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On the characterization of flat metrics by the spectrum

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1. Introduction

Let M be an n-dimensional compact, connected, oriented C^∞ manifold without boundary. Let $\mathcal{R}$ be the space of C^∞ Riemannian metrics on M with the C^∞ topology. For $g \in \mathcal{R}$, Spec(M, g) denotes the spectrum of the Laplace-Beltrami operator $\Delta = -g^{ij}\nabla_i\nabla_j$ acting on C^∞ functions on M, namely,

$$\text{Spec}\,(M, g) = \{0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots\},$$

where each eigenvalue is written as many times as its multiplicity. Then, the Minakshisundaram's formula for Spec (M, g) is given by

$$\sum_{k=0}^{\infty} \exp(-\lambda_k t) \underset{t\downarrow 0}{\sim} \left(\frac{1}{4\pi t}\right)^{n/2} \sum_{s=0}^{\infty} a_s t^s,$$

where the coefficients a_s's are expressed by the metric and its derivatives (curvature) (cf. [1], [2], [3]).

It is obvious that if (M, g) is flat, $a_s = 0$ holds for $s \geqslant 1$. However, $a_s = 0$ $(s \geqslant 1)$ does not imply that (M, g) is flat. In fact, Patodi [2] showed that for the non-flat space $S^3(c) \times [H^3(-c)/\Lambda]$, the coefficients a_s's vanish for $s \geqslant 1$. Here, $S^3(c)$ and $H^3(-c)$ are a Euclidean 3-sphere with constant curvature $c > 0$ and a hyperbolic 3-space with constant curvature $-c$, respectively, and Λ is some discontinuous group of motions of $H^3(-c)$. In the low dimensional cases, the following has been shown.

THEOREM. (1) (Patodi [2]) *For $2 \leqslant n \leqslant 5$, $a_2 \geqslant 0$ holds, and equality holds if and only if (M, g) is flat.*

(2) (Tanno [3]) *For $n = 6$, $a_2 \geqslant 0$ holds, and if $a_2 = a_3 = 0$, then (M, g) is flat or locally Riemannian product $S^3(c) \times H^3(-c)$.*

The purpose of this paper is to prove the following theorem which asserts that the condition $a_2 = 0$ 'locally' characterizes flat metrics.

THEOREM A. *Suppose γ is a C^∞ flat Riemannian metric on M. Then, there is a neighbourhood U of γ in $\mathcal{R}$ such that if $g \in U$ and $a_2(g)=0$, g is also a flat metric.*

Remark. For $2 \leqslant n \leqslant 6$, the neighbourhood U in Theorem A can be taken equally to the whole space $\mathcal{R}$, that is, if M admits a flat metric then $a_2(g)=0$ implies that g is flat (see §7). For $n \geqslant 7$, the author does not know whether there are counterexamples or not.

As a corollary of Theorem A, we have the following theorem.

THEOREM B. *Suppose (M, γ) is a flat manifold. Then, there is a neighbourhood U of γ in $\mathcal{R}$ such that if $g \in U$ and* Spec $(M, g)=$ Spec (M, γ), *then* $(M, g)=(M, \gamma)$ (*isometric*).

In order to derive this theorem, we have only to note the following result of Kneser and Sunada [4].

THEOREM (Kneser, Sunada). *There are only finitely many isometry classes of flat manifolds with a given spectrum.*

Remark. In the previous paper [5] we showed that a metric of flat torus is characterized in the "infinitesimal" sense by its spectrum. Theorem B is an extension of this result.

After giving notations and a fundamental lemma in §2, we review in §3 the properties concerning the space of flat metrics following Fischer and Marsden [6], [7]. In §4 we study the function $a_2(g)$ and calculate its derivatives. In §5 we establish the weak Morse lemma for normed spaces, which gives a basic tool for the proof of the main theorem. Then we prove Theorem A in §6. Finally in §7 we consider the "global" characterization of flat metrics.

Remark. Fischer and Marsden gave a theorem [6, Theorem 1.5.2], [7, Theorem 10] which is of same type as our Theorem A. Our proof is performed on the same lines as in [7], but differently in details.

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2. Preliminaries

Let M be an n-dimensional compact, connected, oriented C^∞ manifold without boundary. Let $T^p_q(M)$ denote the tensor bundle of type (p, q) over M, and

$ST_2(M)$ the bundle of symmetric covariant 2-tensors on M. For a C^∞ Hermitian vector bundle T, let $C^\infty(T)$ be the space of C^∞ cross-sections of T, and $H^s(T)$ the Sobolev space of cross-sections of T with respect to a fixed C^∞ Riemannian metric. The topology of $H^s(T)$ does not depend on the choice of a metric.

We use the following notations.

$V^s = H^s(T_0^1(M))$; the H^s vector fields,
$A^s = H^s(T_1^0(M))$; the 1-forms of class H^s,
$S_2^s = H^s(ST_2(M))$; the symmetric covariant 2-tensor fields of class H^s,
$\mathscr{D}^s$; the group of H^s diffeomorphisms of M, defined for $s > (n/2)+1$ (see Ebin [8]). The group $\mathscr{D}^{s+1}$ acts on S_2^s as follows;

$$S_2^s \times \mathscr{D}^{s+1} \to S_2^s;\ (h, \eta) \mapsto \eta^* h,$$

where $\eta^* h$ denotes the pull-back of h by η.

$\mathscr{R}^s (\subset S_2^s)$; the Hilbert manifold of Riemannian metrics of class H^s. The manifold $\mathscr{R}^s$ is an open convex positive cone in S_2^s, and invariant under the action of $\mathscr{D}^{s+1}$.

$\mathscr{F}^s (\subset \mathscr{R}^s)$; the subset of flat matrics of class H^s, defined for $s > (n/2)+1$.

If the s is omitted, the space is understood to be of C^∞ class and endowed with the C^∞ topology.

