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Steenrod squares in the mod 2 cohomology of a finite H-space

by JAMES P. LIN*

§0. Introduction

In recent years the study of finite H-spaces produced some very surprising theorems. I consider the theorem due to Emery Thomas, published in 1963 [9], to be one of the most spectacular. For the class of H-spaces with primitively generated mod 2 cohomology, the theorem describes a simple pattern of Steenrod algebra connections between the algebra generators that depends only on the dyadic expansion of the degree of the generator. Because most, but not all known finite H-spaces have primitively generated mod 2 cohomology algebras, his results imply a very rigid structure.

Of course, it is a very tantalizing problem to generalize results of Thomas to non-primitively generated finite H-spaces. The purpose of this paper is to begin this study. Thomas' theorems turn out to be corollaries of theorems about truncated polynomial algebras over the Steenrod algebra. In this paper we prove his results by studying a secondary cohomology operation applied to elements of the cohomology of the H-space. By considering Thomas' theorem from this other viewpoint, we also obtain theorems about generators of H-spaces with non-primitively generated mod 2 cohomology. These theorems show there are Steenrod algebra connections similar to those for primitively generated H-spaces for generators of large degree.

Finally, we point out an error in the proof of theorem 1.1 of Thomas' paper. This theorem claims that for primitively generated H-spaces X with $PH^{\text{odd}}(X; \mathbf{Z}_2)$ finite dimensional, the odd primitives are connected by certain Steenrod algebra elements. This theorem is now open to conjecture.

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§1. The theorems

We begin by describing a theorem due to Thomas [9]. His theorem is

THEOREM 1. *Let X be a finite H-space whose mod two cohomology $H^*(X;\mathbf{Z}_2)$ is primitively generated. Then if a primitive element x has degree $2^r+2^{r+1}k-1$ for some $k>0$, $r\geq 0$, then*

$$x=Sq^{2^r}y \quad \text{and} \quad Sq^{2^r}x=0.$$

Remark 1. Note that every integer has a dyadic expansion. Thomas' theorem implies if (degree x) + 1 is not a power of two, then x is in the image of a Steenrod operation.

Remark 2. There is an error in theorem 1.1 of Browder and Thomas [3]. In this theorem, they claim that if X has primitively generated mod 2 cohomology then there is a structure theorem for $H^*(P_2X;\mathbf{Z}_2)\cong\tilde{A}/D^3\tilde{A}\oplus\tilde{S}$ where the splitting is over $A(2)/(Sq^1)$.

We have $\tilde{A}\cong\otimes_i\mathbf{Z}_2[y_i]$ where $i:H^*(P_2X)\to H^*(X)$ has the property that $i(y_i)$ form a basis for the odd primitives of $H^*(X;\mathbf{Z}_2)$.

The proof of the structure theorem is purely algebraic and depends only on the exact triangle

$$\begin{tikzcd} H^*(P_2X) \arrow[r, "i"] & H^*(X) \arrow[dl, "\bar{\Delta}"] \\ \bar{H}^*(X)\otimes\bar{H}^*(X) \arrow[u, "\eta"] & \end{tikzcd}$$

and the fact that $H^*(X;\mathbf{Z}_2)$ is primitively generated.

Therefore, consider the following possible Hopf algebra

$$H^*(X;\mathbf{Z}_2)\cong\mathbf{Z}_2[x_4]\oplus\wedge(x_5) \quad \text{with} \quad Sq^1x_4=x_5.$$

This Hopf algebra is clearly primitively generated. If one performs the construction described in Browder and Thomas, there exist elements $y\in H^6(P_2X{:}\,\mathbf{Z}_2)$ and $z\in H^5(P_2X;\mathbf{Z}_2)$ with $i(z)=x_4$, $i(y)=x_5$ with $Sq^1(z)=y$, and therefore $Sq^2(z^2)=y^2$.

The conclusion of theorem 1.1 implies there is an $\mathfrak{A}(2)/(Sq^1)$ splitting

$$H^*(P_2X;\mathbf{Z}_2)\cong\mathbf{Z}[y]/y^3\oplus\tilde{S},$$

where $z^2\in\tilde{S}$. But $Sq^2(z^2)=y^2$ contradicts this claim. The error in their proof stems from misuse of the Cartan formula mod 2 in proving $\tilde{S}$ is an $A(2)/(Sq^1)$ ideal. In the event that one assumes $Sq^1H^{\text{even}}(X;\mathbf{Z}_2)$ is decomposable, which is true for *finite* H-spaces, one can prove $\tilde{S}$ is an $A(2)/(Sq^1)$ invariant ideal, and Theorem 1.1 holds true.

