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On the characterization of flat metrics by the spectrum

RUISHI KUWABARA

1. Introduction

Let M be an n -dimensional compact, connected, oriented C^∞ manifold without boundary. Let \mathcal{R} be the space of C^∞ Riemannian metrics on M with the C^∞ topology. For $g \in \mathcal{R}$, $\text{Spec}(M, g)$ denotes the spectrum of the Laplace-Beltrami operator $\Delta = -g^{ij}\nabla_i\nabla_j$ acting on C^∞ functions on M , namely,

$$\text{Spec}(M, g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\},$$

where each eigenvalue is written as many times as its multiplicity. Then, the Minakshisundaram's formula for $\text{Spec}(M, g)$ is given by

$$\sum_{k=0}^{\infty} \exp(-\lambda_k t) \underset{t \downarrow 0}{\sim} \left(\frac{1}{4\pi t}\right)^{n/2} \sum_{s=0}^{\infty} a_s t^s,$$

where the coefficients a_s 's are expressed by the metric and its derivatives (curvature) (cf. [1], [2], [3]).

It is obvious that if (M, g) is flat, $a_s = 0$ holds for $s \geq 1$. However, $a_s = 0$ ($s \geq 1$) does not imply that (M, g) is flat. In fact, Patodi [2] showed that for the non-flat space $S^3(c) \times [H^3(-c)/\Lambda]$, the coefficients a_s 's vanish for $s \geq 1$. Here, $S^3(c)$ and $H^3(-c)$ are a Euclidean 3-sphere with constant curvature $c > 0$ and a hyperbolic 3-space with constant curvature $-c$, respectively, and Λ is some discontinuous group of motions of $H^3(-c)$. In the low dimensional cases, the following has been shown.

THEOREM. (1) (Patodi [2]) *For $2 \leq n \leq 5$, $a_2 \geq 0$ holds, and equality holds if and only if (M, g) is flat.*

(2) (Tanno [3]) *For $n = 6$, $a_2 \geq 0$ holds, and if $a_2 = a_3 = 0$, then (M, g) is flat or locally Riemannian product $S^3(c) \times H^3(-c)$.*

The purpose of this paper is to prove the following theorem which asserts that the condition $a_2 = 0$ 'locally' characterizes flat metrics.

THEOREM A. *Suppose γ is a C^∞ flat Riemannian metric on M . Then, there is a neighbourhood U of γ in \mathcal{R} such that if $g \in U$ and $a_2(g) = 0$, g is also a flat metric.*

Remark. For $2 \leq n \leq 6$, the neighbourhood U in Theorem A can be taken equally to the whole space \mathcal{R} , that is, if M admits a flat metric then $a_2(g) = 0$ implies that g is flat (see §7). For $n \geq 7$, the author does not know whether there are counterexamples or not.

As a corollary of Theorem A, we have the following theorem.

THEOREM B. *Suppose (M, γ) is a flat manifold. Then, there is a neighbourhood U of γ in \mathcal{R} such that if $g \in U$ and $\text{Spec}(M, g) = \text{Spec}(M, \gamma)$, then $(M, g) = (M, \gamma)$ (isometric).*

In order to derive this theorem, we have only to note the following result of Kneser and Sunada [4].

THEOREM (Kneser, Sunada). *There are only finitely many isometry classes of flat manifolds with a given spectrum.*

Remark. In the previous paper [5] we showed that a metric of flat torus is characterized in the “infinitesimal” sense by its spectrum. Theorem B is an extension of this result.

After giving notations and a fundamental lemma in §2, we review in §3 the properties concerning the space of flat metrics following Fischer and Marsden [6], [7]. In §4 we study the function $a_2(g)$ and calculate its derivatives. In §5 we establish the weak Morse lemma for normed spaces, which gives a basic tool for the proof of the main theorem. Then we prove Theorem A in §6. Finally in §7 we consider the “global” characterization of flat metrics.

Remark. Fischer and Marsden gave a theorem [6, Theorem 1.5.2], [7, Theorem 10] which is of same type as our Theorem A. Our proof is performed on the same lines as in [7], but differently in details.

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2. Preliminaries

Let \dot{M} be an n -dimensional compact, connected, oriented C^∞ manifold without boundary. Let $T_q^p(M)$ denote the tensor bundle of type (p, q) over M , and

$ST_2(M)$ the bundle of symmetric covariant 2-tensors on M . For a C^∞ Hermitian vector bundle T , let $C^\infty(T)$ be the space of C^∞ cross-sections of T , and $H^s(T)$ the Sobolev space of cross-sections of T with respect to a fixed C^∞ Riemannian metric. The topology of $H^s(T)$ does not depend on the choice of a metric.

We use the following notations.

$V^s = H^s(T_0^1(M))$; the H^s vector fields,
 $A^s = H^s(T_1^0(M))$; the 1-forms of class H^s ,
 $S_2^s = H^s(ST_2(M))$; the symmetric covariant 2-tensor fields of class H^s ,
 \mathcal{D}^s ; the group of H^s diffeomorphisms of M , defined for $s > (n/2) + 1$ (see Ebin [8]). The group \mathcal{D}^{s+1} acts on S_2^s as follows;

$$S_2^s \times \mathcal{D}^{s+1} \rightarrow S_2^s; (h, \eta) \mapsto \eta^* h,$$

where $\eta^* h$ denotes the pull-back of h by η .

$\mathcal{R}^s (\subset S_2^s)$; the Hilbert manifold of Riemannian metrics of class H^s . The manifold \mathcal{R}^s is an open convex positive cone in S_2^s , and invariant under the action of \mathcal{D}^{s+1} .

$\mathcal{F}^s (\subset \mathcal{R}^s)$; the subset of flat metrics of class H^s , defined for $s > (n/2) + 1$.

If the s is omitted, the space is understood to be of C^∞ class and endowed with the C^∞ topology.