We define various inner products of $H^s(T)$ $(s > (n/2)+1)$ by $g \in \mathscr{R}^s$ as follows;

$$\text{(a)}\ \langle T, T' \rangle_g^0 = g_{ii'} \cdots g_{jj'} g^{kk'} \cdots g^{mm'} T^{i \cdots j}_{k \cdots m} T'^{i' \cdots j'}_{k' \cdots m'},$$

$$\text{(b)}\ \langle T, T' \rangle_g^k = \sum_{r=0}^{k} \langle \nabla_g^{(r)} T, \nabla_g^{(r)} T' \rangle_g^0 \quad (k \leq s),$$

where $\nabla_g^{(r)} T$ is the tensor field $\overbrace{\nabla_g \cdots \nabla_g}^{r} T$ and ∇_g is the covariant derivative with respect to g.

$$\text{(c)}\ (T, T')_g^k = \int_M \langle T, T' \rangle_g^k \, dV(g),$$

where $dV(g)$ denotes the volume element induced from g.

Using the above inner product (c), we can introduce the Riemannian structure on $\mathscr{R}^s$ by $g \mapsto (,)_g^k$. This metric is $\mathscr{D}^{s+1}$-invariant, i.e., $\mathscr{D}^{s+1}$ acts by isometry (see [8, pp. 18–21]).

For a metric $g\in\mathscr{R}$, we define a differential operator

$$\delta_g : C^\infty(ST_2(M)) \to C^\infty(T_1^0(M)); \qquad (\delta_g\xi)_j = -\nabla^i_g\xi_{ij}.$$

Then δ_g extends to a continuous linear map $\delta_g^s : S_2^s \to A^{s-1}$. The adjoint operator δ_g^* of δ_g with respect to $(,)^0_g$ extends to a map

$$(\delta_g^s)^* : A^s \to S_2^{s-1}; \qquad \{(\delta_g^s)^*\xi\}_{ij} = \tfrac{1}{2}(\mathscr{L}_X g)_{ij},$$

where $s>(n/2)+1$, and $\mathscr{L}$ is the Lie derivative and $X(\in V^s)$ is dual to ξ.

LEMMA 2.1 (Berger and Ebin [9]). *For $g\in\mathscr{R}$, there is an orthogonal decomposition*

$$S_2^s = (\delta_g^s)^{-1}(0) \oplus (\delta_g^{s+1})^*(A^{s+1}),$$

where the summands are orthogonal with respect to $(,)^0_g$.

3. Space of flat metrics

In [6] and [7] Fischer and Marsden studied the space $\mathscr{F}^s$ of flat metrics of class H^s. We review their results in the first part of this section (Lemma 3.1 and Proposition 3.2).

In Lemma 2.1, g is assumed to be of C^∞ class (more precisely, g is required to be of class H^{s+1}). However, if g is flat, the following is obtained by one of the regularity theorems.

LEMMA 3.1 ([6, p. 237], [7, p. 530]). *Let $g\in\mathscr{F}^s$, $s>(n/2)+1$. Then there is an orthogonal decomposition*

$$S_2^s = (\delta_g^s)^{-1}(0) \oplus (\delta_g^{s+1})^*(A^{s+1}).$$

We denote by $\Gamma(g)$ the Riemannian connection of $g\in\mathscr{R}^s$. Let $\mathscr{K}^s$ be the set of flat Riemannian connections of class H^s. For $\Gamma\in\mathscr{K}^{s-1}$, set

$$\mathscr{F}_1^s = \{g\in\mathscr{F}^s;\ \Gamma(g)=\Gamma\}.$$

Furthermore, for $g\in\mathscr{R}^s$, let us define

$$E_g : S_2^s \to \mathscr{R}^s;\ h \mapsto g\exp(g^{-1}h),$$

where $g^{-1}h$ is an endomorphism of $T_x(M)$ at each $x \in M$, given by $h^i_j = g^{ik}h_{kj}$ in local coordinates. Then E_g is a C^∞ diffeomorphism with $E_g(0) = g$ (see [8, p. 36]).

PROPOSITION 3.2. *Let* $\Gamma \in \mathcal{K}^{s-1}$ *and* $g \in \mathcal{F}^s_\Gamma$, $s > (n/2)+1$. *Set* $PS^s_2(g) = \{h \in S^s_2;\ \nabla_g h = 0\}$. Then,

(a) $\mathcal{F}^s_\Gamma = E_g(PS^s_2(g))$, *and* $\mathcal{F}^s_\Gamma$ *is a finite dimensional closed* C^∞ *submanifold of* $\mathcal{R}^s$. *Moreover, the tangent space of* $\mathcal{F}^s_\Gamma$ *at* g *is*

$$T_g(\mathcal{F}^s_\Gamma) = PS^s_2(g).$$

(b) $\mathcal{F}^s = \mathcal{D}^{s+1}(\mathcal{F}^s_\Gamma) = \{\eta^*\gamma \in \mathcal{R}^s;\ \eta \in \mathcal{D}^{s+1},\ \gamma \in \mathcal{F}^s_\Gamma\}$, *and* $\mathcal{F}^s$ *is a closed* C^∞ *submanifold of* $\mathcal{R}^s$. *Moreover,*

$$T_g(\mathcal{F}^s) = PS^s_2(g) \oplus (\delta^{s+1}_g)^*(A^{s+1}).$$

Proof. See Fischer and Marsden [6, Theorem I.3.3], [7, Theorem 6].

In the remainder of this section, let us prove the following Proposition 3.3. For $g \in \mathcal{F}^s_\Gamma$, set

$$S(g) = E_g((\delta^s_g)^{-1}(0)).$$

Then we have the following.

PROPOSITION 3.3. (a) $S(g)$ *is a closed* C^∞ *submanifold of* $\mathcal{R}^s$, *and* $\mathcal{F}^s_\Gamma$ *is a closed* C^∞ *submanifold of* $S(g)$. *Moreover,*

$$T_g(S(g)) = (\delta^s_g)^{-1}(0).$$

(b) *For any neighbourhood* V *of* g *in* $S(g)$, *there is a neighbourhood* U *of* g *in* $\mathcal{R}^s$ *such that* $U \subset \mathcal{D}^{s+1}(V)$.

Proof. (a) We have $PS^s_2(g) \subset (\delta^s_g)^{-1}(0) \subset S^s_2$, where each subspace is closed. Therefore, the assertion is obvious because E_g is a C^∞ diffeomorphism.