In a later paper, Theorem 1.1 is used by Thomas [9] to prove a theorem for H-spaces that are not finite, but have $PH^{\text{odd}}(X;\mathbf{Z}_2)$ finite dimensional. This theorem is now an open conjecture: Is Theorem 1 true if the word finite is replaced by $PH^{\text{odd}}(X;\mathbf{Z}_2)$ finite dimensional?

We now proceed to prove Theorem 1 using secondary cohomology operations.

The proof is by induction on r. For $r=0$ the theorem amounts to proving $PH^{2k}(X;\mathbf{Z}_2)\subseteq\operatorname{im}Sq^1$.

Consider the following bundle induced over the contractible fibre space

$$\begin{tikzcd}
K(\mathbf{Z}_2,2n-1) \arrow[d] & K(\mathbf{Z}_2,2n-1) \arrow[d] \\
BE_n \arrow[r] \arrow[d] & LK(\mathbf{Z}_2,2n). \arrow[d] \\
K(\mathbf{Z}_2,n) \arrow[r, "Sq^n"'] & K(\mathbf{Z}_2,2n)
\end{tikzcd}$$

$\Omega BE_n=E_n$ has the homotopy type of $K(\mathbf{Z}_2,n-1)\otimes K(\mathbf{Z}_2,2n-2)$. It is shown in Zabrodsky [10] that the coproduct of the $2n-2$ dimensional fundamental class is

$$\bar{\Delta}\iota_{2n-2}=\iota_{n-1}\otimes\iota_{n-1}.$$

It follows that if $\hat{E}$ is a space which maps to E_n

$$\hat{E}\xrightarrow{f}E_n$$

with $f^*(\iota_{n-1})=u\neq0$ and f is an H-map, then $\bar{\Delta}f^*(\iota_{2n-2})=u\otimes u\neq0$.

In our case, let $n=2k$ and let $\hat{E}$ be the bundle induced by

$$\begin{tikzcd}
\hat{E} \arrow[d] & \\
K(\mathbf{Z}_2, 2k)\times K(\mathbf{Z}_2, 4k-1) \arrow[r, "g"] & K(\mathbf{Z}_2, 4k)
\end{tikzcd}$$

$$g^*(i_{4n}) = Sq^{2n} i_{2n} - Sq^1 i_{4n-1}.$$

There is an H-map (in fact infinite loop map)

$$\hat{E} \xrightarrow{f} E_{2k},$$

because

$$\begin{tikzcd}
K(\mathbf{Z}_2, 4k-1) \arrow[r] \arrow[d, "\hat{\jmath}"'] & K(\mathbf{Z}_2, 4k) \arrow[d] & \\
\hat{E} \arrow[r, "f"] \arrow[d, "\hat{\pi}"'] & E_{2k} \arrow[d, "\pi"] & \\
K(\mathbf{Z}_2, 2k)\times K(\mathbf{Z}_2, 4k-1) \arrow[r] \arrow[d] & K(\mathbf{Z}_2, 2k) \arrow[dr] & \\
K(\mathbf{Z}_2, 4k) \arrow[rr, "Sq^1"'] & & K(\mathbf{Z}_2, 4k+1)
\end{tikzcd}$$

is a diagram of infinite loop maps.

Therefore if $v = f^*(i_{4k})$ then

$$\bar{\Delta} v = u \otimes u \quad \text{where} \quad \hat{\pi}^*(i_{2k}) = u$$

and

$$\hat{\jmath}^*(i_{4k}) = Sq^1 i_{4k-1}.$$

Now to prove the theorem, $PH^{2k}(X; \mathbf{Z}_2) \subseteq \text{image } Sq^1$, assume by induction that the theorem holds for all even primitives of degree greater than $2k$.

Given $x \in PH^{2k}(X; \mathbf{Z}_2)$, $x^2 \in PH^{4k}(X; \mathbf{Z}_2)$. Therefore $x^2 = Sq^1 y_1$ for some y_1. By a simple argument (see Browder [2] p. 365, bottom) there exists $x' = Sq^1 y' \in$

$PH^{2k}(X; \mathbf{Z}_2)$ and $(x-x')^2 = Sq^1 w$ where w is primitive. If $x = x'$ we are done. Otherwise $x - x'$ and w are primitive and there exists an H-map

$$h : X \to K(\mathbf{Z}_2, 2\mathrm{k}) \times K(\mathbf{Z}_2, 4k-1)$$

with $h^*(i_{2k}) = x - x'$, $h^*(i_{4k-1}) = w$.

