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# Homology of $SL_2(Z[\omega])$

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In this article we shall describe a simplicial complex which is a natural structure for the action of $GL_2(R)$, the group of $2\times 2$ invertible matrices over the ring $R$. With strong conditions on $R$ this complex is contractible; it is then possible to give a presentation of $GL_2(R)$ and to compute the homology of $GL_2(R)$ in terms of stabilizer subgroups of the simplices in a fundamental domain for the action. We shall work in detail with the ring $Z[\omega]$, $\omega^2=\omega-1$; similar methods apply to the rings $Z[\theta]$, $\theta^2=\theta+1$, and $Z[\lambda]$, $\lambda^3=\lambda+1$ but the details are quite elaborate and will be left for a later time. Initial motivation came from Quillen's construction of the tree for $SL_2(Z)$ (compare Serre [3]).

## §1

Let $R$ be a ring. Consider the set $\mathcal{L}$ of free direct summands of $R^2$. Elements of $\mathcal{L}$ are called lines.

DEFINITION. $L_1, L_2 \in \mathcal{L}$ are independent if $L_1+L_2=L_1\oplus L_2=R^2$.

Let $\mathcal{U}(R)$ be the simplicial complex whose vertices are the elements of $\mathcal{L}$ and whose $q$-simplices are determined by a set $\{L_0,\ldots,L_q\}$, $L_i\in\mathcal{L}$ where $L_i$, $L_j$ are independent for $0\le i\ne j\le q$.

Let $R(a,b)$ be a vertex of $\mathcal{U}(R)$ and suppose $R(c,d)$ is independent of $R(a,b)$.

LEMMA 1. *Any line in $R^2$ independent of $R(a,b)$ is of the form*

$$L=R(ra+c,\ rb+d)$$

*for some $r\in R$.*

*Proof.* Suppose $L=R(c',d')$ is independent of $R(a,b)$. Put $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$,

$B=\begin{pmatrix} a & b \\ c' & d' \end{pmatrix}$; $A, B$ are in $GL_2(R)$. Let $A^{-1}=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ Then

$$BA^{-1}=\begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}$$

for $s\in R$, $t\in R^*$ (units of $R$). Thus $c'=sa+tc$, $d'=sb+td$; hence $R(c',d')=R(ra+c, rb+d)$ with $r=t^{-1}s$.

LEMMA 2. *The lines* $L_1=R(r_1a+c, r_1b+d)$ $L_2=R(r_2a+c, r_2b+d)$ *are independent iff* $r_1-r_2\in R^*$.

*Proof.* Let $C=\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$; then

$$\begin{pmatrix} r_1a+c & r_1b+d \\ r_2a+c & r_2b+d \end{pmatrix} C=\begin{pmatrix} r_1 & r_2 \\ 1 & 1 \end{pmatrix}.$$

Hence $L_1, L_2$ are independent iff $r_1-r_2\in R^*$.

Consider the link of a vertex $R(a,b)$ in $\mathcal{U}(R)$, Link $R(a,b)$; this is the full subcomplex of $\mathcal{U}(R)$ containing all lines which are independent of $R(a,b)$. Let $\mathcal{R}$ be the simplicial complex whose vertices are given by the elements of $R$ and in which a $q$-simplex is given by a set $\{r_0,\ldots,r_q\}$, $r_i\in R$ with $r_i-r_j\in R^*$ for $0\le i\ne j\le q$. The next lemma follows easily from the previous discussion.

LEMMA 3. *Link* $R(a,b)\cong\mathcal{R}$.

Put $M_R=\sup\{m \mid \exists r_1,\ldots,r_m\in R \ni \forall i,j,\ 1\le i<j\le m,\ r_1-r_j\in R^*\}$.

LEMMA 4. (Lenstra [2]) $M_R$ *is finite if* $R$ *has an ideal* $(\ne R)$ *of finite index.*

COROLLARY. $\mathcal{U}(R)$ *is finite dimensional if* $R$ *is the ring of integers in a number field and* $\dim\mathcal{U}(R)=M_R$.

*Proof.* The ring of integers in a number field has an ideal of finite index, for example (2). It follows easily that the dimension of a simplex in $\mathcal{U}(R)$ is $\le 1+\dim\mathcal{R}=M_R$.

When $R$ is the ring of integers in an algebraic field, and $R$ has a unit of infinite order then according to a result of Vaserstein [2], $SL_2(R)$ is generated by elementary matrices. If $R$ is a Euclidean ring then $SL_2(R)$ is generated by elementary matrices. It follows then in case $R$ is Euclidean or $R$ has a unit of infinite order that $\mathcal{U}(R)$ is connected. The cases excluded by this are the non-Euclidean rings of integers in imaginary quadratic number fields.