(b) By the regularity theorem ([6, Theorem I.3.1], [7, Theorem 5]), there is $\eta \in \mathcal{D}^{s+1}$ such that $\eta^* g = g'$ belongs to $\mathcal{F}$. Hence, the orbit $O^s(g)$ through g is equal to $O^s(g')$ and is a C^∞ submanifold of $\mathcal{R}^s$. Let $N = N(O^s(g))$ be the normal bundle with respect to the weak Riemannian metric $\gamma \mapsto (\,,)^0_\gamma$ ([8, *pp.* 30–31]). We define $E: N \to \mathcal{R}^s$ by $E(\gamma, h) = E_\gamma(h)$, where $\gamma \in O^s(g)$ and $h \in N_\gamma = (\delta^s_\gamma)^{-1}(0)$, N_γ being the fibre of N at γ. Then, it is easily shown that E is a C^∞ map and $E(\eta^*\gamma, \eta^*h) = \eta^*E(\gamma, h)$ holds for $\eta \in \mathcal{D}^{s+1}$. Moreover, the first derivative of E

at $(g, 0)$ is given by

$$dE(g, 0)\ (h', h'') = h' + h'',$$

where $h' \in T_g(O^s(g)) = (\delta_g^{s+1})^*(A^{s+1})$ and $h'' \in N_g = (\delta_g^s)^{-1}(0)$. Thus, $dE(g, 0)$ is an isomorphism (Lemma 3.1). Therefore, there are a neighbourhood U' of g in $\mathcal{R}^s$ and a neighbourhood W of $(g, 0)$ in N such that $E: W \to U'$ is a diffeomorphism. Let $\gamma \mapsto (,)_\gamma^s$ be the strong Riemannian metric of $\mathcal{R}^s$. Then, the neighbourhood W is given by

$$W = \{(\gamma, h) \in N;\ \gamma \in W',\ (h, h)_\gamma^s < \varepsilon,\ \varepsilon > 0\},$$

W' being a neighbourhood of g in $O^s(g)$. For given $V(\subset S(g))$ there is $\varepsilon'(\leq \varepsilon)$ such that if $V' = \{(g, h) \in N_g;\ (h, h)_g^s < \varepsilon'\}$, $E_g(V') \subset V$ holds. Set

$$V'' = \{(\gamma, h) \in N;\ \gamma \in W',\ (h, h)_\gamma^s < \varepsilon'\} \subset W,$$

and $U = E(V'')$. Then U is open in $\mathcal{R}^s$ and satisfies $U \subset \mathcal{D}^{s+1}(V)$. In fact, if γ is in U and $\gamma = E(\eta^* g, h)$, then $(\eta^{-1})^* h = h'$ belongs to V' because $(\eta^{-1})^*: S_2^s \to S_2^s$ is an isometry with respect to the metric $(,)^s$. Thus, $\gamma = E(\eta^* g, \eta^* h') = \eta^* E(g, h') = \eta^* E_g(h') \subset \mathcal{D}^{s+1}(V)$. □

4. Derivatives of $a_2(g)$

For $g \in \mathcal{R}$, let $\{^{\ i}_{jk}\}$, R^i_{jkm}, R_{ij} and τ denote the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature, respectively. The curvature tensor is defined by

$$R^i_{jkm} = \frac{\partial}{\partial x^k}\left\{ {i \atop jm} \right\} - \frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} + \left\{ {s \atop jm} \right\}\left\{ {i \atop sk} \right\} - \left\{ {s \atop jk} \right\}\left\{ {i \atop sm} \right\},$$

in terms of the local coordinates (x^i).

It is known that the Minakshisundaram's coefficient a_2 is given by

$$a_2 = \frac{1}{360}\int_M (2|R|^2 - 2|\rho|^2 + 5\tau^2)\,dV(g) = \frac{1}{360}F(g),$$

Where $|R|^2 = R_{ijkm}R^{ijkm}$ and $|\rho|^2 = R_{ij}R^{ij}$ (cf. [1], [2], [3]).

It is easily shown that $\mathrm{Spec}\,(M, \eta^*g) = \mathrm{Spec}\,(M, g)$ for $\eta \in \mathcal{D}$ and $g \in \mathcal{R}$, hence $F(\eta^*g) = F(g)$ holds.

The function F can be regarded to be defined on $\mathcal{R}^s$ if $s > (n/2) + 4$. We write this function F^s.

PROPOSITION 4.1. *The function F^s on $\mathcal{R}^s$ is of C^∞ class.*

We need the following lemma which was proved in [10, 11.3].

LEMMA 4.2. *If ξ and η are C^∞ vector bundles over M and $f : \xi \to \eta$ is a C^∞ fibre preserving map, then for $s > n/2$ the map $f_* : H^s(\xi) \to H^s(\eta)$ defined by $f_*(\alpha) = f \circ \alpha$ is of C^∞ class.*

Proof of Proposition 4.1. We prove that $g \mapsto \int_M |R|^2 dV(g)$ is a C^∞ function. The proof is done in two steps.

First step: $\phi : g \mapsto |R|^2$ is a C^∞ map of $\mathcal{R}^s$ into $H^{s-2}(M, \boldsymbol{R})$, the Hilbert space of all H^{s-2} functions. In fact, we have

$$|R|^2 = R^a_{bcd} R^i_{jkm} g_{ai} g^{bj} g^{ck} g^{dm}.$$

Thus, as is easily shown, $|R|^2$ is a rational combinations of g, dg, d^2g, so that $|R|^2 : J^2(\xi) \to M \times \boldsymbol{R}$ is a C^∞ fibre preserving map, where ξ is the fibre subbundle of $ST_2(M)$ consisting of positive definite forms on each tangent space and $J^2(\xi)$ the second jet bundle of ξ. Noting that $\mathcal{R}^s = H^s(\xi) \subset H^{s-2}(J^2(\xi))$, we can conclude from Lemma 4.2 that ϕ is a C^∞ map of $\mathcal{R}^s$ into $H^{s-2}(M, \boldsymbol{R})$.