The relation $Sq^{2k}(x-x') = (x-x')^2 = Sq^1 w$ implies there is a lifting $\tilde{f} : X \to \hat{E}$ that makes the following diagram commutative:

$$\begin{tikzcd}
 & K(\mathbf{Z}_2, 4k-1) \arrow[d] & \\
 & \hat{E} \arrow[d] & \\
X \arrow[r, "h"'] \arrow[ur, dashed, bend left, "\tilde{f}"] & K(\mathbf{Z}_2, 2k) \times K(\mathbf{Z}_2, 4k-1) \arrow[dr] & \\
 & & K(\mathbf{Z}_2, 4k)
\end{tikzcd}$$

Now the H-deviation of $\tilde{f}$ is a map $D : X \wedge X \to \hat{E}$.

Since $\hat{\pi} D$ is the H-deviation of h and h is an H-map, it follows that D factors through the fibre

$$\begin{tikzcd}
 & K(\mathbf{Z}_2, 4k-1) \arrow[d, "\hat{j}"] \\
X \wedge X \arrow[r, "D"'] \arrow[ur, dashed, "\hat{D}"] & \hat{E}
\end{tikzcd}$$

Hence

$$\begin{aligned}
\bar{\Delta} \tilde{f}^*(v) &= \tilde{f}^* \otimes \tilde{f}^*(\bar{\Delta} v) + \hat{D}^* \hat{j}^*(v) = \tilde{f}^* \otimes \tilde{f}(u \otimes u) + \hat{D}^*(Sq^1 i_{4k-1}) \\
&= (x - x') \otimes (x - x') + Sq^1 \hat{D}^*(i_{4k-1}).
\end{aligned}$$

At this point, we use the fact that $H^*(X; \mathbf{Z}_2)$ is primitively generated if and only if $H_*(X; \mathbf{Z}_2)$ is commutative, associative, and has no squares [7].

If $x-x' \notin \text{image } Sq^1$, there exists a $t \in H_*(X; \mathbf{Z}_2)$ with $\langle t, x-x'\rangle \neq 0$ and $tSq^1 = 0$. Hence

$$\langle t^2, \tilde{f}^*(v)\rangle = \langle t \otimes t, \bar{\Delta}\tilde{f}^*(v)\rangle = \langle t \otimes t,\ x-x' \otimes x-x'\rangle + \langle t \otimes t,\ \text{im } Sq^1\rangle$$
$$= \langle t, x-x'\rangle^2 \neq 0.$$

Therefore if $x-x' \notin$ image Sq^1, $t^2 \neq 0$ which contradicts the fact that $H^*(X; \mathbf{Z}_2)$ is primitively generated.

We conclude $x = x' + (x-x') \in \text{image } Sq^1$. This proves the theorem in the case $v = 0$.

Before proceeding to the induction step, we will need a few more lemmas concerning Hopf algebras and factorizations in the Steenrod algebra.

LEMMA 1. *If A is a commutative Hopf algebra over $\mathbf{Z}_p$, p a prime, there is an exact sequence* [7]

$$0 \to P(\xi A) \to P(A) \to Q(A).$$

COROLLARY 2. *Let x and y be odd degree primitives in a commutative Hopf algebra A over $\mathbf{Z}_p$. Then if $x-y$ is decomposable, then $x-y=0$.*

Proof. $x-y$ is decomposable primitive, hence $x-y \in P(\xi A)$. But $P(\xi A)$ is even dimensional. Hence $x-y=0$.

The following relations hold in the Steenrod algebra.

LEMMA 3

$$Sq^{2^j} Sq^{2^j} = \sum_{l=0}^{j-1} Sq^{2^l} \beta_l, \qquad j > 0, \qquad Sq^1 Sq^1 = 0. \tag{a}$$

$$Sq^{2^r} Sq^{2^{r+1}k} = Sq^{2^r + 2^{r+1}k} + \sum_{i=0}^{r-1} Sq^{2^i} b_i. \tag{b}$$

Proof. These relations are easily proven by induction, using the Adem relations.