We define various inner products of $H^s(T)$ ($s > (n/2) + 1$) by $g \in \mathcal{R}^s$ as follows;

$$(a) \langle T, T' \rangle_g^0 = g_{ii'} \cdots g_{jj'} g^{kk'} \cdots g^{mm'} T_{k \cdots m}^{i \cdots j} T_{k' \cdots m'}^{i' \cdots j'},$$

$$(b) \langle T, T' \rangle_g^k = \sum_{r=0}^k \langle \nabla_g^{(r)} T, \nabla_g^{(r)} T' \rangle_g^0 \quad (k \leq s),$$

where $\nabla_g^{(r)} T$ is the tensor field $\overbrace{\nabla_g \cdots \nabla_g}^r T$ and ∇_g is the covariant derivative with respect to g .

$$(c) (T, T')_g^k = \int_M \langle T, T' \rangle_g^k dV(g),$$

where $dV(g)$ denotes the volume element induced from g .

Using the above inner product (c), we can introduce the Riemannian structure on \mathcal{R}^s by $g \mapsto (\cdot)_g^k$. This metric is \mathcal{D}^{s+1} -invariant, i.e., \mathcal{D}^{s+1} acts by isometry (see [8, pp. 18–21]).

For a metric $g \in \mathcal{R}$, we define a differential operator

$$\delta_g : C^\infty(ST_2(M)) \rightarrow C^\infty(T_1^0(M)); \quad (\delta_g \xi)_j = -\nabla_g^i \xi_{ij}.$$

Then δ_g extends to a continuous linear map $\delta_g^s : S_2^s \rightarrow A^{s-1}$. The adjoint operator δ_g^* of δ_g with respect to $(\cdot, \cdot)_g^0$ extends to a map

$$(\delta_g^s)^* : A^s \rightarrow S_2^{s-1}; \quad \{(\delta_g^s)^* \xi\}_{ij} = \frac{1}{2}(\mathcal{L}_X g)_{ij},$$

where $s > (n/2) + 1$, and \mathcal{L} is the Lie derivative and $X(\in V^s)$ is dual to ξ .

LEMMA 2.1 (Berger and Ebin [9]). *For $g \in \mathcal{R}$, there is an orthogonal decomposition*

$$S_2^s = (\delta_g^s)^{-1}(0) \oplus (\delta_g^{s+1})^*(A^{s+1}),$$

where the summands are orthogonal with respect to $(\cdot, \cdot)_g^0$.

3. Space of flat metrics

In [6] and [7] Fischer and Marsden studied the space \mathcal{F}^s of flat metrics of class H^s . We review their results in the first part of this section (Lemma 3.1 and Proposition 3.2).

In Lemma 2.1, g is assumed to be of C^∞ class (more precisely, g is required to be of class H^{s+1}). However, if g is flat, the following is obtained by one of the regularity theorems.

LEMMA 3.1 ([6, p. 237], [7, p. 530]). *Let $g \in \mathcal{F}^s$, $s > (n/2) + 1$. Then there is an orthogonal decomposition*

$$S_2^s = (\delta_g^s)^{-1}(0) \oplus (\delta_g^{s+1})^*(A^{s+1}).$$

We denote by $\Gamma(g)$ the Riemannian connection of $g \in \mathcal{R}^s$. Let \mathcal{K}^s be the set of flat Riemannian connections of class H^s . For $\Gamma \in \mathcal{K}^{s-1}$, set

$$\mathcal{F}_1^s = \{g \in \mathcal{F}^s; \Gamma(g) = \Gamma\}.$$

Furthermore, for $g \in \mathcal{R}^s$, let us define

$$E_g : S_2^s \rightarrow \mathcal{R}^s; \quad h \mapsto g \exp(g^{-1}h),$$

where $g^{-1}h$ is an endomorphism of $T_x(M)$ at each $x \in M$, given by $h_j^i = g^{ik}h_{kj}$ in local coordinates. Then E_g is a C^∞ diffeomorphism with $E_g(0) = g$ (see [8, p. 36]).

PROPOSITION 3.2. *Let $\Gamma \in \mathcal{H}^{s-1}$ and $g \in \mathcal{F}_\Gamma^s$, $s > (n/2) + 1$. Set $PS_2^s(g) = \{h \in S_2^s; \nabla_g h = 0\}$. Then,*

(a) $\mathcal{F}_\Gamma^s = E_g(PS_2^s(g))$, and \mathcal{F}_Γ^s is a finite dimensional closed C^∞ submanifold of \mathcal{R}^s . Moreover, the tangent space of \mathcal{F}_Γ^s at g is

$$T_g(\mathcal{F}_\Gamma^s) = PS_2^s(g).$$

(b) $\mathcal{F}^s = \mathcal{D}^{s+1}(\mathcal{F}_\Gamma^s) = \{\eta^* \gamma \in \mathcal{R}^s; \eta \in \mathcal{D}^{s+1}, \gamma \in \mathcal{F}_\Gamma^s\}$, and \mathcal{F}^s is a closed C^∞ submanifold of \mathcal{R}^s . Moreover,

$$T_g(\mathcal{F}^s) = PS_2^s(g) \oplus (\delta_g^{s+1})^*(A^{s+1}).$$

Proof. See Fischer and Marsden [6, Theorem I.3.3], [7, Theorem 6].

In the remainder of this section, let us prove the following Proposition 3.3. For $g \in \mathcal{F}_\Gamma^s$, set

$$S(g) = E_g((\delta_g^s)^{-1}(0)).$$

Then we have the following.

PROPOSITION 3.3. (a) $S(g)$ is a closed C^∞ submanifold of \mathcal{R}^s , and \mathcal{F}_Γ^s is a closed C^∞ submanifold of $S(g)$. Moreover,

$$T_g(S(g)) = (\delta_g^s)^{-1}(0).$$

(b) For any neighbourhood V of g in $S(g)$, there is a neighbourhood U of g in \mathcal{R}^s such that $U \subset \mathcal{D}^{s+1}(V)$.

Proof. (a) We have $PS_2^s(g) \subset (\delta_g^s)^{-1}(0) \subset S_2^s$, where each subspace is closed. Therefore, the assertion is obvious because E_g is a C^∞ diffeomorphism.