Second step: The function $\psi : H^{s-2}(M, \boldsymbol{R}) \times \mathcal{R}^s \to \boldsymbol{R}$ defined by $(f, g) \mapsto \int_M f\, dV(g)$ is of C^∞ *class*. In fact, fix $g_0 \in \mathcal{R}^s$ and define $\mu : \mathcal{R}^s \to H^s(M, \boldsymbol{R})$ by the equation $\mu(g)\, dV(g_0) = dV(g)$. Then it is easy to see that the map μ is of C^∞ class (see [8]). The map ψ is decomposed as $\psi = \psi_0 \circ (\mathrm{id} \times \mu)$, where $\psi_0 : H^{s-2}(M, \boldsymbol{R}) \times H^2(M, \boldsymbol{R}) \to \boldsymbol{R}$ is defined by $(f, f') \mapsto \int_M ff'\, dV(g_0)$. Since μ and ψ_0 are C^∞ maps, ψ is of C^∞ class.

Finally, the function $g \mapsto \int_M |R|^2\, dV(g)$ is decomposed as follows:

$$\begin{array}{ccccc}
 & & H^{s-2}(M, \boldsymbol{R}) \times H^s(M, \boldsymbol{R}) & & \\
 & \nearrow^{\mathrm{id} \times \mu} & & \searrow^{\psi_0} & \\
\mathcal{R}^s & \xrightarrow{\phi \times \mathrm{id}} & H^{s-2}(M, \boldsymbol{R}) \times \mathcal{R}^s & \xrightarrow{\psi} & \boldsymbol{R} \\
\psi \downarrow & & \psi \downarrow & & \psi \downarrow \\
g & \longmapsto & (|R|^2, g) & \longmapsto & \int_M |R|^2\, dV(g)
\end{array}$$

Since ϕ and ψ are C^∞ maps, $g \mapsto \int_M |R|^2\, dV(g)$ is of C^∞ class.

It is similarly shown that the functions $g \mapsto \int_M |\rho|^2\, dV(g)$ and $g \mapsto \int_M \tau^2\, dV(g)$ are of C^∞ class. □

PROPOSITION 4.3. *$F^s(\eta^*g)=F^s(g)$ holds for $\eta\in\mathscr{D}^{s+1}$.*

Proof. The action $S_2^s\times\mathscr{D}^{s+1}\to S_2^s$ is continuous ([8, pp. 17–18]), and F^s is of C^∞ class. Hence, the proposition follows from $F(\eta^*g)=F(g)$ for $g\in\mathscr{R}$ and $\eta\in\mathscr{D}$.

Now, we give the formulas about the derivatives of F^s, which have been calculated in the previous paper [5].

PROPOSITION 4.4. *For $g\in\mathscr{R}^s$ and $h\in S_2^s$, the first derivative of F^s is given by*

$$dF^s(g)(h)=\int_M\langle T(g),h\rangle_g^0\,dV(g)=\int_M T_{ij}(g)h^{ij}\,dV(g), \tag{4.1}$$

where

$$T_{ij}(g)=12\nabla_i\nabla_j\tau-6\nabla_k\nabla^kR_{ij}+8R_{ik}R_j^k-4R_{kimj}R^{km}+4R_{ikms}R_j^{kms}$$
$$+9(\Delta\tau)g_{ij}-10\tau R_{ij}+|R|^2g_{ij}-|\rho|^2g_{ij}+\tfrac{5}{2}\tau^2g_{ij},$$

∇ and the curvatures being induced from g. Therefore, if $g\in\mathscr{F}^s$, then $dF^s(g)=0$, i.e., a flat metric is a critical point of F^s.

Proof. This is a direct but tedious calculation (cf. [5]).

Remark. $T(g)$ is an element of S_2^{s-4}, and $g\mapsto T(g)$ is a C^∞ map of $\mathscr{R}^s$ into S_2^{s-4}. This is proved on the same lines as Proposition 4.1.

PROPOSITION 4.5. *The second derivative of F^s at $g\in\mathscr{R}^s$ is given by*

$$d^2F^s(g)(h,k)=\int_M\langle[dT(g)+\tfrac{1}{2}T(g)\mathrm{tr}(g)]h,k\rangle_g^0\,dV(g), \tag{4.2}$$

where $\mathrm{tr}(g)h=g^{ij}h_{ij}$. In particular, at $g\in\mathscr{F}^s$,

$$\begin{aligned}d^2F^s(g)(h,h)=3\int_M&[6(\Delta h_s^s)(\nabla_i\nabla_jh^{ji})+3(\Delta h_s^s)^2\\&+4(\nabla^k\nabla^mh_{km})(\nabla_i\nabla_jh^{ji})-2(\nabla_k\nabla_ih^{ji})(\nabla^k\nabla_mh_j^m)\\&+(\nabla_k\nabla^kh^{ji})(\nabla_m\nabla^mh_{ji})]\,dV(g).\end{aligned} \tag{4.3}$$

Proof. This is obtained by straightforward calculation starting from (4.1).

Remark. $dT(g)+(1/2)T(g)\mathrm{tr}(g)$ is an element of $L(S_2^s;S_2^{s-4})$, the space of all continuous linear maps of S_2^s into S_2^{s-4}.

5. Weak Morse lemma for normed spaces

In this section we establish the weak Morse lemma for normed spaces. This work is motivated by Tromba's paper [11], in which the Morse lemma for almost-Riemannian manifolds is considered.

Let $X_1, X_2, \cdots$ be normed vector spaces, and define $L(X_1, \ldots, X_k; X_{k+1})$ as the normed vector space of all continuous k-linear maps of $X_1 \ldots X_k$ into X_{k+1}.

Let β be a continuous bilinear form on a normed vector space X, i.e., $\beta \in L(X, X; \boldsymbol{R})$. β is called the weak inner product of X if (a) $\beta(x, y) = \beta(y, x)$, (b) $\beta(x, x) > 0$ for $x \neq 0$. The space X with β is regarded as a pre-Hilbert space denoted by X_β. Let $\hat{X}_\beta$ be the completion of X_β, and $\hat{\beta}$ the continuous extension of β to $\hat{X}_\beta$. Thus the space $\hat{X}_\beta b$ is a Hilbert space with inner product $\hat{\beta}$. The canonical injection $X \to X_\beta(\hat{X}_\beta)$ is continuous.

Let $f : X \to \boldsymbol{R}$ be a C^k function, $k \geq 2$.

DEFINITION. The C^k function f is of C^k_β *class* if

(a) for each $x \in X$, the second derivative $d^2 f(x)$ belongs to $L(X_\beta, X_\beta; \boldsymbol{R})$.