LEMMA 4. *If $x \in H^*(X; \mathbf{Z}_2)$ has nonzero projection in $QH^*(X; \mathbf{Z}_2)$ and $x = \theta y$ for $\theta \in \mathfrak{A}(2)$, then y has nonzero projection in $QH^*(X; \mathbf{Z}_2)$.*

Proof. This follows from the Cartan formula.

Now, armed with these lemmas, we return to the induction step of the theorem.

We may assume by induction that

$$PH^{2^l+2^{l+1}k-1}(X;\mathbf{Z}_2)\subseteq \text{image } Sq^{2^l}, \tag{1}$$

and $Sq^{2^l}PH^{2^l+2^{l+1}k-1}(X;\mathbf{Z}_2)=0$ for $l<r$.

By downward induction, we may assume

$$PH^{2^r+2^{r+1}k'-1}(X;\mathbf{Z}_2)\subseteq \text{image } Sq^{2^r}, \tag{2}$$

and $Sq^{2^r}PH^{2^r+2^{r+1}k'-1}(X;\mathbf{Z}_2)=0$ for $k'>k$.

Then let $x\in PH^{2^r+2^{r+1}k-1}(X;\mathbf{Z}_2)$. By Lemma 3

$$Sq^{2^r+2^{r+1}k}=Sq^{2^r}Sq^{2^{r+1}k}+\sum_{i=0}^{r-1} Sq^{2^i}\alpha_i.$$

We have $Sq^{2^{r+1}k}x$ is primitive. Therefore

$$Sq^{2^{r+1}k}x=Sq^{2^r}y_r \quad \text{by (1)} \quad \alpha_i x=Sq^{2^i}y_i \quad \text{by (2)}.$$

For $i>0$, $\alpha_i x$ and $Sq^{2^{r+1}k}x$ are odd primitives. By Lemma 1 they are indecomposable. By Lemma 4, the y_i are indecomposable. Since $H^*(X;\mathbf{Z}_2)$ is primitively generated, Corollary 2 implies we may choose the y_i for $i>0$ to be primitive.

Finally consider $\alpha_0 x=Sq^1 y_0$. If $\alpha_0 x$ has nonzero projection in $QH^*(X;\mathbf{Z}_2)$ then there is a primitive indecomposable w with $\alpha_0 x\equiv Sq^1 w$ modulo decomposables. But then $\alpha_0 x-Sq^1 w$ is primitive decomposable in degree $\equiv 2 \bmod 4$. Consider the exact sequence for $A=H^*(X;\mathbf{Z}_2)$

$$0\longrightarrow P(\xi^2 A)\longrightarrow P(\xi A)\xrightarrow{\theta} Q(\xi A).$$

We must have $\theta(\alpha_0 x-Sq^1 w)\neq 0$ because $P(\xi^2 A)$ is concentrated in degree $\equiv 0 \bmod 4$.

Hence there is a primitive generator z in degree $2^r+2^{r+1}k-1$ with $\alpha_0 x-Sq^1 w=z^2=Sq^1(Sq^{2^r+2^{r+1}k-2}z)$. This implies $\alpha_0 x=Sq^1(w+Sq^{2^r+2^{r+1}k-2}z)$. Hence we may choose $y_0=w+Sq^{2^r+2^{r+1}k-2}z$ to be primitive.

We now have relations

$$Sq^{2^{r+1}k}x=Sq^{2^r}y_r, \qquad \alpha_i x=Sq^{2^i}y_i,$$

where y_i are all chosen to be primitive. Hence

$$Sq^{2^r+2^{r+1}k}x = Sq^{2^r}Sq^{2^{r+1}k}x + \sum_{i=0}^{r-1} Sq^{2^i}\alpha_i x = Sq^{2^r}Sq^{2^r}y_r + \sum_{i=0}^{r-1} Sq^{2^i}Sq^{2^i}y_i.$$

Now by Lemma 3

$$Sq^{2^j}Sq^{2^j}y_j = \sum_{l=0}^{j-1} Sq^{2^l}\alpha_{j,l}y_j,$$

and by induction $\alpha_{j,l}y_j = Sq^{2^l}y_j$, where $y_{j,l}$ may be chosen primitive.