## §2

We suppose now that $R$ is a Euclidean ring with respect to the function $|\ |: R \to N$. Suppose also that $|\ |$ is multiplicative and thus gives rise to a function on the quotients field of $R$, $K$. Define

$$|\ |: \mathcal{L} \to N \quad \text{via}$$

$|R(a, b)| = |b|$. This is independent of the particular representation of the line since units have value one under the Euclidean function. Let $\mathcal{U}(n, R)$ be the full subcomplex of $\mathcal{U}(R)$ containing all vertices $L$ of $\mathcal{L}$ with $|L| \leq n$.

Consider now the link of a vertex $R(a, b)$, $|R(a, b)| = n$, in $\mathcal{U}(n, R)$, denoted $\text{Link}_n\, R(a, b)$. This link is the full subcomplex of $\mathcal{U}(n, R)$ containing lines $L$ independent of $R(a, b)$ with $|L| \leq n$. Let $R(c, d)$ be independent of $R(a, b)$. It follows from Lemma 1 that this link contains only vertices $R(ra + c, rb + d)$ with $|rb + d| \leq n$. Using the Euclidean algorithm we write $d = qb + d_0$ with $|d_0| < |b| = n$; let $c_0 = c - qa$. Thus the link contains only those lines $R(c_0 + ra, d_0 + rb)$ with $|d_0 + rb| \leq n$.

Now if $x \in K$, put $R_x = \{r \in R \mid |x - r| \leq 1\}$. Let $\mathcal{R}_x$ be the simplicial complex in which a $q$-simplex is determined by a set $\{r_0, \ldots, r_q\}$, $r_i \in R_x$, $r_1 - r_j \in R^*$, $0 \leq i \neq j \leq q$; this is the full subcomplex of $\mathcal{R}$ containing the vertices $R_x$.

LEMMA 5. $\text{Link}_n\, R(a, b) \cong \mathcal{R}_x$, $x = d_0/b$.

*Proof.* The vertices in $\text{Link}_n\ R(a, b)$ are $R(c_0 + ra, d_0 + rb)$, $|d_0 + rb| \leq |b|$ or equivalently $|d_0/b + r| \leq 1$. Thus there is a $1-1$ correspondence between the simplices of the link and the simplices of $\mathcal{R}_x$, $x = d_0/b$. The incidence relation on $\mathcal{R}_x$ is designed so as to agree with that for the link.

We make the observations below which will be of use later.

LEMMA 6. $\mathcal{R}_x \cong \mathcal{R}_{x+a}$ $x \in K$, $a \in R$.

LEMMA 7. $\mathcal{R}_x \cong \mathcal{R}_{ux}$ $x \in K$, $u \in R^*$.

There are two types of elements of $K$ which we need to distinguish. If $x \in K$ and $R_x = \{r \mid |x-r|<1\}$ then $x$ will be called of type I; otherwise $x$ is of type II.

## §3

In this section we analyze the structure of the complexes $\mathcal{R}$ and $\mathcal{R}_x$ for the ring $R = Z[\omega]$, $\omega^2 = \omega - 1$. The simplices of $\mathcal{R}$ are given by sets $\{r_0, \ldots, r_q\}$; we shall at times denote this by $r + u\{s_0, \ldots, s_q\}$, $r \in R$, $u \in R^*$, $r_i = r + us_i$, $0 \le i \le q$.

LEMMA 8. (a) *Every 1-simplex of $\mathcal{R}$ is uniquely of the form $r + \omega^i\{0, 1\}$, $0 \le i < 3$.*

(b) *Every 2-simplex of $\mathcal{R}$ is uniquely of the form $r + \omega^i\{0, 1, \omega\}$, $0 \le i < 2$.*

*Proof.* Given a 1-simplex, $\{r_0, r_1\}$, we may write this as $r_0 + (r_1 - r_0)\{0, 1\}$, $r_1 - r_0 \in R^*$. Notice that $\{0, -1\} = -1 + \{0, 1\}$ and this provides the required form. For a 2-simplex we may suppose that it has the form $r + u\{0, 1, \eta\}$ with $\eta$, $\eta - 1 \in R^*$. Then it follows easily that $\eta = \omega$ or $\eta = \omega^5$. Notice that $\{0, 1, \omega^5\} = \omega^{-1}\{0, 1, \omega\}$. Thus every 2-simplex has the form $r + u\{0, 1, \omega\}$; in order to get the proper restriction on $u$ notice the relations:

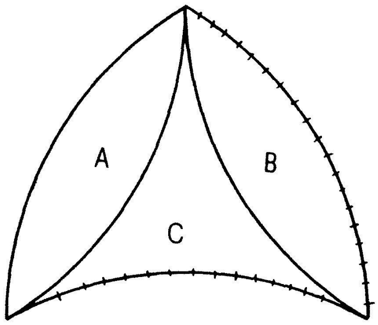
$$-\{0, 1, \omega\} = -\omega + \omega\{0, 1, \omega\}. \qquad \omega^2\{0, 1, \omega\} = -1 + \{0, 1, \omega\}.$$

For the uniqueness part suppose that $r + u\sigma = s + v\sigma$ where $\sigma$ is $\{0, 1\}$ in part (a) or $\sigma = \{0, 1, \omega\}$ in part (b) and $u, v$ are restricted suitably. We obtain then a relation $\sigma = v^{-1}(r-s) + v^{-1}u \cdot \sigma$. Thus we may suppose that there is a relation $\sigma = \rho + \tau\sigma$ and show that $\tau = 1$ and $\rho = 0$. This is quite simple in case (a). In case (b) we observe that $\rho$ must be one of $0, 1, \omega$. If $\rho = 1$ then either $\tau = -1$ or $\tau = \omega^2$; both of these are excluded by the form. If $\rho = \omega$ then either $\omega + \tau = 0$ or $\omega + \tau\omega = 0$; one checks that this is impossible.

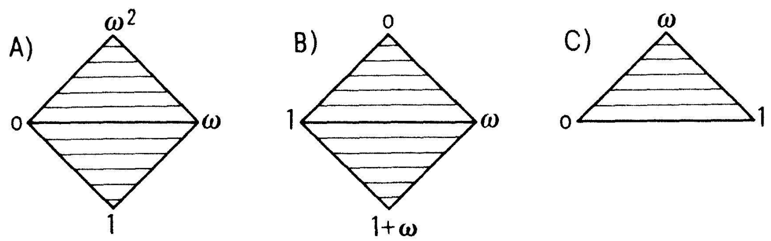
COROLLARY. *If $R = Z[\omega]$ then $\mathcal{R}$ is contractible.*

*Proof.* View $R$ embedded in $\mathbb{C}$ as a lattice, then the simplices $r + \omega\{0, 1, \omega\}$, $r + \{0, 1, \omega\}$ provide $\mathbb{C}$ with a simplicial structure tessellated by these two types of simplices.

Now for the structure of $\mathcal{R}_x$ we may using Lemma 6 assume that $0 \in R_x$; our only concern is with $x \in K - R$. View $R$ embedded in $\mathbb{C}$ and hence also $K$. The norm $N: K \to Q$, which is the square of the usual absolute value on $\mathbb{C}$, provides the multiplicative Euclidean function on $R$. Using Lemma 7 we may assume that $x$ belongs to the region below.




If $x \in K-R$ belongs to this region and is not on one of three solid arcs then it is of type I. Following the labeling of the three regions we describe $R_x$. For region $A$ which includes the two solid arcs we have $R_x=\{0, 1, \omega, \omega^2\}$; for region $B$ which includes the third solid arc $R_x=\{0, 1, \omega, 1+\omega\}$; for region $C$, $R_x=\{0, 1, \omega\}$. The complexes $\mathcal{R}_x$ have the following structure:




Notice that for $x \in K-R$ of type II, $R_x$ is a union of two 2-simplices.

## §4

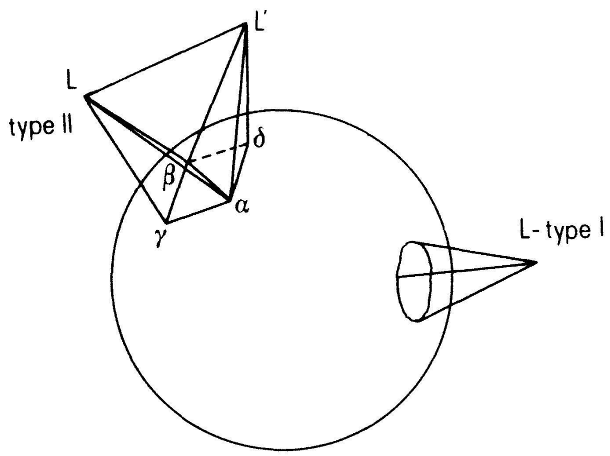
THEOREM. $\mathcal{U}(Z[\omega])$ *is contractible.*