(b) $x \mapsto d^2 f(x)$ is a C^{k-2} map of X into $L(X_\beta, X_\beta; \boldsymbol{R})$.

Suppose $X = Y \times Z$ (the product normed space), and $f : X \to \boldsymbol{R}$ is a C^k_β function ($k \geq 2$). We have

$$d^2 f(x)((u, v), (u', v')) = D_1^2 f(x)(u, u') + D_1 D_2 f(x)(u, v') + D_2 D_1 f(x)(v, u') + D_2^2 f(x)(v, v').$$

where (u,v), $(u', v') \in Y \times Z$, and $D_i f(x)$ $(i = 1, 2)$ is the partial derivative of f at x with respect to the i-th variable. Since f is of C^k_β class, there is a unique $B(x) \in L(Z_\beta; \hat{Z}_\beta)$ such that

$$D_2^2 f(x)(u, v) = \hat{\beta}(B(x)u, v),$$

for $u, v \in Z$. Moreover, $x \mapsto B(x)$ is a C^{k-2} map of X into $L(Z_\beta; \hat{Z}_\beta)$.

DEFINITION. Let K be a subset of Y. The subset $K \times \{0\}$ of X is called the $\beta-$*nondegenerate critical subset* of f, if for each $x \in K \times \{0\}$,

(a) $df(x) = 0$, and

(b) $\hat{B}(x)$, the continuous extension of $B(x)$ to $\hat{Z}_\beta$, is invertible.

We are now ready to state and prove the following.

PROPOSITION 5.1(weak Morse lemma). *Let* $f : X = Y \times Z \to \boldsymbol{R}$ *be a* C^k_β *function,* $k \geq 2$. *Suppose* K *is a compact subset of* Y. *If the subset* $K \times \{0\}$ *is a*

β-nondegenerate critical subset of f and $f(K\times\{0\})=0$, then there are a neighbourhood V of the origin in Z and C^{k-2} map $\phi: K\times V\to \hat{Z}_\beta$ such that

(a) *$\phi(x)=0$ if and only if $x=(y,0)$, and*

(b) *$f(x)=\frac{1}{2}\widehat{D_2^2f}((y,0))(\phi(x),\phi(x))$, $x=(y,z)\in K\times V$,*

where $\widehat{D_2^2f}(x)$ is the continuous extension of $D_2^2f(x)$ to $\hat{Z}_\beta\times\hat{Z}_\beta$.

Proof. By the Taylor's formula we have

$$f((y,z))=\int_0^1(1-\lambda)D_2^2f((y,\lambda z))(z,z)\,d\lambda.$$

Set

$$J(y,z)(u,v)=\int_0^1(1-\lambda)D_2^2f((y,\lambda z))(u,v)\,d\lambda.$$

Then, $J(y,z)$ belongs to $L(Z_\beta, Z_\beta;\boldsymbol{R})$ since f is of C_β^k class. Therefore, we can write $J(y,z)(u,v)=\hat{\beta}(B(y,z)u,v)$ and $D_2^2f((y,0))(u,v)=2\hat{\beta}(B(y,0)u,v)$ where $B(y,z)\in L(Z_\beta;\hat{Z}_\beta)$. Let $\hat{B}(y,z)$ be the continuous extension of $B(y,z)$ to $\hat{Z}_\beta$. Then, $(y,z)\mapsto\hat{B}(y,z)$ is a C^{k-2} map of X into $L(\hat{Z}_\beta;\hat{Z}_\beta)$. Moreover, $\hat{B}(y,z)$ is self-adjoint for each (y,z). Since $\hat{B}(y,0)$ is invertible and K is compact, so $\hat{B}(y,z)$ is invertible in $K\times V'$, V' being a neighbourhood of the origin. Define $Q(y,z)=\hat{B}(y,z)^{-1}\hat{B}(y,0)$ and Q is a C^{k-2} map of $K\times V'$ into $L(\hat{Z}_\beta;\hat{Z}_\beta)$. Now $Q(y,0)=$ identity and since a square root function is defined in a neighbourhood of the identity operator by a convergent power series with real coefficients, we can define a C^{k-2} map $R: K\times V(\subset K\times V')\to L(\hat{Z}_\beta;\hat{Z}_\beta)$ with each $R(y,z)$ invertible and $Q(y,z)=[R(y,z)]^2$. We see easily from the definition of Q that $Q(y,z)^*\hat{B}(y,z)=\hat{B}(y,z)Q(y,z)$ hence $R(y,z)^*\hat{B}(y,z)=\hat{B}(y,z)R(y,z)$ holds. Thus, we have $R(y,z)^*\hat{B}(y,z)R(y,z)=\hat{B}(y,0)$, or $\hat{B}(y,z)=R_1(y,z)^*\hat{B}(y,0)R_1(y,z)$, where $R_1(y,z)=R(y,z)^{-1}$. Now, set $\phi((y,z))=R_1(y,z)z$, and we have

$$\begin{aligned}f((y,z))&=\hat{\beta}(R_1(y,z)^*\hat{B}(y,0)R_1(y,z)z,z)\\&=\hat{\beta}(\hat{B}(y,0)\phi((y,z)),\phi((y,z))).\end{aligned}$$

Finally, $\phi((y,z))=R_1(y,z)z=0$ holds if and only if $z=0$, because $R_1(y,z)$ is invertible. □

COROLLARY 5.2. *Besides assumptions in Proposition* 5.1, *assume that*

$$D_2^2f((y,0))(u,u)>0$$

holds for $y\in K$ and $u(\in Z)\neq 0$. If $f(x)=0$ and $x\in K\times V$, then x belongs to $K\times\{0\}$.

Proof. From Proposition 5.1, we have only to prove that $\widehat{D_2^2 f}((y,0))(u,u)>0$ holds for any $u(\in \hat{Z}_\beta)\neq 0$. Suppose there is $u\neq 0$ such that $\widehat{D_2^2 f}((y,0))(u,u)=\hat{\beta}(\hat{B}(y,0)u,u)=0$. Then, $\inf_{\beta(u,u)=1}\hat{\beta}(\hat{B}(y,0)u,u)=0$, hence zero belongs to the spectrum of $\hat{B}(y,0)$, which is absurd because $\hat{B}(y,0)$ is invertible. □

In the remainder of this section we give a supplement.