(b) By the regularity theorem ([6, Theorem I.3.1], [7, Theorem 5]), there is $\eta \in \mathcal{D}^{s+1}$ such that $\eta^* g = g'$ belongs to \mathcal{F} . Hence, the orbit $O^s(g)$ through g is equal to $O^s(g')$ and is a C^∞ submanifold of \mathcal{R}^s . Let $N = N(O^s(g))$ be the normal bundle with respect to the weak Riemannian metric $\gamma \mapsto (\cdot, \cdot)_\gamma^0$ ([8, pp. 30–31]). We define $E: N \rightarrow \mathcal{R}^s$ by $E(\gamma, h) = E_\gamma(h)$, where $\gamma \in O^s(g)$ and $h \in N_\gamma = (\delta_\gamma^s)^{-1}(0)$, N_γ being the fibre of N at γ . Then, it is easily shown that E is a C^∞ map and $E(\eta^* \gamma, \eta^* h) = \eta^* E(\gamma, h)$ holds for $\eta \in \mathcal{D}^{s+1}$. Moreover, the first derivative of E

at $(g, 0)$ is given by

$$dE(g, 0)(h', h'') = h' + h'',$$

where $h' \in T_g(O^s(g)) = (\delta_g^{s+1})^*(A^{s+1})$ and $h'' \in N_g = (\delta_g^s)^{-1}(0)$. Thus, $dE(g, 0)$ is an isomorphism (Lemma 3.1). Therefore, there are a neighbourhood U' of g in \mathcal{R}^s and a neighbourhood W of $(g, 0)$ in N such that $E: W \rightarrow U'$ is a diffeomorphism. Let $\gamma \mapsto (\cdot)_\gamma^s$ be the strong Riemannian metric of \mathcal{R}^s . Then, the neighbourhood W is given by

$$W = \{(\gamma, h) \in N; \gamma \in W', (h, h)_\gamma^s < \varepsilon, \varepsilon > 0\},$$

W' being a neighbourhood of g in $O^s(g)$. For given $V(\subset S(g))$ there is $\varepsilon'(\leq \varepsilon)$ such that if $V' = \{(g, h) \in N_g; (h, h)_g^s < \varepsilon'\}$, $E_g(V') \subset V$ holds. Set

$$V'' = \{(\gamma, h) \in N; \gamma \in W', (h, h)_\gamma^s < \varepsilon'\} \subset W,$$

and $U = E(V'')$. Then U is open in \mathcal{R}^s and satisfies $U \subset \mathcal{D}^{s+1}(V)$. In fact, if γ is in U and $\gamma = E(\eta^*g, h)$, then $(\eta^{-1})^*h = h'$ belongs to V' because $(\eta^{-1})^*: S_2^s \rightarrow S_2^s$ is an isometry with respect to the metric $(\cdot)_\gamma^s$. Thus, $\gamma = E(\eta^*g, \eta^*h') = \eta^*E(g, h') = \eta^*E_g(h') \subset \mathcal{D}^{s+1}(V)$. \square

4. Derivatives of $a_2(g)$

For $g \in \mathcal{R}$, let $\{^i_{jk}\}$, R^i_{jkm} , R_{ij} and τ denote the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature, respectively. The curvature tensor is defined by

$$R^i_{jkm} = \frac{\partial}{\partial x^k} \left\{ \begin{matrix} i \\ jm \end{matrix} \right\} - \frac{\partial}{\partial x^m} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \left\{ \begin{matrix} s \\ jm \end{matrix} \right\} \left\{ \begin{matrix} i \\ sk \end{matrix} \right\} - \left\{ \begin{matrix} s \\ jk \end{matrix} \right\} \left\{ \begin{matrix} i \\ sm \end{matrix} \right\},$$

in terms of the local coordinates (x^i) .

It is known that the Minakshisundaram's coefficient a_2 is given by

$$a_2 = \frac{1}{360} \int_M (2|R|^2 - 2|\rho|^2 + 5\tau^2) dV(g) = \frac{1}{360} F(g),$$

Where $|R|^2 = R_{ijkm} R^{ijkm}$ and $|\rho|^2 = R_{ij} R^{ij}$ (cf. [1], [2], [3]).

It is easily shown that $\text{Spec}(M, \eta^*g) = \text{Spec}(M, g)$ for $\eta \in \mathcal{D}$ and $g \in \mathcal{R}$, hence $F(\eta^*g) = F(g)$ holds.

The function F can be regarded to be defined on \mathcal{R}^s if $s > (n/2) + 4$. We write this function F^s .

PROPOSITION 4.1. *The function F^s on \mathcal{R}^s is of C^∞ class.*

We need the following lemma which was proved in [10, 11.3].

LEMMA 4.2. *If ξ and η are C^∞ vector bundles over M and $f: \xi \rightarrow \eta$ is a C^∞ fibre preserving map, then for $s > n/2$ the map $f_*: H^s(\xi) \rightarrow H^s(\eta)$ defined by $f_*(\alpha) = f \circ \alpha$ is of C^∞ class.*

Proof of Proposition 4.1. We prove that $g \mapsto \int_M |R|^2 dV(g)$ is a C^∞ function. The proof is done in two steps.

First step: $\phi: g \mapsto |R|^2$ is a C^∞ map of \mathcal{R}^s into $H^{s-2}(M, \mathbf{R})$, the Hilbert space of all H^{s-2} functions. In fact, we have

$$|R|^2 = R_{bcd}^a R_{jkm}^i g_{ai} g^{bj} g^{ck} g^{dm}.$$

Thus, as is easily shown, $|R|^2$ is a rational combinations of g , dg , d^2g , so that $|R|^2: J^2(\xi) \rightarrow M \times \mathbf{R}$ is a C^∞ fibre preserving map, where ξ is the fibre subbundle of $ST_2(M)$ consisting of positive definite forms on each tangent space and $J^2(\xi)$ the second jet bundle of ξ . Noting that $\mathcal{R}^s = H^s(\xi) \subset H^{s-2}(J^2(\xi))$, we can conclude from Lemma 4.2 that ϕ is a C^∞ map of \mathcal{R}^s into $H^{s-2}(M, \mathbf{R})$.

