar{t}^2; \quad \bar{p} = p t p^{-1} t^{-1} p^{-1}, \quad \bar{t} = p t \quad (7)$$

and of the form (6), we get a presentation (3) or (4) for the pair $(G; S_0, S_1, \dots, S_m)$.

The passage from the two geometric pairs $(G_1; \dots)$ and $(G_2; \dots)$ to $(G; \underline{S})$ in Case 1), or from $(G_1; \dots)$ to $(G; \underline{S})$ in Case 2) can, of course, be replaced by a geometric procedure on the corresponding surfaces-with-boundary.

5. Proof of Theorem 1'

5.1. We recall that surface groups have canonical presentations

$$G = \left\langle x_1, y_1, \dots, x_g, y_g \left| \prod_{j=1}^g [x_j, y_j] = 1 \right. \right\rangle, \quad g \geq 1 \quad (8)$$

in the orientable, and

$$G = \left\langle z_0, \dots, z_g \left| \prod_{j=0}^g z_j^2 = 1 \right. \right\rangle, \quad g \geq 1 \quad (9)$$

in the non-orientable case.

Let G be a PD^2 -group which splits over a finitely generated group L as (α) $G = G_1 *_L G_2$, $G_1 \neq L \neq G_2$ or (β) $G = G_1 *_L G_2$. Since L has infinite index in G it is free [13].

If $rk(L) = 1$, $L = C$, we consider the pairs $(G_1; C)$ and $(G_2; C)$ in case (α) , or $(G_1; C, p^{-1}Cp)$ in case (β) . By [3], Theorem 8.1 and 8.3 these pairs are PD^2 -pairs and hence geometric; they have presentations (3) or (4), and by amalgamation or HNN-extension these yield presentations of the form (8) or (9) (by using, if necessary, transformations (6) and (7)). Thus G is a surface group.

Of course, the appropriate surface can also be obtained geometrically from the surfaces-with-boundary corresponding to the group pairs.

5.2. If $rk(L) \geq 2$, we will obtain from Theorem A a new splitting of G over a subgroup M with $rk(M) < rk(L)$. This reduces the problem to the case $rk(L) = 1$ above.

(α) Assume first that $G = G_1 *_L G_2$. We consider the Mayer-Vietoris sequence

$$\begin{aligned} \dots \rightarrow 0 \rightarrow H^1(G_1; \mathbf{Z}G) \oplus H^1(G_2; \mathbf{Z}G) &\xrightarrow{(\text{res}_1, -\text{res}_2)} \\ &H^1(L; \mathbf{Z}G) \xrightarrow{\delta} H^2(G; \mathbf{Z}G) \rightarrow \dots \end{aligned}$$

and show the following:

(10) If the weight of L with respect to both $(G_1; \emptyset)$ and $(G_2; \emptyset)$ is greater

than one, then $H^1(L; \mathbf{Z}L) \cap \text{im}(\text{res}_1, -\text{res}_2) = 0$. (Here we consider $H^1(L; \mathbf{Z}L)$ as submodule of $H^1(L; \mathbf{Z}G)$.)

Proof. Let C_L denote $H^1(L; \mathbf{Z}L)$ and $C_i = H^1(G_i; \mathbf{Z}G_i)$, $i = 1, 2$. Choose sets $\{x_i; i \in I\}$ and $\{y_j; j \in J\}$ of representatives of the (right) cosets $\in G_1/L$ and G_2/L (both sets containing e). We then have the following sets of representatives:

$$\Sigma_1 = \{e\} \cup \{y_{j_1}x_{i_2} \cdots; y_{j_l} \neq e \neq x_{i_1}\} \quad \text{for } G/G_1;$$

$$\Sigma_2 = \{e\} \cup \{x_{i_1}y_{j_2} \cdots; y_{j_l} \neq e \neq x_{i_1}\} \quad \text{for } G/G_2;$$