*Proof.* We filter $\mathcal{U}(Z[\omega])$ by the subcomplexes $\mathcal{U}(n, Z[\omega])$ according the norm. Let $\mathcal{L}_n=\{L \in \mathcal{L} \mid |L|=n\}$. We shall establish that $\mathcal{U}(n, Z[\omega])$ is contractible to $Z[\omega](1,0)$ by induction on $n$. Notice first that $\mathcal{U}(0)=Z[\omega](1,0)$ and that $\mathcal{U}(1)$ has $Z[\omega](1,0)$ as a cone point; suppose inductively then that $\mathcal{U}(n-1, Z[\omega])$ is contractible for $n>1$. We have

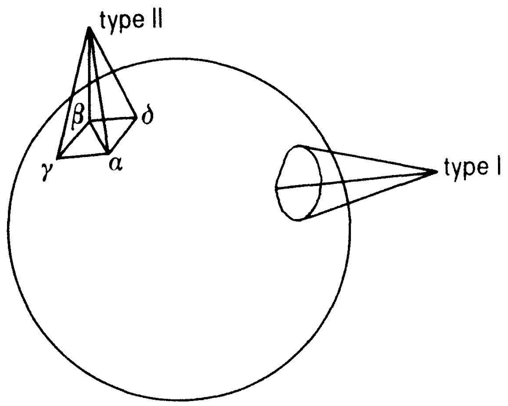
$$\mathcal{U}(n-1, Z[\omega] \cong \mathcal{U}(n, Z[\omega]) - \bigcup_{L \in \mathcal{L}_n} \mathrm{st}\,(L).$$

where $\mathrm{st}\,(L)$ is the open star of $L$ in $\mathcal{U}(n, Z[\omega])$. Thus $\mathcal{U}(n, Z[\omega])$ is obtained from $\mathcal{U}(n-1)$ by attaching the Cone $(\mathrm{Link}_n\, L)$, $L \in \mathcal{L}_n$ to $\mathcal{U}(n-1)$ along the

$\text{Link}_n L$. Now the $\text{Link}_n L$ corresponds to one of the complexes $\mathcal{R}_x$, $x \in K - R$, which if $x$ is of type I has all of its vertices in $\mathcal{U}(n-1)$; however if $x$ is of type II there is a unique vertex $L'$ in $\text{Link}_n L$ which belongs to $\mathcal{L}_n$. We diagram this by the picture:




We have notices here that if $L$ if of type II then for the vertex $L'$ in $\text{Link}_n L$ there must be exactly two 1-simplices meeting at $L'$ in the link. Now to complete the picture we examine $\text{Link}_n L'$; the link of $L'$ contains $\alpha$, $\beta$, $L$ and another vertex $\delta \in \mathcal{U}(n-1)$, $\delta \neq \gamma$, arranged as in the diagram. Now to contract $\mathcal{U}(n)$ we first contract $L$ to $L'$ along the edge joining them for every pair $L, L' \in \mathcal{L}_n$ which are in each other's links. We obtain then a complex with the same homotopy type as $\mathcal{U}(n)$, $\mathcal{V}(n)$. Now




$$\mathcal{V}(n) \cong \mathcal{U}(n-1) \bigcup_{L \in \mathcal{L}'_n} \text{Cone}\,(\text{Link}_n\,(L))$$

where $\mathcal{L}'_n$ is the subset of $\mathcal{L}_n$ containing all type I vertices and one from each pair of type II vertices as above. The $\text{Link}_n\,(L)$ is unchanged for type I and for the type II the link has the same homotopy type. Now $\mathcal{U}(n-1)$ is contractible so we obtain

$$\mathcal{U}(n) \cong \mathcal{V}(n) \cong \bigvee_{L \in \mathcal{L}'_n} \text{Susp}\,(\text{Cone}\,(\text{Link}_n\,(L)).$$

Hence, since each link is contractible we have that $\mathcal{U}(n)$ is contractible. It follows then that $\mathcal{U}$ is contractible.

We denote by $\mathcal{U}'(Z[\omega])$ the first barycentric subdivision of $\mathcal{U}(Z[\omega])$.

COROLLARY. $\mathcal{U}(Z[\omega] \cong \mathcal{U}'(Z[\omega]) - \bigcup_{L \in \mathcal{L}} \text{st}\,(L)$

*Proof.* According to the corollary of Lemma 8, the simplicial complex $\mathcal{R}$ is contractible. We have $\mathcal{R} \cong \text{Link}\,(L)$ for any $L \in \mathcal{L}$. Notice:

$$\mathcal{U} \cong \mathcal{U}' \cong \left(\mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}\,(L)\right) \cup \bigcup_{L \in \mathcal{L}} \text{Cone}\,(\text{Link}\,(L))$$

Now the Link $(L)$ above is computed in $\mathcal{U}'$ but its homotopy type is unchanged, i.e., it's contractible. Thus

$$\mathcal{U} \cong \mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}\,(L)$$

is contractible.