Second step: The function $\psi: H^{s-2}(M, \mathbf{R}) \times \mathcal{R}^s \rightarrow \mathbf{R}$ defined by $(f, g) \mapsto \int_M f dV(g)$ is of C^∞ class. In fact, fix $g_0 \in \mathcal{R}^s$ and define $\mu: \mathcal{R}^s \rightarrow H^s(M, \mathbf{R})$ by the equation $\mu(g) dV(g_0) = dV(g)$. Then it is easy to see that the map μ is of C^∞ class (see [8]). The map ψ is decomposed as $\psi = \psi_0 \circ (\text{id} \times \mu)$, where $\psi_0: H^{s-2}(M, \mathbf{R}) \times H^2(M, \mathbf{R}) \rightarrow \mathbf{R}$ is defined by $(f, f') \mapsto \int_M f f' dV(g_0)$. Since μ and ψ_0 are C^∞ maps, ψ is of C^∞ class.

Finally, the function $g \mapsto \int_M |R|^2 dV(g)$ is decomposed as follows:

$$\begin{array}{ccccc} & & H^{s-2}(M, \mathbf{R}) \times H^s(M, \mathbf{R}) & & \\ & \nearrow \text{id} \times \mu & & \searrow \psi_0 & \\ \mathcal{R}^s & \xrightarrow{\phi \times \text{id}} & H^{s-2}(M, \mathbf{R}) \times \mathcal{R}^s & \xrightarrow{\psi} & \mathbf{R} \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\ g & \longmapsto & (|R|^2, g) & \longmapsto & \int_M |R|^2 dV(g) \end{array}$$

Since ϕ and ψ are C^∞ maps, $g \mapsto \int_M |R|^2 dV(g)$ is of C^∞ class.

It is similarly shown that the functions $g \mapsto \int_M |\rho|^2 dV(g)$ and $g \mapsto \int_M \tau^2 dV(g)$ are of C^∞ class. \square

PROPOSITION 4.3. $F^s(\eta^*g) = F^s(g)$ holds for $\eta \in \mathcal{D}^{s+1}$.

Proof. The action $S_2^s \times \mathcal{D}^{s+1} \rightarrow S_2^s$ is continuous ([8, pp. 17–18]), and F^s is of C^∞ class. Hence, the proposition follows from $F(\eta^*g) = F(g)$ for $g \in \mathcal{R}$ and $\eta \in \mathcal{D}$.

Now, we give the formulas about the derivatives of F^s , which have been calculated in the previous paper [5].

PROPOSITION 4.4. For $g \in \mathcal{R}^s$ and $h \in S_2^s$, the first derivative of F^s is given by

$$dF^s(g)(h) = \int_M \langle T(g), h \rangle_g^0 dV(g) = \int_M T_{ij}(g) h^{ij} dV(g), \quad (4.1)$$

where

$$\begin{aligned} T_{ij}(g) = & 12\nabla_i \nabla_j \tau - 6\nabla_k \nabla^k R_{ij} + 8R_{ik} R_j^k - 4R_{kimj} R^{km} + 4R_{ikms} R_j^{kms} \\ & + 9(\Delta\tau)g_{ij} - 10\tau R_{ij} + |R|^2 g_{ij} - |\rho|^2 g_{ij} + \frac{5}{2}\tau^2 g_{ij}, \end{aligned}$$

∇ and the curvatures being induced from g . Therefore, if $g \in \mathcal{F}^s$, then $dF^s(g) = 0$, i.e., a flat metric is a critical point of F^s .

Proof. This is a direct but tedious calculation (cf. [5]).

Remark. $T(g)$ is an element of S_2^{s-4} , and $g \mapsto T(g)$ is a C^∞ map of \mathcal{R}^s into S_2^{s-4} . This is proved on the same lines as Proposition 4.1.

PROPOSITION 4.5. The second derivative of F^s at $g \in \mathcal{R}^s$ is given by

$$d^2F^s(g)(h, k) = \int_M \langle [dT(g) + \frac{1}{2}T(g)\text{tr}(g)]h, k \rangle_g^0 dV(g), \quad (4.2)$$

where $\text{tr}(g)h = g^{ij}h_{ij}$. In particular, at $g \in \mathcal{F}^s$,

$$\begin{aligned} d^2F^s(g)(h, h) = & 3 \int_M [6(\Delta h_s^s)(\nabla_i \nabla_j h^{ji}) + 3(\Delta h_s^s)^2 \\ & + 4(\nabla^k \nabla^m h_{km})(\nabla_i \nabla_j h^{ji}) - 2(\nabla_k \nabla_i h^{ji})(\nabla^k \nabla_m h_j^m) \\ & + (\nabla_k \nabla^k h^{ji})(\nabla_m \nabla^m h_{ji})] dV(g). \end{aligned} \quad (4.3)$$

Proof. This is obtained by straightforward calculation starting from (4.1).

Remark. $dT(g) + (1/2)T(g)\text{tr}(g)$ is an element of $L(S_2^s; S_2^{s-4})$, the space of all continuous linear maps of S_2^s into S_2^{s-4} .

5. Weak Morse lemma for normed spaces

In this section we establish the weak Morse lemma for normed spaces. This work is motivated by Tromba's paper [11], in which the Morse lemma for almost-Riemannian manifolds is considered.

Let X_1, X_2, \dots be normed vector spaces, and define $L(X_1, \dots, X_k; X_{k+1})$ as the normed vector space of all continuous k -linear maps of $X_1 \dots X_k$ into X_{k+1} .

Let β be a continuous bilinear form on a normed vector space X , i.e., $\beta \in L(X, X; \mathbf{R})$. β is called the weak inner product of X if (a) $\beta(x, y) = \beta(y, x)$, (b) $\beta(x, x) > 0$ for $x \neq 0$. The space X with β is regarded as a pre-Hilbert space denoted by X_β . Let \hat{X}_β be the completion of X_β , and $\hat{\beta}$ the continuous extension of β to \hat{X}_β . Thus the space \hat{X}_β is a Hilbert space with inner product $\hat{\beta}$. The canonical injection $X \rightarrow X_\beta(\hat{X}_\beta)$ is continuous.

Let $f: X \rightarrow \mathbf{R}$ be a C^k function, $k \geq 2$.

DEFINITION. The C^k function f is of C_β^k class if

- (a) for each $x \in X$, the second derivative $d^2f(x)$ belongs to $L(X_\beta, X_\beta; \mathbf{R})$.
- (b) $x \mapsto d^2f(x)$ is a C^{k-2} map of X into $L(X_\beta, X_\beta; \mathbf{R})$.