Let us define a C^∞ map $x(\in X)\mapsto \beta(x)$ (the weak inner product of X) such that the topology of $X_{\beta(x)}$ does not depend on x. We call this map the weak C^∞ Riemannian structure of X. Let $\beta=\beta(0)$. Then, for each $x\in X$, there is $C(x)\in L(\hat{X}_\beta;\hat{X}_\beta)$ such that

$$\hat{\beta}(x)(y,z)=\hat{\beta}(C(x)y,z),\qquad y,z\in \hat{X}_{\beta(x)}(=\hat{X}_\beta),$$

and $x\mapsto C(x)$ is of C^∞ class. Moreover, we can easily prove the following.

PROPOSITION 5.3. *Let $f:X\to \boldsymbol{R}$ be a C^k function $(k\geq 2)$. f is of C^k_β class if and only if*

(a) *for each $x\in X$, $d^2f(x)\in L(X_{\beta(x)},X_{\beta(x)};\boldsymbol{R})$, and*

(b) *if $B(x)$ is given by $d^2f(x)(u,v)=\hat{\beta}(x)(B(x)u,v)$, then $x\mapsto B(x)$ is a C^{k-2} map of X into $L(X_\beta;\hat{X}_\beta)$.*

6. Proof of the main theorem

In this section we prove the following theorem and Theorem A.

THEOREM A′. *Let $\gamma\in\mathscr{F}$ and s be sufficiently large. Then, there is a neighbourhood $U\subset\mathscr{R}^s$ of γ such that if $g\in U$ and $F^s(g)=0$, g is in $\mathscr{F}^s$.*

We define $f:S^s_2\to\boldsymbol{R}$ by $f=F^s\circ E_\gamma$. Let $\tilde{f}$ be the restriction of f to $X=(\delta^s_\gamma)^{-1}(0)(\subset S^s_2)$. Then, $\tilde{f}$ is a C^∞ function (Proposition 4.1). Let $Y=PS^s_2(\gamma)$. We have the following from Propositions 3.2 and 4.4.

PROPOSITION 6.1. *$\tilde{f}(y)=d\tilde{f}(y)=0$ holds for each $y\in Y$.*

We apply Corollary 5.2 to the function $\tilde{f}$ on the Hilbert space X.

Let us introduce a weak C^∞ Riemannian structure on X. First, we define a weak Riemannian metric on $\mathscr{R}^s$ as follows;

$$\begin{aligned}(h,k)_g&=\int_M[\langle h,k\rangle^0_g+2\langle\nabla h,\nabla k\rangle^0_g+\langle\nabla\nabla h,\nabla\nabla k\rangle^0_g]\,dV(g)\\&=((1+\bar{\Delta}_g)^2h,k)^0_g,\end{aligned}\qquad(6.1)$$

where $\bar{\Delta}_g$ is the rough Laplacian defined by $(\bar{\Delta}_g h)_{ij} = -g^{st}\nabla_s\nabla_t h_{ij}$ in local coordinates.

LEMMA 6.2. *Let* $L_g = (1+\bar{\Delta}_g)^2$. *Then, the maps*

$$\mathscr{R}^s \times S_2^s \to S_2^{s-4}; (g, h) \mapsto L_g h,$$

and

$$\mathscr{R}^s \times S_2^{s-4} \to S_2^s; (g, h) \mapsto L_g^{-1} h$$

are of C^∞ *class.*

Proof. First, we note that for each $g \in \mathscr{R}^s$, L_g has a continuous linear inverse L_g^{-1}. In fact, the differential operator $(1+\bar{\Delta}_g)^2$ is an injective self-adjoint elliptic operator. Therefore, L_g is surjective by the decomposition theorem (e.g. [12, Ch. XI]). Furthermore, by the open mapping theorem L_g has a continuous inverse.

Now, it is easily shown that $(g, h) \mapsto L_g h$ is C^∞ (cf. [13, Lemma 2.11]). Moreover, it follows that $g \mapsto L_g$ is a C^∞ map of $\mathscr{R}^s$ into $L(S_2^s; S_2^{s-4})$. On the other hand, $L_g \mapsto L_g^{-1}$ is a C^∞ map (e.g. [14, Ch. 8]). Therefore, $g \mapsto L_g^{-1}$ is C^∞ and accordingly $(g, h) \mapsto L_g^{-1} h$ is C^∞. □

PROPOSITION 6.3. *The Riemannian structure defined by* (6.1) *is of* C^∞ *class.*

Proof. The proposition follows from Lemma 6.2 and the proof of Proposition 4.1. □

Now, we define a C^∞ Riemannian structure $\beta(x)$ on S_2^s as the pull-back of $(\,,)_g$ by E_γ. Namely,

$$\beta(x)(y, z) = (dE_\gamma(x)(y), dE_\gamma(x)(z))_g,$$

where $g = E_\gamma(x)$.

Let $\beta = \beta(0)$. Obviously, $\widehat{(S_2^s)}_\beta = S_2^2$ holds.

PROPOSITION 6.4. *The function* $\tilde{f}: X \to \boldsymbol{R}$ *is of* C_β^∞ *class.*

For the proof we first prove the following lemmas.

LEMMA 6.5. *The first and the second derivatives of E_γ are given by*

$$dE_\gamma(x)(y)=\gamma\left[\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\{(\gamma^{-1}x)^k(\gamma^{-1}y)\}\right],$$

and

$$d^2E_\gamma(x)(y,z)=\gamma\left[\sum_{k=0}^{\infty}\frac{1}{(k+2)!}\{(\gamma^{-1}x)^k(\gamma^{-1}y)(\gamma^{-1}z)\}\right],$$

respectively, where $\{A_1 A_2 \cdots A_k\}=\sum_\sigma A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(k)}$, *the summation being taken over all permutations* σ *of* $(1,2,\ldots,k)$.

Proof. These are straightforward calculations.