The complex $\mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}\,(L)$ may be described as follows: Consider the partially ordered set (by inclusion) of subsets of $\mathcal{L}$ of the type

$$\{L_0, \ldots, L_q\}, \qquad q \geq 1$$

for which $L_i$, $L_j$ are independent for $0 \leq i \neq j \leq q$; then $\mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}\,(L)$ has the homotopy type of the realization of this poset, say $\mathcal{Y}(Z[\omega])$).

## §5

LEMMA 9. *For the complex* $\mathcal{U}(R)$, $R = Z[\omega]$,

(a) *every vertex is* $GL_2(R)$ *equivalent to* $\{R(1, 0)\}$;

(b) *every* 1-*simplex is* $GL_2(R)$ *equivalent to* $\{R(1, 0), R(0, 1)\}$;

(c) *every* 2-*simplex is* $GL_2(R)$ *equivalent to* $\{R(1, 0), R(0, 1), R(1, 1)\}$;

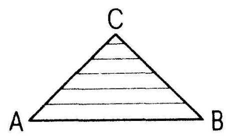
(d) *every* 3-*simplex is* $GL_2(R)$ *equivalent to* $\{R(1, 0),\ R(0, 1),\ R(1, 1), R(1, \omega)\}$.

*Proof.* The first two parts of the lemma are easy. For part (c), any 2-simplex is equivalent via $GL_2$ to a simplex which must have the form $\{R(0,1), R(1,0), R(1,\alpha)\}$, $\alpha \in R^*$. Multiplication by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ converts this 2-simplex to the required form. For 3-simplices we may by the action of $GL_2$ bring this to the simplex

$$\{R(0,1), R(1,0), R(1,1), R(1,\alpha)\}.$$

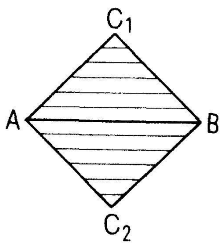
with $\alpha, \alpha - 1 \in R^*$. According to the proof of lemma 8, $\alpha = \omega$ or $\omega^5$. If $\alpha = \omega^5$ then multiplication by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on this simplex converts it to the required form.

COROLLARY. *The fundamental domain for the action of* $GL_2(Z[\omega])$ *on* $\mathcal{Y}(Z[\omega])$ *is a single* 2-*simplex.*




*Proof.* Recall the description of $\mathcal{Y}(Z[\omega])$ at the end of the previous section. Using the previous lemma now, the fundamental domain for $GL_2$ on $\mathcal{Y}(Z[\omega])$ has vertices $A = \{(0,1),(1,0)\}$, $B = \{(0,1),(1,0),(1,1)\}$ and $C = \{(1,0),(0,1),(1,1),(1,\omega)\}$. (We have given only the generators for the lines in $A$, $B$, $C$.)

COROLLARY. *The fundamental domain for the action of* $SL_2(Z[\omega])$ *on* $\mathcal{Y}(Z[\omega])$ *is the* 2-*complex*




*Proof.* Given a vertex $\{R(a,b), R(c,d)\}$ we may multiplying $a, b, c, d$, by $u = (ad - bc)^{-1}$ assume that the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is in $SL_2(Z[\omega])$. Hence this vertex is $SL_2$ equivalent to $A = \{(0,1),(1,0)\}$. For any vertex containing exactly three lines there is an $SL_2$ equivalent having the form $\{(0,1),(1,0),(1,\alpha)\}$, $\alpha \in Z[\omega]^*$. Multiplication by $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & \omega\\ \omega^2 & 0\end{pmatrix}$ or $\begin{pmatrix}-\omega & 0\\ 0 & \omega^2\end{pmatrix}$ converts $(1,-1)$, $(1,\omega)$ or $(1,\omega^2)$ to $(1,1)$ and preserves $A$. Thus any vertex containing three lines is $SL_2$ equivalent to $B=\{(0,1),(1,0),(1,1)\}$. Finally any vertex containing four independent lines is $SL_2$ equivalent to $\{(0,1),(1,0),(1,1),(1,\alpha)\}$, $\alpha, \alpha-1 \in Z[\omega]^*$. Hence $\alpha=\omega$ or $-\omega^2$; these two vertices $C_1, C_2$ corresponding to $\alpha=\omega$ and $\alpha=-\omega^2$ respectively are easily seen to be inequivalent.