Suppose $X = Y \times Z$ (the product normed space), and $f: X \rightarrow \mathbf{R}$ is a C_β^k function ($k \geq 2$). We have

$$d^2f(x)((u, v), (u', v')) = D_1^2f(x)(u, u') + D_1D_2f(x)(u, v') + D_2D_1f(x)(v, u') + D_2^2f(x)(v, v').$$

where $(u, v), (u', v') \in Y \times Z$, and $D_i f(x)$ ($i = 1, 2$) is the partial derivative of f at x with respect to the i -th variable. Since f is of C_β^k class, there is a unique $B(x) \in L(Z_\beta; \hat{Z}_\beta)$ such that

$$D_2^2f(x)(u, v) = \hat{\beta}(B(x)u, v),$$

for $u, v \in Z$. Moreover, $x \mapsto B(x)$ is a C^{k-2} map of X into $L(Z_\beta; \hat{Z}_\beta)$.

DEFINITION. Let K be a subset of Y . The subset $K \times \{0\}$ of X is called the β -nondegenerate critical subset of f , if for each $x \in K \times \{0\}$,

- (a) $df(x) = 0$, and
- (b) $\hat{B}(x)$, the continuous extension of $B(x)$ to \hat{Z}_β , is invertible.

We are now ready to state and prove the following.

PROPOSITION 5.1 (weak Morse lemma). Let $f: X = Y \times Z \rightarrow \mathbf{R}$ be a C_β^k function, $k \geq 2$. Suppose K is a compact subset of Y . If the subset $K \times \{0\}$ is a

β -nondegenerate critical subset of f and $f(K \times \{0\}) = 0$, then there are a neighbourhood V of the origin in Z and C^{k-2} map $\phi: K \times V \rightarrow \hat{Z}_\beta$ such that

(a) $\phi(x) = 0$ if and only if $x = (y, 0)$, and

(b) $f(x) = \frac{1}{2} \widehat{D}_2^2 f((y, 0))(\phi(x), \phi(x))$, $x = (y, z) \in K \times V$,

where $\widehat{D}_2^2 f(x)$ is the continuous extension of $D_2^2 f(x)$ to $\hat{Z}_\beta \times \hat{Z}_\beta$.

Proof. By the Taylor's formula we have

$$f((y, z)) = \int_0^1 (1 - \lambda) D_2^2 f((y, \lambda z))(z, z) d\lambda.$$

Set

$$J(y, z)(u, v) = \int_0^1 (1 - \lambda) D_2^2 f((y, \lambda z))(u, v) d\lambda.$$

Then, $J(y, z)$ belongs to $L(Z_\beta, Z_\beta; \mathbf{R})$ since f is of C_β^k class. Therefore, we can write $J(y, z)(u, v) = \hat{\beta}(B(y, z)u, v)$ and $D_2^2 f((y, 0))(u, v) = 2\hat{\beta}(B(y, 0)u, v)$ where $B(y, z) \in L(Z_\beta; \hat{Z}_\beta)$. Let $\hat{B}(y, z)$ be the continuous extension of $B(y, z)$ to \hat{Z}_β . Then, $(y, z) \mapsto \hat{B}(y, z)$ is a C^{k-2} map of X into $L(\hat{Z}_\beta; \hat{Z}_\beta)$. Moreover, $\hat{B}(y, z)$ is self-adjoint for each (y, z) . Since $\hat{B}(y, 0)$ is invertible and K is compact, so $\hat{B}(y, z)$ is invertible in $K \times V'$, V' being a neighbourhood of the origin. Define $Q(y, z) = \hat{B}(y, z)^{-1} \hat{B}(y, 0)$ and Q is a C^{k-2} map of $K \times V'$ into $L(\hat{Z}_\beta; \hat{Z}_\beta)$. Now $Q(y, 0) = \text{identity}$ and since a square root function is defined in a neighbourhood of the identity operator by a convergent power series with real coefficients, we can define a C^{k-2} map $R: K \times V (\subset K \times V') \rightarrow L(\hat{Z}_\beta; \hat{Z}_\beta)$ with each $R(y, z)$ invertible and $Q(y, z) = [R(y, z)]^2$. We see easily from the definition of Q that $Q(y, z)^* \hat{B}(y, z) = \hat{B}(y, z) Q(y, z)$ hence $R(y, z)^* \hat{B}(y, z) = \hat{B}(y, z) R(y, z)$ holds. Thus, we have $R(y, z)^* \hat{B}(y, z) R(y, z) = \hat{B}(y, 0)$, or $\hat{B}(y, z) = R_1(y, z)^* \hat{B}(y, 0) R_1(y, z)$, where $R_1(y, z) = R(y, z)^{-1}$. Now, set $\phi((y, z)) = R_1(y, z)z$, and we have

$$\begin{aligned} f((y, z)) &= \hat{\beta}(R_1(y, z)^* \hat{B}(y, 0) R_1(y, z) z, z) \\ &= \hat{\beta}(\hat{B}(y, 0) \phi((y, z)), \phi((y, z))). \end{aligned}$$

Finally, $\phi((y, z)) = R_1(y, z)z = 0$ holds if and only if $z = 0$, because $R_1(y, z)$ is invertible. \square

COROLLARY 5.2. Besides assumptions in Proposition 5.1, assume that

$$D_2^2 f((y, 0))(u, u) > 0$$

holds for $y \in K$ and $u \in Z \neq 0$. If $f(x) = 0$ and $x \in K \times V$, then x belongs to $K \times \{0\}$.

Proof. From Proposition 5.1, we have only to prove that $\widehat{D_2^2 f}((y, 0))(u, u) > 0$ holds for any $u \in \hat{Z}_\beta \neq 0$. Suppose there is $u \neq 0$ such that $\widehat{D_2^2 f}((y, 0))(u, u) = \hat{\beta}(\hat{B}(y, 0)u, u) = 0$. Then, $\inf_{\beta(u, u)=1} \hat{\beta}(\hat{B}(y, 0)u, u) = 0$, hence zero belongs to the spectrum of $\hat{B}(y, 0)$, which is absurd because $\hat{B}(y, 0)$ is invertible. \square

In the remainder of this section we give a supplement.