From this lemma we immediately obtain

LEMMA 6.6. *For each* $x\in X$, $dE_\gamma(x)\in L((S_2^s)_\beta;(S_2^s)_\beta)$ *and* $d^2E_\gamma(x)\in L((S_2^s)_\beta,(S_2^s)_\beta;(S_2^s)_\beta)$. *Moreover, the maps*

$$S_2^s\to L((S_2^s)_\beta;(S_2^s)_\beta);\ x\mapsto dE_\gamma(x),$$

and

$$S_2^s\to L((S_2^s)_\beta,(S_2^s)_\beta;(S_2^s)_\beta);\ x\mapsto d^2E_\gamma(x)$$

are of C^∞ *class.*

Lemma 6.7. *For each* $g\in\mathcal{R}^s$, $dF^s(g)\in L((S_2^s)_\beta;\boldsymbol{R})$ *and* $d^2F^s(g)\in L((S_2^s)_\beta;\boldsymbol{R})$. *Moreover, the maps*

$$\mathcal{R}^s\to L((S_2^s)_\beta;\boldsymbol{R});\ g\mapsto dF^s(g),$$

and

$$\mathcal{R}^s\to L((S_2^s)_\beta,(S_2^s)_\beta;\boldsymbol{R});\ g\mapsto d^2F^s(g)$$

are of C^∞ *class.*

Proof. From Proposition 4.4 and 4.5 we obtain

$$dF^s(g)(h) = (T(g), h)^0_g = (L_g^{-1}T(g), h)_g,$$
$$d^2F^s(g)(h, k) = (L_g^{-1}[dT(g) + \tfrac{1}{2}T(g)\,\mathrm{tr}\,(g)]h, k)_g.$$

Hence, using Proposition 5.3, we have $dF^s(g) \in L((S_2^s)_\beta; \boldsymbol{R})$ and $d^2F^s(g) \in L((S_2^s)_\beta, (S_2^s)_\beta; \boldsymbol{R})$. Moreover, it is easy to check that $g \mapsto dF^2(g)$ and $g \mapsto d^2F^s(g)$ are C^∞. □

Proof of Proposition 6.4. We have

$$d^2\tilde{f}(x)(y, z) = d^2F^s(E_\gamma(x))(dE_\gamma(x)(y), dE_\gamma(x)(z)) + dF^s(E_\gamma(x))(d^2E_\gamma(x)(y, z)).$$

Therefore, the proposition follows from Lemmas 6.6 and 6.7. □

At the origin of X we have $d^2\tilde{f}(0)(x, x) = d^2F^s(\gamma)(dE_\gamma(0)(x), dE_\gamma(0)(x)) = d^2F^s(\gamma)(x, x)$. Since $x \in (\delta_\gamma^s)^{-1}(0)$, we have the following from Proposition 4.5.

$$\begin{aligned} d^2\tilde{f}(0)(x, x) &= 3\int_M [3(\Delta x_s^s)^2 + (\nabla_s\nabla^s x^{ji})(\nabla_t\nabla^t x_{ji})]\, dV(\gamma) \\ &= 3(L_\gamma^{-1}[\bar{\Delta}_\gamma^2 + 3\gamma\,\mathrm{tr}\,(\gamma)\bar{\Delta}_\gamma^2]x, x)_\gamma \qquad (6.2) \\ &= 3\hat{\beta}(L_\gamma^{-1}[\bar{\Delta}_\gamma^2 + 3\gamma\,\mathrm{tr}\,(\gamma)\bar{\Delta}_\gamma^2]x, x). \end{aligned}$$

Set $D = \bar{\Delta}_\gamma^2 + 3\gamma\,\mathrm{tr}\,(\gamma)\bar{\Delta}_\gamma^2$. The symbol of the differential operator D is given by $\sigma(D)(v)x = (\|v\|^4 + 3\gamma\|v\|^4\,\mathrm{tr}\,(\gamma))x$, for $v \in T_1^0(M)$ and $x \in ST_2(M)$. Thus $\sigma(D)(v)(v \neq 0)$ is injective. Hence, by the decomposition theroem ([9, Theorem 4.11]), we have

$$S_2^s = \mathrm{range}(D) \oplus \ker(D), \qquad (6.3)$$

because $D = D^*$(the L^2-adjoint of D). Moreover, it follows that $D^2 = D^*D$ is elliptic, and $D^2: S_2^s \to S_2^{s-8}$ is a Fredholm operator.

LEMMA 6.8. $\ker(D) = Y(=PS^s(\gamma))$.

Proof. From (6.2), $Dx = 0$ holds if and only if $\nabla_s\nabla^s x_{ij} = \Delta x_s^s = 0$. This condition is equivalent to $\nabla x = 0$, i.e., $x \in Y$, because M is connected and compact. □

Set $Z = \text{range }(D) \cap X$, *and we have a decomposition,*

$$S_2^s = (\delta_\gamma^{s+1})^*(A^{s+1}) \oplus Y \oplus Z.$$

We immediately obtain the following from (6.2).

PROPOSITION 6.9. $d^2\tilde{f}(0)(z, z) > 0$ *holds for* $z(\in Z) \neq 0$.

Since $\nabla_\gamma(L_\gamma^{-1}D) = (L_\gamma^{-1}D)\nabla_\gamma$ for $\gamma \in \mathscr{F}$, we have

$$L_\gamma^{-1}D((\delta_\gamma^{s+1})^*(A^{s+1})) \subset (\delta_\gamma^{s+1})^*(A^{s+1}),$$
$$L_\gamma^{-1}D(X) \subset X, \quad L_\gamma^{-1}D(Z) \subset Z. \tag{6.4}$$

Hence, we get from (6.2),

$$\hat{B}(0) = 3L_\gamma^{-1}D : \hat{Z}_\beta(\subset S_2^2) \to \hat{Z}_\beta.$$

LEMMA 6.10. $\hat{B}(0)$ *is invertible*

Proof. Obviously, $\hat{B}(0)$ is injective, hence, by the open mapping theorem we have only to show it to be surjective. From (6.3) (by replacing s with s-4), we have

$$S_2^s = \text{range }(L_\gamma^{-1}D) + L_\gamma^{-1}\ (\ker(D)).$$

Since L_γ^{-1} $(\ker(D)) = Y$, we conclude that $Z = L_\gamma^{-1}D(Z) = (L_\gamma^{-1}D)^2(Z)$ by noting (6.4). Hence $(L_\gamma^{-1}D)^2(\hat{Z}_\beta)$ is dense in $\hat{Z}_\beta$. On the other hand, $(L_\gamma^{-1}D)^2(\hat{Z}_\beta) = (L_\gamma^{-1})^2D^2(\hat{Z}_\beta)$ is closed because $(L_\gamma^{-1})^2D^2 : S_2^2 \to S_2^2$ is Fredholm. Therefore, $(L_\gamma^{-1}D)^2(\hat{Z}_\beta) = \hat{Z}_\beta$, which leads to $\hat{B}(0)(\hat{Z}_\beta) = (3L_\gamma^{-1}D)(\hat{Z}_\beta) = \hat{Z}_\beta$. □

From this lemma we have the following.