$$\Sigma_L = \Sigma_1 \cup \Sigma_2 \quad \text{for } G/L.$$

Hence we get decompositions

$$H^1(G_i; \mathbf{Z}G) = \bigoplus_{z \in \Sigma_i} C_i z, \quad i = 1, 2;$$

$$H^1(L; \mathbf{Z}G) = \bigoplus_{z \in \Sigma_L} C_L z.$$

The “length” of a summand $C_i z$ or $C_L z$ is defined as the number of representatives $x_i, y_j \neq e$ occurring in z . Consider now $0 \neq (c_1, c_2) \in H^1(G_1; \mathbf{Z}G) \oplus H^1(G_2; \mathbf{Z}G)$. We want to show that $\text{res}_1(c_1) - \text{res}_2(c_2) \notin C_L$. For this we consider a non-trivial component d of (c_1, c_2) lying in a summand (of the above decompositions) of maximal length; say $d = cz_1$ in $C_1 z_1$ of length l . Let $\text{res}_1(c)$ be $\sum_{i \in I} b_i x_i$, $b_i \in C_L$. Because the weight of L with respect to $(G_1; \emptyset)$ is greater than one, there is at least one i_0 with $x_{i_0} \neq e$, $b_{i_0} \neq 0$. So $\text{res}_1(cz_1)$ contains the summand $b_{i_0} x_{i_0} z_1$ in $C_L x_{i_0} z_1$ of length $l+1$, and because of the maximality of l there is no other contribution in $\text{res}_1(c_1) - \text{res}_2(c_2)$ to the component $C_L x_{i_0} z_1$. So indeed $\text{res}_1(c_1) - \text{res}_2(c_2) \notin C_L$, which proves (10).

By assumption, $H^2(G; \mathbf{Z}G)$ is free abelian of rank one and L has infinitely many ends. Therefore the restriction of δ to $H^1(L; \mathbf{Z}L)$ cannot be injective. Because of the exactness of the Mayer—Vietoris sequence, $H^1(L; \mathbf{Z}L) \cap \text{im}(\text{res}_1, -\text{res}_2) \neq 0$. By (10), L has weight *one* with respect to $(G_1; \emptyset)$ or $(G_2; \emptyset)$, say $(G_1; \emptyset)$. (Note that L cannot have weight 0, since res_1 and res_2 are injective.) By Theorem A, we have one of the following two cases:

$$1) \quad G_1 = H_1 * H_2, \quad L = L_1 * L_2, \quad e \neq L_i \subset H_i, \quad i = 1, 2;$$

$$2) \quad G_1 = H_1 * \langle t \rangle, \quad L = L_1 * tL_2t^{-1}, \quad e \neq L_1, L_2 \subset H_1.$$

In Case 1), we have $G = H_1 *_{L_1} (H_2 *_{L_2} G_2)$. If $L_1 \neq H_1$, G splits over L_1 ; if $L_1 = H_1$, then $L_2 \neq H_2$ and $G = H_2 *_{L_2} G_2$ splits over L_2 .

In Case 2), $G = (H_1 *_{L_1} G_2) *_{L_2, t^{-1}}$ splits over L_2 .

So in both cases we have a splitting of G over a group M with $rk(M) < rk(L)$.

(β) The case $G = G_1 *_{L, p}$ is treated similarly. If L is not cyclic, one can show that (by changing the notation if necessary) $n(L) = 1$ with respect to $(G_1; p^{-1}Lp)$; to prove that the pair is adapted and to compute the weight one proceeds by methods analogous to those in the proof of (10). By Theorem A we have again the cases 1) or 2) above, where moreover $p^{-1}Lp$ is conjugate to a subgroup of H_1 . By changing the stable letter p we can get $p^{-1}Lp \subset H_1$.

In Case 1), $G = (H_1 *_{L_{1,p}}) *_{L_2} H_2$ splits over L_2 if $L_2 \neq H_2$; or else over L_1 .

In Case 2), $G = (H_1 *_{L_{1,p}}) *_{L_2, t^{-1}}$ splits over L_2 . This completes the proof of Theorem 1'.

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