Let us define a C^∞ map $x \in X \mapsto \beta(x)$ (the weak inner product of X) such that the topology of $X_{\beta(x)}$ does not depend on x . We call this map the weak C^∞ Riemannian structure of X . Let $\beta = \beta(0)$. Then, for each $x \in X$, there is $C(x) \in L(\hat{X}_\beta; \hat{X}_\beta)$ such that

$$\hat{\beta}(x)(y, z) = \hat{\beta}(C(x)y, z), \quad y, z \in \hat{X}_{\beta(x)} (= \hat{X}_\beta),$$

and $x \mapsto C(x)$ is of C^∞ class. Moreover, we can easily prove the following.

PROPOSITION 5.3. *Let $f: X \rightarrow \mathbf{R}$ be a C^k function ($k \geq 2$). f is of C_β^k class if and only if*

- (a) *for each $x \in X$, $d^2 f(x) \in L(X_{\beta(x)}, X_{\beta(x)}; \mathbf{R})$, and*
- (b) *if $B(x)$ is given by $d^2 f(x)(u, v) = \hat{\beta}(x)(B(x)u, v)$, then $x \mapsto B(x)$ is a C^{k-2} map of X into $L(X_\beta; \hat{X}_\beta)$.*

6. Proof of the main theorem

In this section we prove the following theorem and Theorem A.

THEOREM A'. *Let $\gamma \in \mathcal{F}$ and s be sufficiently large. Then, there is a neighbourhood $U \subset \mathcal{R}^s$ of γ such that if $g \in U$ and $F^s(g) = 0$, g is in \mathcal{F}^s .*

We define $f: S_2^s \rightarrow \mathbf{R}$ by $f = F^s \circ E_\gamma$. Let \tilde{f} be the restriction of f to $X = (\delta_\gamma^s)^{-1}(0) (\subset S_2^s)$. Then, \tilde{f} is a C^∞ function (Proposition 4.1). Let $Y = PS_2^s(\gamma)$. We have the following from Propositions 3.2 and 4.4.

PROPOSITION 6.1. *$\tilde{f}(y) = d\tilde{f}(y) = 0$ holds for each $y \in Y$.*

We apply Corollary 5.2 to the function \tilde{f} on the Hilbert space X .

Let us introduce a weak C^∞ Riemannian structure on X . First, we define a weak Riemannian metric on \mathcal{R}^s as follows;

$$\begin{aligned} (h, k)_g &= \int_M [\langle h, k \rangle_g^0 + 2\langle \nabla h, \nabla k \rangle_g^0 + \langle \nabla \nabla h, \nabla \nabla k \rangle_g^0] dV(g) \\ &= ((1 + \bar{\Delta}_g)^2 h, k)_g^0, \end{aligned} \tag{6.1}$$

where $\bar{\Delta}_g$ is the rough Laplacian defined by $(\bar{\Delta}_g h)_{ij} = -g^{st} \nabla_s \nabla_t h_{ij}$ in local coordinates.

LEMMA 6.2. *Let $L_g = (1 + \bar{\Delta}_g)^2$. Then, the maps*

$$\mathcal{R}^s \times S_2^s \rightarrow S_2^{s-4}; (g, h) \mapsto L_g h,$$

and

$$\mathcal{R}^s \times S_2^{s-4} \rightarrow S_2^s; (g, h) \mapsto L_g^{-1} h$$

are of C^∞ class.

Proof. First, we note that for each $g \in \mathcal{R}^s$, L_g has a continuous linear inverse L_g^{-1} . In fact, the differential operator $(1 + \bar{\Delta}_g)^2$ is an injective self-adjoint elliptic operator. Therefore, L_g is surjective by the decomposition theorem (e.g. [12, Ch. XI]). Furthermore, by the open mapping theorem L_g has a continuous inverse.

Now, it is easily shown that $(g, h) \mapsto L_g h$ is C^∞ (cf. [13, Lemma 2.11]). Moreover, it follows that $g \mapsto L_g$ is a C^∞ map of \mathcal{R}^s into $L(S_2^s; S_2^{s-4})$. On the other hand, $L_g \mapsto L_g^{-1}$ is a C^∞ map (e.g. [14, Ch. 8]). Therefore, $g \mapsto L_g^{-1}$ is C^∞ and accordingly $(g, h) \mapsto L_g^{-1} h$ is C^∞ . \square

PROPOSITION 6.3. *The Riemannian structure defined by (6.1) is of C^∞ class.*

Proof. The proposition follows from Lemma 6.2 and the proof of Proposition 4.1. \square

Now, we define a C^∞ Riemannian structure $\beta(x)$ on S_2^s as the pull-back of $(\cdot)_g$ by E_γ . Namely,

$$\beta(x)(y, z) = (dE_\gamma(x)(y), dE_\gamma(x)(z))_g,$$

where $g = E_\gamma(x)$.

Let $\beta = \beta(0)$. Obviously, $(\widehat{S_2^s})_\beta = S_2^2$ holds.

PROPOSITION 6.4. *The function $\tilde{f}: X \rightarrow \mathbf{R}$ is of C_β^∞ class.*

For the proof we first prove the following lemmas.

LEMMA 6.5. *The first and the second derivatives of E_γ are given by*

$$dE_\gamma(x)(y) = \gamma \left[\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \{(\gamma^{-1}x)^k(\gamma^{-1}y)\} \right],$$

and

$$d^2E_\gamma(x)(y, z) = \gamma \left[\sum_{k=0}^{\infty} \frac{1}{(k+2)!} \{(\gamma^{-1}x)^k(\gamma^{-1}y)(\gamma^{-1}z)\} \right],$$

respectively, where $\{A_1 A_2 \cdots A_k\} = \sum_{\sigma} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(k)}$, the summation being taken over all permutations σ of $(1, 2, \dots, k)$.

Proof. These are straightforward calculations.