PROPOSITION 6.11. *There is a compact* β*-nondegenerate critical subset* $K \subset Y$ *of* $\tilde{f} : X(= Y \oplus Z) \to \boldsymbol{R}$, *which contains the origin.*

Proof. Noting Lemma 6.10 and that $\tilde{f}$ is of C_β^∞ class, we see that there is a neighbourhood $W \subset Y$ of the origin such that $\hat{B}(y)$ is invertible for $y \in W$. Since Y is of finite dimension, so locally compact, there is a compact subset $K = \bar{U}' \subset W$ ($\bar{U}'$ being the closure of the open set U') which contains the origin. □

We are now in a position to prove Theorem A′.

Proof of Theorem A′. From Propositions 6.1, 6.4, 6.9 and 6.11, the function $\tilde{f} : X(= Y \oplus Z) \to \boldsymbol{R}$ satisfies the assumptions of Corollary 5.2. Let $K = \bar{U}'$ and V

be the sets mentioned in Corollary 5.2. Since $E_\gamma : X \to S(\gamma)$ is a C^∞ diffeomorphism, there is a neighbourhood $W = E_\gamma(U' + V)$ of γ in $S(\gamma)$ such that $F^s(g) = 0$ implies $g \in \mathcal{F}^s_\Gamma$ $(\Gamma = \Gamma(\gamma))$ if $g \in W$. From Proposition 3.3, (b), there is a neighbourhood U of γ in $\mathcal{R}^s$ such that $U \subset \mathcal{D}^{s+1}(W)$. Then U satisfies the assertion of the theorem because $\mathcal{F}^s = \mathcal{D}^{s+1}(\mathcal{F}^s_\Gamma)$, and $F^s(\eta^* g) = F^s(g)$ holds for $\eta \in \mathcal{D}^{s+1}$ (Proposition 4.3). □

By virtue of Theorem A′ we prove Theorem A.

Proof of Theorem A. Let $\gamma \in \mathcal{F}$ and $U(\subset \mathcal{R}^s)$ be the neighbourhood mentioned in Theorem A′. Then, $U' = U \cap \mathcal{R}$ is a neighbourhood of γ in $\mathcal{R}$ because the inclusion map $\mathcal{R} \to \mathcal{R}^s$ is continuous (Sobolev lemma). This neighbourhood U' satisfies the assertion of Theorem A. □

Remark. The space $\mathcal{R}$ is an ILH-manifold [13]. Moreover, it is easy to see that $\mathcal{F}$ is an ILH-submanifold of $\mathcal{R}$.

7. Supplementary discussions

The purpose of this section is to prove the following theorem, which "globally" characterizes flat metrics.

THEOREM 7.1. *Suppose $n = \dim M \leq 6$ and $\mathcal{F} \neq \phi$. Then,*

$$\mathcal{F} = F^{-1}(0).$$

The theorem for $n \leq 5$ was proved by Patodi [2]. We give the proof for $n = 6$. Hereafter, we assume $n = \dim M = 6$.

The following is due to Tanno [3, Lemma 1].

LEMMA 7.2. *If $F(g) = 0$, then (M, g) is conformally flat and the scalar curvature τ is vanishing.*

The Gauss-Bonnet-Chern formula for $n = 6$ is given by

$$\chi(M) = \frac{1}{384\pi^3} \int_M [\tau^3 - 12\tau|\rho|^2 + 3\tau|R|^2 + 16 R^i{}_j R^j{}_k R^k{}_i - 24 R^{ik} R^{jm} R_{ijkm} + 24 R^{st} R_s^{jkm} R_{tjkm} - 8 R^{ijkm} R^s_{jkt} R^t_{ims} - 2 R^{ij}{}_{..km} R^{km}{}_{..st} R^{st}{}_{..ij}]\, dV(g).$$

When (M, g) is conformally flat and $\tau=0$, this reduces to

$$\chi(M)=\frac{1}{256\pi^3}\int_M R_j^i R_k^j R_i^k \, dV(g). \tag{7.1}$$

LEMMA 7.3. *Suppose (M, g) is conformally flat and $\tau=0$. If $\chi(M)=0$, then* $\nabla_i R_{jk}=0$.

Proof. By Tanno [3, Lemma 2], if (M, g) is conformally flat and $\tau=0$, we have

$$\int_M (\nabla_i R_{jk})(\nabla^i R^{jk}) dV(g) = -\tfrac{3}{2}\int_M R_j^i R_k^j R_i^k \, dV(g).$$

Using (7.1), we get $\nabla_i R_{jk}=0$ if $\chi(M)=0$. □

Proof of Theorem 7.1. Since $\mathcal{F}\neq\phi$, $\chi(M)=0$ holds. Tanno [3, Proposition 5] showed that if (M, g) is conformally flat and $\tau=\nabla_i R_{jk}=0$, then (M, g) is either (1) locally flat, or (2) Riemannian product $S^3(c)\times[H^3(-c)/\Lambda]$, Λ being some discontinuous group of isometries of $H^3(-c)$. On the other hand, the homotopy group $\pi_3(S^3(c)\times[H^3(-c)/\Lambda])=\mathbf{Z}$, hence the manifold $S^3(c)\times[H^3(-c)/\Lambda]$ has no flat metrics (Cartan-Hardamard Theorem). Now, the proof is completed by virtue of Lemmas 7.2 and 7.3. □

Remark. For $n\geqslant 7$, the author does not know wether there is such a manifold that satisfies

$$F^{-1}(0)\neq\mathcal{F}\neq\phi.$$

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