Continuing this process as far as possible, we produce relations among primitives of degrees $2^l+2^{l+1}k-1$ for $l<r$. Let $K,\bar{K}$ be generalized Eilenberg Maclane spaces $K=\Pi\, K(\mathbf{Z}_2, n_i)$ and $K \xrightarrow{g} \bar{K}$ be a map that describes all the relations. If $\hat{E}$ is the bundle induced over this map,

$$\begin{tikzcd}
\Omega K_0 \arrow[d] & \\
\hat{E} \arrow[d] & \\
K \arrow[dr] & \\
 & \bar{K}.
\end{tikzcd}$$

There exists a map of infinite loop spaces

$$\begin{tikzcd}
\Omega K_0 \arrow[d, "\hat{\jmath}"'] & \\
\hat{E} \arrow[r] \arrow[d, "\hat{\pi}"'] & E_n \arrow[d] \\
K \arrow[r] & K_n
\end{tikzcd}$$

where $n=2^r+2^{r+1}k-1$ and there is a $v\in H^{2n}(\hat{E};\mathbf{Z}_2)$ with $\hat{\jmath}^*(v)\in$ image $(Sq^{2^r}, Sq^{2^{r-1}},\dots, Sq^1)$ and $\bar{\Delta}v=u\otimes u$ where $\hat{\pi}^*(i_n)=u$.

By construction, there is an H-map $h:X\to K$

$$\begin{tikzcd} & \Omega K_0 \arrow[d] \\ & \hat{E} \arrow[d] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "h"'] & K \end{tikzcd}$$

and a lifting $\tilde{f}:X\to\hat{E}$. As before,

$$\bar{\Delta}\tilde{f}^*(v)=\tilde{f}^*\otimes\tilde{f}^*(u\otimes u)+\sum_{l=0}^{r}\text{ image }Sq^{2^l}=x\otimes x+\sum_{l=0}^{r}\text{ image }Sq^{2^l}.$$

Now $x\notin$ image Sq^{2^l} for $l<r$. Because if so, then $x=Sq^{2^l}y$ and we may assume by Lemma 4 and Corollary 2 that $y\in PH^{2^l+2^{l+1}k'-1}(X;\mathbf{Z}_2)$ for some $k'>0$. Hence by induction $Sq^{2^l}y=0$.

Therefore if $x\otimes x\in\sum_{l=0}^{r}$ image Sq^{2^l}, we conclude

$$x=Sq^{2^r}y\quad\text{for some}\quad y\in PH^{2^{r+1}k-1}(X;\mathbf{Z}_2).$$

It remains to show $Sq^{2^r}x=0$. By Lemma 3,

$$Sq^{2^r}x=Sq^{2^r}Sq^{2^r}y=\sum_{l=0}^{r-1}Sq^{2^l}\beta_l y.$$

By induction $Sq^{2^l}\beta_l y=0$. This completes the proof. Q.E.D.

It may be instructive to trace through the proof for $PH^{4k+1}(X;\mathbf{Z}_2)\subseteq$ image Sq^2.

Note

$$Sq^{4k+2}=Sq^2Sq^{4k}+Sq^1Sq^{4k}Sq^1.$$

The relations are for $x\in PH^{4k+1}(X;\mathbf{Z}_2)$:

$$Sq^{4k}x=Sq^2y_1,\qquad Sq^{4k}Sq^1x=Sq^1y_0,\qquad \alpha_0=Sq^{4k}Sq^1,\qquad \alpha_1=Sq^{4k}$$
$$Sq^2Sq^2y_1=Sq^1Sq^2Sq^1y_1,\qquad \alpha_{1,1}=Sq^2Sq^1,\qquad Sq^2Sq^1y_1=Sq^1y_{1,0}$$

All y_j, $y_{j,l}$ are primitive. Let $K_n = K(\mathbf{Z}_2, n)$. Then

$$K = K_{4k+1} \times K_{8k-1} \times K_{8k+1} \times K_{8k+1} \qquad \bar{K} = K_{8k+1} \times K_{8k+2} \times K_{8k+2} \qquad g: K \to \bar{K}$$

$$g^* i_{8k+1} = Sq^{4k} i_{4k+1} - Sq^2 i_{8k-1} \qquad g^* i_{8k+2} = Sq^{4k} Sq^1 i_{4k+1} - Sq^1 i_{8k+1}$$

$$g^* i_{8k+2} = Sq^2 Sq^1 i_{8k-1} - Sq^1 \bar{i}_{8k+1}.$$

Let $g_1 : \bar{K} \to K_{8k+3}$ be defined by

$$g_1^*(i_{8k+3}) = Sq^2 i_{8k+1} + Sq^1 i_{8k+2} + Sq^1 i_{8k+2}.$$

Then stably

$$\begin{aligned} g^* g_1^*(i_{8k+3}) &= Sq^2 g^*(i_{8k+1}) + Sq^1 g^*(i_{8k+2}) + Sq^1 g^*(\bar{i}_{8k+2}) \\ &= Sq^2[Sq^{4k} i_{4k+1} - Sq^2 i_{8k-1}] \\ &\quad + Sq^1[Sq^{4k} Sq^1 i_{4k+1} - Sq^1 i_{8k+1}. \\ &\quad + Sq^1[Sq^2 Sq^1 i_{8k-1} - Sq^1 \bar{i}_{8k+1}] \\ &= Sq^{4k+2} i_{4k+1}. \end{aligned}$$

g, g_1 are assumed to be infinite loop maps.