From this lemma we immediately obtain

LEMMA 6.6. *For each $x \in X$, $dE_\gamma(x) \in L((S_2^s)_\beta; (S_2^s)_\beta)$ and $d^2E_\gamma(x) \in L((S_2^s)_\beta, (S_2^s)_\beta; (S_2^s)_\beta)$. Moreover, the maps*

$$S_2^s \rightarrow L((S_2^s)_\beta; (S_2^s)_\beta); x \mapsto dE_\gamma(x),$$

and

$$S_2^s \rightarrow L((S_2^s)_\beta, (S_2^s)_\beta; (S_2^s)_\beta); x \mapsto d^2E_\gamma(x)$$

are of C^∞ class.

Lemma 6.7. *For each $g \in \mathcal{R}^s$, $dF^s(g) \in L((S_2^s)_\beta; \mathbf{R})$ and $d^2F^s(g) \in L((S_2^s)_\beta; \mathbf{R})$. Moreover, the maps*

$$\mathcal{R}^s \rightarrow L((S_2^s)_\beta; \mathbf{R}); g \mapsto dF^s(g),$$

and

$$\mathcal{R}^s \rightarrow L((S_2^s)_\beta, (S_2^s)_\beta; \mathbf{R}); g \mapsto d^2F^s(g)$$

are of C^∞ class.

Proof. From Proposition 4.4 and 4.5 we obtain

$$\begin{aligned} dF^s(g)(h) &= (T(g), h)_g^0 = (L_g^{-1}T(g), h)_g, \\ d^2F^s(g)(h, k) &= (L_g^{-1}[dT(g) + \frac{1}{2}T(g) \operatorname{tr}(g)]h, k)_g. \end{aligned}$$

Hence, using Proposition 5.3, we have $dF^s(g) \in L((S_2^s)_\beta; \mathbf{R})$ and $d^2F^s(g) \in L((S_2^s)_\beta, (S_2^s)_\beta; \mathbf{R})$. Moreover, it is easy to check that $g \mapsto dF^2(g)$ and $g \mapsto d^2F^s(g)$ are C^∞ . \square

Proof of Proposition 6.4. We have

$$d^2\tilde{f}(x)(y, z) = d^2F^s(E_\gamma(x))(dE_\gamma(x)(y), dE_\gamma(x)(z)) + dF^s(E_\gamma(x))(d^2E_\gamma(x)(y, z)).$$

Therefore, the proposition follows from Lemmas 6.6 and 6.7. \square

At the origin of X we have $d^2\tilde{f}(0)(x, x) = d^2F^s(\gamma)(dE_\gamma(0)(x), dE_\gamma(0)(x)) = d^2F^s(\gamma)(x, x)$. Since $x \in (\delta_\gamma^s)^{-1}(0)$, we have the following from Proposition 4.5.

$$\begin{aligned} d^2\tilde{f}(0)(x, x) &= 3 \int_M [3(\Delta x_s^s)^2 + (\nabla_s \nabla^s x^{ji})(\nabla_i \nabla^t x_{ji})] dV(\gamma) \\ &= 3(L_\gamma^{-1}[\bar{\Delta}_\gamma^2 + 3\gamma \operatorname{tr}(\gamma)\bar{\Delta}_\gamma^2]x, x)_\gamma \\ &= 3\hat{\beta}(L_\gamma^{-1}[\bar{\Delta}_\gamma^2 + 3\gamma \operatorname{tr}(\gamma)\bar{\Delta}_\gamma^2]x, x). \end{aligned} \tag{6.2}$$

Set $D = \bar{\Delta}_\gamma^2 + 3\gamma \operatorname{tr}(\gamma)\bar{\Delta}_\gamma^2$. The symbol of the differential operator D is given by $\sigma(D)(v)x = (\|v\|^4 + 3\gamma\|v\|^4 \operatorname{tr}(\gamma))x$, for $v \in T_1^0(M)$ and $x \in ST_2(M)$. Thus $\sigma(D)(v)(v \neq 0)$ is injective. Hence, by the decomposition theorem ([9, Theorem 4.11]), we have

$$S_2^s = \operatorname{range}(D) \oplus \ker(D), \tag{6.3}$$

because $D = D^*$ (the L^2 -adjoint of D). Moreover, it follows that $D^2 = D^*D$ is elliptic, and $D^2: S_2^s \rightarrow S_2^{s-8}$ is a Fredholm operator.

LEMMA 6.8. $\ker(D) = Y(=PS^s(\gamma))$.

Proof. From (6.2), $Dx = 0$ holds if and only if $\nabla_s \nabla^s x_{ij} = \Delta x_s^s = 0$. This condition is equivalent to $\nabla x = 0$, i.e., $x \in Y$, because M is connected and compact. \square

Set $Z = \text{range } (D) \cap X$, and we have a decomposition,

$$S_2^s = (\delta_\gamma^{s+1})^*(A^{s+1}) \oplus Y \oplus Z.$$

We immediately obtain the following from (6.2).

PROPOSITION 6.9. $d^2\tilde{f}(0)(z, z) > 0$ holds for $z(\in Z) \neq 0$.

Since $\nabla_\gamma(L_\gamma^{-1}D) = (L_\gamma^{-1}D)\nabla_\gamma$ for $\gamma \in \mathcal{F}$, we have

$$\begin{aligned} L_\gamma^{-1}D((\delta_\gamma^{s+1})^*(A^{s+1})) &\subset (\delta_\gamma^{s+1})^*(A^{s+1}), \\ L_\gamma^{-1}D(X) &\subset X, \quad L_\gamma^{-1}D(Z) \subset Z. \end{aligned} \tag{6.4}$$

Hence, we get from (6.2),

$$\hat{B}(0) = 3L_\gamma^{-1}D : \hat{Z}_\beta (\subset S_2^2) \rightarrow \hat{Z}_\beta.$$

LEMMA 6.10. $\hat{B}(0)$ is invertible

Proof. Obviously, $\hat{B}(0)$ is injective, hence, by the open mapping theorem we have only to show it to be surjective. From (6.3) (by replacing s with $s-4$), we have

$$S_2^s = \text{range } (L_\gamma^{-1}D) + L_\gamma^{-1}(\ker(D)).$$

Since $L_\gamma^{-1}(\ker(D)) = Y$, we conclude that $Z = L_\gamma^{-1}D(Z) = (L_\gamma^{-1}D)^2(Z)$ by noting (6.4). Hence $(L_\gamma^{-1}D)^2(\hat{Z}_\beta)$ is dense in \hat{Z}_β . On the other hand, $(L_\gamma^{-1}D)^2(\hat{Z}_\beta) = (L_\gamma^{-1})^2D^2(\hat{Z}_\beta)$ is closed because $(L_\gamma^{-1})^2D^2 : S_2^2 \rightarrow S_2^2$ is Fredholm. Therefore, $(L_\gamma^{-1}D)^2(\hat{Z}_\beta) = \hat{Z}_\beta$, which leads to $\hat{B}(0)(\hat{Z}_\beta) = (3L_\gamma^{-1}D)(\hat{Z}_\beta) = \hat{Z}_\beta$. \square

From this lemma we have the following.