## §6

Let $R=Z[\omega]$; the vertices in the fundamental domain for $\mathcal{Y}(R)/SL_2(R)$ are $A=\{R(0,1),\ R(1,0)\}$ $B=\{R(0,1),\ R(1,0),\ R(1,1)\}$, $C_1=\{R(1,0),\ R(0,1),\ R(1,1),\ R(1,\omega)\}$ and $C_2=\{R(1,0),\ R(0,1),\ R(1,1),\ R(1,-\omega^2)\}$. Put $\Gamma=SL_2(R)$; denote by $\Gamma_v$ the stabilizer of the vertex $v$. Each vertex is determined by a collection of pairwise independent lines $\mathcal{L}(v)$. Consequently we have homomorphisms

$$\Gamma_v \to \Sigma_{\mathcal{L}(v)}$$

($\Sigma_S$ denotes the symmetric group on the set $S$) with kernel denoted $K_v$.

In case $v=A$ then $\Gamma_A$ contains $\begin{pmatrix}\omega & 0\\ 0 & -\omega^2\end{pmatrix}=\sigma$ and $\tau=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ which induces the transposition of the elements of $A$. Consequently there is an exact sequence

$$0 \to Z_6 \to \Gamma_A \to \Sigma_2 \to 0.$$

It is easy to see then that $\Gamma_A$ is the dicyclic group of order 12:

$$\Gamma_A = \langle \sigma, \tau \mid \tau^2=\sigma^3=(\sigma\tau)^2\rangle$$

In case $v=B$ then an analysis yields the fact that $\Gamma_B$ contains the matrices $s=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ together with $t=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}$ which induces a 3-cycle on the lines in $B$. We have an exact sequence

$$0 \to Z_2 \to \Gamma_B \to Z_3 \to 0$$

so that $\Gamma_B$ is a cyclic group generated by $t$ of order 6.

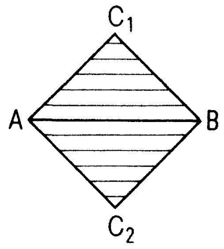
In the last case where $v = C_1$ or $C_2$ then it is easy to see that $K_{C_1}$ is cyclic of order 2 generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It is not difficult to see that the image of $\Gamma_{C_1}$ in $\Sigma_{\mathcal{L}(C_1)}$ contains no transpositions; however there are 3 cycles and double transpositions. In $\Gamma_{C_1}$ a 3 cycle is afforded by $t = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and a double transposition by $r = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$. We find that $\Gamma_{C_1}$ is the binary tetrahedral group:

$$\Gamma_{C_1} = \langle t, r \mid t^3 = r^2 = (t^{-1}r)^3 \rangle$$

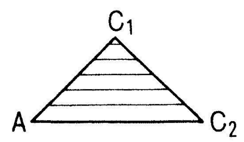
The group $\Gamma_{C_2}$ is $\alpha \Gamma_{C_1} \alpha^{-1}$, $\alpha = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$.

From this information it is then easy to describe the stabilizers of the edges and 2-simplices in the fundamental domain for $SL_2(R)$. We summarize this data: ($\Gamma_{xy} = \Gamma \cap \Gamma_y$, etc.) $\Gamma_{AB} = \langle t^3 \rangle$ is cyclic of order 2, $\Gamma_{AC_1} = \langle r \rangle$ is cyclic of order 4, $\Gamma_{BC_1} = \langle t \rangle$ is cyclic of order 6, $\Gamma_{AC_2} = \langle \alpha r \alpha^{-1} \rangle = \langle \tau\sigma^4 \rangle$ is cyclic of order 4, $\Gamma_{BC_2} = \langle \sigma\tau\alpha^{-1} \rangle = \langle t^{-1} \rangle$ is cyclic of order 6, $\Gamma_{ABC_1} = \Gamma_{ABC_2} = \langle t^3 \rangle$ is cyclic of order 2. Observe that $\alpha r \alpha^{-1} = t\sigma^4$, and $\alpha\tau\alpha^{-1} = t^{-1}$.