In view of our proof of Thomas' theorem, it now becomes clear that there exist certain generalizations of this theorem to H-spaces whose mod 2 cohomology is not primitively generated. Certainly, if the variables that appear in the domain of the operation are primitive, there is no obstruction to obtaining a secondary operation that detects the dual of a square.

Work of Browder [1] shows that for *any* finite H-space X, the square of an odd homology primitive is zero.

In Lin [5] it is shown that generators of the cohomology ring may be chosen so that they have a "primitive degree." This means roughly for each generator x there exists an $\mathfrak{A}(2)$ subHopf algebra B_x with $\bar{\Delta} x \in B_x \otimes B_x$ and $x \notin B_x$. B_x therefore measures the deviation from primitivity of x.

The following theorem is proved in Lin [5]:

THEOREM 2. *Let x be a generator in $H^{n-1}(X; \mathbf{Z}_2)$. Let B be an $\mathfrak{A}(2)$ subHopf algebra with $\bar{\Delta} x \in B \otimes B$ and $x \notin B$.*

Suppose $Sq^n = \sum a_i b_i$ and $b_i x$ is decomposable in B for each i. Then there is a secondary operation ϕ defined on x and $\bar{\Delta}\phi(x) = x \otimes x + \sum \operatorname{im} a_i + I(B) H^* \otimes H^* +$

$H^*\otimes I(B)H^*$. Hence if $t\in PH_{n-1}(X;\mathbf{Z}_2)$ with $\langle t,B\rangle=0$, $t\otimes t\in\ker\sum a_i$ then $t^2\neq 0$.

Theorem 2 can be used to prove the following theorem about H-spaces with possibly non-primitive cohomology rings:

THEOREM 3. *Let X be a finite H-space and let n be the largest integer such that $QH^n(X;\mathbf{Z}_2)\neq 0$. Expand $n+1$ dyadicly*

$$n+1=2^{r_0}+2^{r_1}+\cdots+2^{r_s},\qquad 0\leq r_0<r_1<\cdots<r_s.$$

Then for all indecomposables of degree $2^l+2^{l+1}k-1>2^{r_s}$,

$$Q^{2^l+2^{l+1}k-1}=Sq^{2^l}Q^{2^{l+1}k-1}\quad\text{and}\quad Sq^{2^l}Q^{2^l+2^{l+1}k-1}=0.$$

In particular $Q^n=Sq^{2^{r_0}}Sq^{2^{r_1}}\cdots Sq^{2^{r_{s-1}}}Q^{2^{r_s}-1}$.

Before proving this theorem, we remark that this proves that there are the following Steenrod squares connecting generators of the exceptional groups

(1) $H^*(G_2):Sq^2x_3=x_5$

(2) $H^*(F_4):Sq^8x_{15}=x_{23}$

(3) $H^*(E_6):Sq^8x_{15}=x_{23},\ Sq^2x_{15}=x_{17}$

(4) $H^*(E_7):Sq^8x_{15}=x_{23},\ Sq^2x_{15}=x_{17},\ Sq^4x_{23}=x_{27}$

(5) $H^*(E_8):Sq^8x_{15}=x_{23},\ Sq^2x_{15}=x_{17},\ Sq^4x_{23}=x_{27},\ Sq^2x_{27}=x_{29}$.

Note that E_6, E_7 and E_8 are not primitively generated; in fact, x_{15} may be chosen such that $\bar{\Delta}x_{15}$ contains $x_3^2\otimes x_9$ as a summand [8].

We now prove Theorem 5.

Proof. Our restrictions imply $n\geq 2^l+2^{l+1}k-1>2^{r_s}-1$. Therefore $k>0$.