PROPOSITION 6.11. There is a compact β -nondegenerate critical subset $K \subset Y$ of $\tilde{f} : X (= Y \oplus Z) \rightarrow \mathbf{R}$, which contains the origin.

Proof. Noting Lemma 6.10 and that \tilde{f} is of C_β^∞ class, we see that there is a neighbourhood $W \subset Y$ of the origin such that $\hat{B}(y)$ is invertible for $y \in W$. Since Y is of finite dimension, so locally compact, there is a compact subset $K = \bar{U}' \subset W$ (\bar{U}' being the closure of the open set U') which contains the origin. \square

We are now in a position to prove Theorem A'.

Proof of Theorem A'. From Propositions 6.1, 6.4, 6.9 and 6.11, the function $\tilde{f} : X (= Y \oplus Z) \rightarrow \mathbf{R}$ satisfies the assumptions of Corollary 5.2. Let $K = \bar{U}'$ and V

be the sets mentioned in Corollary 5.2. Since $E_\gamma : X \rightarrow S(\gamma)$ is a C^∞ diffeomorphism, there is a neighbourhood $W = E_\gamma(U' + V)$ of γ in $S(\gamma)$ such that $F^s(g) = 0$ implies $g \in \mathcal{F}_\Gamma^s$ ($\Gamma = \Gamma(\gamma)$) if $g \in W$. From Proposition 3.3, (b), there is a neighbourhood U of γ in \mathcal{R}^s such that $U \subset \mathcal{D}^{s+1}(W)$. Then U satisfies the assertion of the theorem because $\mathcal{F}^s = \mathcal{D}^{s+1}(\mathcal{F}_\Gamma^s)$, and $F^s(\eta^*g) = F^s(g)$ holds for $\eta \in \mathcal{D}^{s+1}$ (Proposition 4.3). \square

By virtue of Theorem A' we prove Theorem A.

Proof of Theorem A. Let $\gamma \in \mathcal{F}$ and $U(\subset \mathcal{R}^s)$ be the neighbourhood mentioned in Theorem A'. Then, $U' = U \cap \mathcal{R}$ is a neighbourhood of γ in \mathcal{R} because the inclusion map $\mathcal{R} \rightarrow \mathcal{R}^s$ is continuous (Sobolev lemma). This neighbourhood U' satisfies the assertion of Theorem A. \square

Remark. The space \mathcal{R} is an ILH-manifold [13]. Moreover, it is easy to see that \mathcal{F} is an ILH-submanifold of \mathcal{R} .

7. Supplementary discussions

The purpose of this section is to prove the following theorem, which “globally” characterizes flat metrics.

THEOREM 7.1. *Suppose $n = \dim M \leq 6$ and $\mathcal{F} \neq \emptyset$. Then,*

$$\mathcal{F} = F^{-1}(0).$$

The theorem for $n \leq 5$ was proved by Patodi [2]. We give the proof for $n = 6$. Hereafter, we assume $n = \dim M = 6$.

The following is due to Tanno [3, Lemma 1].

LEMMA 7.2. *If $F(g) = 0$, then (M, g) is conformally flat and the scalar curvature τ is vanishing.*

The Gauss-Bonnet-Chern formula for $n = 6$ is given by

$$\begin{aligned} \chi(M) = & \frac{1}{384\pi^3} \int_M [\tau^3 - 12\tau|\rho|^2 + 3\tau|R|^2 + 16R^i_j R^j_k R^k_i \\ & - 24R^{ik} R^{jm} R_{ijkm} + 24R^{st} R^{jkm}_s R_{tjkm} - 8R^{ijkm} R^s_{jkt} R^t_{ims} \\ & - 2R^{ij}_{..km} R^{km}_{..st} R^{st}_{..ij}] dV(g). \end{aligned}$$

When (M, g) is conformally flat and $\tau = 0$, this reduces to

$$\chi(M) = \frac{1}{256\pi^3} \int_M R_j^i R_k^j R_i^k dV(g). \quad (7.1)$$

LEMMA 7.3. *Suppose (M, g) is conformally flat and $\tau = 0$. If $\chi(M) = 0$, then $\nabla_i R_{jk} = 0$.*

Proof. By Tanno [3, Lemma 2], if (M, g) is conformally flat and $\tau = 0$, we have

$$\int_M (\nabla_i R_{jk})(\nabla^i R^{jk}) dV(g) = -\frac{3}{2} \int_M R_j^i R_k^j R_i^k dV(g).$$

Using (7.1), we get $\nabla_i R_{jk} = 0$ if $\chi(M) = 0$. \square

Proof of Theorem 7.1. Since $\mathcal{F} \neq \phi$, $\chi(M) = 0$ holds. Tanno [3, Proposition 5] showed that if (M, g) is conformally flat and $\tau = \nabla_i R_{jk} = 0$, then (M, g) is either (1) locally flat, or (2) Riemannian product $S^3(c) \times [H^3(-c)/\Lambda]$, Λ being some discontinuous group of isometries of $H^3(-c)$. On the other hand, the homotopy group $\pi_3(S^3(c) \times [H^3(-c)/\Lambda]) = \mathbf{Z}$, hence the manifold $S^3(c) \times [H^3(-c)/\Lambda]$ has no flat metrics (Cartan-Hadamard Theorem). Now, the proof is completed by virtue of Lemmas 7.2 and 7.3. \square

Remark. For $n \geq 7$, the author does not know whether there is such a manifold that satisfies

$$F^{-1}(0) \neq \mathcal{F} \neq \phi.$$

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