We have the fundamental domain as below.




so that $\chi(SL_2(R)) = \frac{1}{12} + \frac{1}{6} + \frac{1}{24} + \frac{1}{24} - \frac{1}{4} - \frac{1}{4} - \frac{1}{6} - \frac{1}{6} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$. Since $\Gamma_{C_1C_2} = \Gamma_B$ we may regard the fundamental domain as a single 2 simplex




Using the presentation of $\Gamma_A, \Gamma_{C_1}, \Gamma_{C_2}$ we may obtain a presentation for $SL_2$(Soule [5]) viz.,

$$SL_2 = \langle \sigma, \tau, t \mid \tau^2 = \sigma^3 = (\sigma\tau)^2 = t^3 = (t^{-1}\sigma\tau)^3 = (t^{-1})^3 = (t\tau^{-1}\sigma)^3 \rangle.$$

## §7

If $X$ is an acyclic space on which a group $\Gamma$ acts there is a spectral sequence

$$E^1_{p,q} = H_q(\Gamma, C_p) \Rightarrow H_{p+q}(\Gamma, Z)$$

where $C_p$ are $p$-chains on $X$ (Serre [4]). If $\Gamma$ acts with fundamental domain $\bar{X}$ then

$$C_p = \bigoplus_i Z\Gamma \otimes_{Z\Gamma_{p_i}} Z$$

where $p_i$ are the $p$-simplices in $\bar{X}$ and $\Gamma_{p_i}$ is the stabilizer of $p_i$ in $\Gamma$. Thus

$$E^1_{p,q} = \bigoplus_i H_q(\Gamma_{p_i}, Z).$$

Now in the case $X = \mathcal{Y}(Z[\omega])$ with fundamental domain as above, $\Gamma_A$, $\Gamma_{C_1}$, $\Gamma_{C_2}$ are all subgroups of the three sphere $S^3$ and hence have periodic homology of period 4.

PROPOSITION. *If $G$ is a finite subgroup of $S^3$ then $H_{4l+k}(G) = H_k(G)$ $k = 1, 2, 3$ $l \geq 0$; $H_{4l}(G) = 0$ $l \geq 1$; $H_3(G) = Z_{|G|}$, $H_2(G) = 0$.*

*Proof.* The action of $G$ on $S^3$ implies that the homology of $G$ is periodic of period 4. The determination of $H_3(G)$ is well known (See Cartan-Eilenberg [1]). If $G$ is a cyclic or dicyclic then $H_2(G) = 0[1]$. Otherwise $G$ is one of the binary polyhedral groups. In this case $S^3/G$ is an orientable 3-manifold; using Poincare duality $H_2(G) \cong$ torsion $H_0(G) = 0$.

PROPOSITION. *If $G$ is a cyclic group of order $n$ then*

$$H_0(G) = Z, \qquad H_k(G) = Z_n, \qquad H_{k+1}(G) = 0, \qquad k \text{ odd}.$$

COROLLARY. *The homology of $SL_2(Z[\omega])$ in dimensions greater than zero is annihilated by* 24.

Recall from §6 that $\Gamma_A = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = (\sigma\tau)^2 \rangle$,

$$\Gamma_{C_1} = \langle t, r \mid t^3 = r^2 = (t^{-1}r)^3 \rangle \quad \text{and} \quad \Gamma_{C_2} = \alpha \Gamma_{C_1} \alpha^{-1}.$$

LEMMA. *$\Gamma_A/\Gamma'_A$ is cyclic of order 4 generated by the image of $\tau$, say $\bar{\tau}$ and $2\bar{\tau}=\bar{\sigma}$. $\Gamma_{C_1}/\Gamma_{C'_1}$ is cyclic of order 3 generated by the image of $\bar{t}$ and $2\bar{t}=\overline{t^{-1}\tau t}$.*

The table below indicates the effects on the first homology of the indicated maps

| *Inclusion Map* | 1*st Homology Map* |
|---|---|
| $\Gamma_{AC_1}\to\Gamma_A$ | $Z_4\xrightarrow{3}Z_4$ |
| $\Gamma_{AC_1}\to\Gamma_{C_1}$ | $Z_4\xrightarrow{0}Z_3$ |
| $\Gamma_{AC_2}\to\Gamma_A$ | $Z_4\xrightarrow{1}Z_4$ |
| $\Gamma_{AC_2}\to\Gamma_{C_2}$ | $Z_4\xrightarrow{0}Z_3$ |
| $\Gamma_{C_1C_2}\to\Gamma_{C_1}$ | $Z_6\xrightarrow{1}Z_3$ |
| $\Gamma_{C_1C_1}\to\Gamma_{C_2}$ | $Z_6\xrightarrow{2}Z_3$ |
| $\Gamma_{AC_1C_2}\to\Gamma_A$ | $Z_2\xrightarrow{2}Z_4$ |
| $\Gamma_{AC_1C_2}\to\Gamma_{C_1}$ | $Z_2\xrightarrow{0}Z_3$ |
| $\Gamma_{AC_1C_2}\to\Gamma_{C_2}$ | $Z_2\xrightarrow{0}Z_3$ |
| $\Gamma_{AC_1C_2}\to\Gamma_{AC_1}$ | $Z_2\xrightarrow{2}Z_4$ |
| $\Gamma_{AC_1C_2}\to\Gamma_{AC_2}$ | $Z_2\xrightarrow{2}Z_4$ |
| $\Gamma_{AC_1C_2}\to\Gamma_{C_1C_2}$ | $Z_2\xrightarrow{3}Z_6$ |

We analyze the spectral sequence in the steps below.