We induct on l. If $l=0$ and k is even, Kane [4] proves $Q^{2k}=0$. If $l=0$ and k is odd, $k=2m+1$, then we must prove $Q^{4m+2}\subseteq\operatorname{im}Sq^1$, if $4m+2>2^{r_s}-1$. We need the following Lemma:

LEMMA 5. *Let A be a commutative finite dimensional Hopf algebra over $\mathfrak{A}(2)$ with Sq^1A^{even} decomposable. Then given a generator $y\in A^{\text{odd}}$ with $y^2\neq 0$, we have $y^2\notin\operatorname{im}Sq^1D$ where $D=I(A)^2$.*

Proof. By the Borel structure theorem A is isomorphic as algebras to a tensor product of truncated polynomial algebras of heights a power of 2. Hence, $A/I(A)^3$ is isomorphic to a tensor product of truncated polynomial algebras of heights 2 and 3. Let

$$\pi : A \to A/I(A)^3$$

be the projection. Then π is an algebra map over the Steenrod algebra. Hence if $y^2 \in Sq^1(I(A)^2)$, then $0 \neq \pi(y)^2 \in Sq^1(\pi I(A))^2$. Since degree $\pi(y)^2$ is even, we may write

$$\pi(y)^2 = Sq^1 \sum_{i=1}^{l} \pi(b_i)\pi(c_i),$$

where deg $\pi(b_i)$ is even, deg $\pi(c_i)$ is odd. Then $Sq^1 A^{\text{even}}$ decomposable implies

$$\pi(y)^2 = \sum_{i=1}^{l} \pi(b_i) Sq^1 \pi(c_i).$$

Hence $\pi(y)^2$ belongs to the ideal generated by the even generators. This is a contradiction. Q.E.D.

We now turn to the proof of Theorem 3. For $l = 0$ we must prove $Q^{4m+2} \subseteq \operatorname{im} Sq^1$.

Let $B(n)$ be the $\mathfrak{A}(2)$ subHopf algebra generated by elements of degree $\leq n$. Following Torsion I [5], an element $\bar{x} \in Q^{4m+2}$ has primitive degree r if there is a representative $x \in B(r+1)$, $x \notin B(r)$, $\bar{\Delta}x \in B(r) \otimes B(r)$.

By induction assume all $4m+2$ dimensional generators of primitive degree less than r lie in image Sq^1. Given $\bar{x}$ of primitive degree r with representative x, note that $Sq^{4m}x$ is decomposable because degree $Sq^{4m}x$ is greater than the degree of the highest generator.

Hence the projection $Sq^{4m}[x]$ in $H^*/B(r)$ is decomposable primitive. Hence $Sq^{4m}[x] = [y]^2 = Sq^1 Sq^{4m}[y]$. But for degree reason $Sq^{4m}[y]$ is decomposable. It follows that $[y]^2 \in Sq^1 D$. This contradicts the lemma unless $[y]^2 = 0$.

We conclude $Sq^{4m}[x]$ is zero. Therefore $Sq^{4m}x \in H^* I(B(r))$ and $\bar{\Delta} Sq^{4m} x \in B(r) \otimes B(r)$. By an argument in Torsion I [5], $Sq^{4m}x$ is decomposable in $B(r)$.

Applying Theorem 2 to the factorization $Sq^{4m+3} = Sq^3 Sq^{4m}$ there is a secondary operation ϕ defined on x with

$$\bar{\Delta}\phi(x) = x \otimes x + \operatorname{im} Sq^3 + I(B(r))H^* \otimes H^* + H^* \otimes I(B(r))H^*.$$

If $x \notin \text{im } Sq^1 + B(r)$ there is a $t \in PH_{4m+2}$ with $\langle t, x \rangle \neq 0$ $\langle t, \text{im } Sq^1 + B(r) \rangle = 0$. Hence $\langle t^2, \phi(x) \rangle \neq 0$ which is a contradiction.

Therefore $\bar{x} \in \text{im } Sq^1 + \overline{B(r)}$. Now by induction $QB(r) \subseteq \text{im } Sq^1$, hence $\bar{x} \in \text{im } Sq^1$, and $Sq^1 \bar{x} = 0$. This completes the case $l = 0$.

Assume by induction that for $n \geq 2^l + 2^{l'+1}k' - 1 > 2^{r_s} - 1$, $l' < l$ that

$$Q^{2^{l'}+2^{l'+1}k'-1} \subseteq \text{im } Sq^{2^{l'}},$$

and

$$Sq^{2^{l'}} Q^{2^{l'}+2^{l'+1}k'-1} = 0.$$

Let $\bar{x} \in Q^{2^l+2^{l+1}k-1}$, $n \geq 2^l + 2^{l+1}k - 1 > 2^{r_s} - 1$.
There is a factorization

$$Sq^{2^l+2^{l+1}k} = \sum_{i=0}^{l} Sq^{2^i} \alpha_i.$$

Now because $n \geq 2^l + 2^{l+1}k - 1 > 2^{r_s} - 1$, $2^{l+1}k$ must have 2^{r_s} in its dyadic expansion. Therefore since degree $\alpha_i \geq 2^{l+1}k$ it follows that $\alpha_i \bar{x}$ has degree $> 2^{r_s+1} - 1 > n$. We conclude $\alpha_i \bar{x}$ is decomposable for each i.

Let x be an r primitive representative for $\bar{x}$ and assume by induction that all $2^l + 2^{l+1}k - 1$ dimensional generators of primitive degree less than r are in the image of Sq^{2^l}. Then $\alpha_i[x] \in H^*/B(r)$ is primitive decomposable. For $i > 0$ $\alpha_i[x]$ is of odd degree, hence $\alpha_i[x] = 0$. For $i = 0$ $\alpha_0[x]$ is of even degree, hence

$$\alpha_0[x] = [y]^2 = Sq^1 Sq^{2^l+2^{l+1}k-2}[y].$$

But deg $Sq^{2^l+2^{l+1}k-2}[y] > n$.

Therefore $Sq^{2^l+2^{l+1}k-2}[y]$ is decomposable, and $[y]^2 \in Sq^1 D$. By Lemma 5, $[y]^2 = 0$.

For all i, therefore $\alpha_i[x] = 0$. Hence $\alpha_i x \in H^* I(B(r))$ and $\bar{\Delta} \alpha_i x \in B(r) \otimes B(r)$. By the argument of theorem 2.2.1 of [5], $\alpha_i x$ is decomposable in $B(r)$.

Now suppose $x \notin \text{im } Sq^{2^l} + B(r)$. Then there is a primitive $t \in H_*(X; \mathbf{Z}_2)$ with $\langle t, x \rangle \neq 0$ and $\langle t, \text{im } Sq^{2^l} + B(r) \rangle = 0$.

We claim $tSq^{2^i} = 0$ for $i < l$. Suppose not. Let $t_i = tSq^{2^i}$, and let $x_i \in H^{2^l+2^{l+1}k-2^i-1}$ be dual to t_i. $\langle t_i, x_i \rangle \neq 0$. Then $\bar{x}_i \neq 0$ and $0 \neq \langle t_i, x_i \rangle = \langle tSq^{2^i}, x_i \rangle = \langle t, Sq^{2^i} x_i \rangle$. This implies $Sq^{2^i} Q^{2^i+2^{i+1}k-1} \neq 0$ which contradicts our inductive assumption. Hence the claim is proven.

Applying Theorem 2, there is a secondary operation ϕ defined on x with

$$\bar{\Delta}\phi(x)=x\otimes x+\sum_{i=0}^{l}\text{im } Sq^{2^i}+H^*\otimes H^*I(B(r))+H^*I(B(r))\otimes H^*.$$

It follows that $\langle t^2, \phi(x)\rangle\neq 0$ which is a contradiction. Therefore $x\in\text{im } Sq^{2^l}+B(r)$ which implies

$$\bar{x}\in\text{im } Sq^{2^l}+QB(r)^{2^l+2^{l+1}k-1}.$$

By induction on the primitive degree

$$(Q(B(r))^{2^l+2^{l+1}k-1}\subseteq\text{im } Sq^{2^l},$$

so $\bar{x}\in\text{im } Sq^{2^l}Q^{2^{l+1}k-1}$.

It follows that

$$Sq^{2^l}\bar{x}\in\text{im } Sq^{2^l}Sq^{2^l}Q^{2^{l+1}k-1}=\sum_{i=0}^{l-1} Sq^{2^l}\beta_i Q^{2^{l+1}k-1},$$

and deg $\beta_i Q^{2^{l+1}k-1}\equiv 2^l \bmod 2^{l+1}$. By induction $Sq^{2^l}\beta_i Q^{2^{l+1}k-1}=0$. We have shown

$$Q^{2^l+2^{l+1}k-1}\subseteq\text{im } Sq^{2^l},$$

and

$$Sq^{2^l}Q^{2^l+2^{l+1}k-1}=0.$$

This completes the proof of Theorem 3.

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