(1) $E^1_{2,p}\xrightarrow{d_1}E^1_{1,p}$. This map corresponds to $H_p(Z_2)\to H_p(Z_4)\oplus H_p(Z_4)\oplus H_p(Z_6)$ from the stabilizer of the 2 simplex to the stabilizers of the edges. This is injective; hence $E^2_{2,p}=0$.

(2) Since all the edges have cyclic stabilizers $E^1_{1,p}\xrightarrow{d_1}E^1_{0,p}$ is zero for $p$ even. Thus $E^2_{1,p}=0$ for $p$ even.

(3) If $p=1$ (4) then $E^1_{1,p}\xrightarrow{d_1}E^1_{0,p}$ corresponds to the map

$$H_p(Z_6)\oplus H_p(Z_4)\oplus H_p(Z_4)\to H_p(\Gamma_{C_1})\oplus H_p(\Gamma_{C_2}\oplus H_p(\Gamma_A)$$

or

$$Z_6\oplus Z_4\oplus Z_4\to Z_3\oplus Z_3\oplus Z_4$$
$$(a,b,c)\to(a,2a,b-c)$$

so that the kernel is of order 8 generated by $(3,0,0)$ and $(0,1,1)$. The image of $d_1:E^1_{2,p}\to E^1_{1,p}$ is generated by $(3,2,2)$ so that $E^2_{1,p}=Z_2\oplus Z_4/(1,2)\cong Z_4$ generated by $(0,1,1)$.

(4) If $p=3$ (4) then $E^1_{1,p}\xrightarrow{d_1}E^1_{0,p}$ corresponds to the map

$$Z_4\oplus Z_6\oplus Z_4\xrightarrow{d_1}Z_{24}\oplus Z_{24}\oplus Z_{12}$$
$$(a,b,c)\longrightarrow(4b+6c,-6a-4b,3a+3c).$$

If $(a, b, c)$ is in the kernel then $4 \mid a+c$, $12 \mid 2b+3c$, $12 \mid 3a+2b$. One finds then that the kernel is generated by $(2, 3, 2)$ which is precisely the image of $E^1_{2,p} \to E^1_{1,p}$. Hence $E^2_{1,p}=0$ for $p=3$ (4).

(5) $E^2_{0,4k}=0$ $k \geq 1$.

(6) If $p=1$ (4) then $E^1_{1,p} \xrightarrow{d_1} E^1_{0,p}$ is the same as in the map in step 3

$$Z_6 \oplus Z_4 \oplus Z_4 \xrightarrow{d_1} Z_3 \oplus Z_3 \oplus Z_4$$

The kernel is of order 8 so that the image is of order 12, hence $E^2_{0,p}=$ cokernel of $d_1 \cong Z_3$.

(7) If $p=3$ (4) then $E^2_{0,p}$ is the cokernel of

$$E^1_{1,p} \xrightarrow{d_1} E^1_{0,p}$$

which from step 4 is the cokernel of

$$Z_4 \oplus Z_6 \oplus Z_4 \to Z_{24} \oplus Z_{24} \oplus Z_{12}$$
$$(a, b, c \to (4b+6c, -6a-4b, 3a+3c)$$

If $x, y, z$ generate $Z_{24}, Z_{24}, Z_{12}$ respectively then the cokernel has relations $-6y+3z$, $4x-4y$, $6x-3z$ or equivalently $6x+3z$ and $10x+2y$. Consequently, $E^2_{0,p} \cong Z_{24} \oplus Z_2 \oplus Z_3$ for $p=3$ (4).

(8) If $p=2$ (4) then $E^1_{1,p} \longrightarrow E^1_{0,p}$ is zero. It follows now that $E^2=E^\infty$ and the next result is then immediate.

THEOREM. *For $\Gamma = SL_2(Z[\omega])$,*

$$H_n(\Gamma) = \begin{cases} Z & n=0 \\ Z_3 & n=1 \ (4) \\ Z_4 & n=2 \ (4) \\ Z_{24} \oplus Z_6 & n=3 \ (4) \\ 0 & n=0 \ (4),\ n \neq 0 \end{cases}$$

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