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## Homology of $SL_2(\mathbb{Z}[\omega])$

ROGER ALPERIN

In this article we shall describe a simplicial complex which is a natural structure for the action of  $GL_2(R)$ , the group of  $2 \times 2$  invertible matrices over the ring  $R$ . With strong conditions on  $R$  this complex is contractible; it is then possible to give a presentation of  $GL_2(R)$  and to compute the homology of  $GL_2(R)$  in terms of stabilizer subgroups of the simplices in a fundamental domain for the action. We shall work in detail with the ring  $\mathbb{Z}[\omega]$ ,  $\omega^2 = \omega - 1$ ; similar methods apply to the rings  $\mathbb{Z}[\theta]$ ,  $\theta^2 = \theta + 1$ , and  $\mathbb{Z}[\lambda]$ ,  $\lambda^3 = \lambda + 1$  but the details are quite elaborate and will be left for a later time. Initial motivation came from Quillen's construction of the tree for  $SL_2(\mathbb{Z})$  (compare Serre [3]).

### §1

Let  $R$  be a ring. Consider the set  $\mathcal{L}$  of free direct summands of  $R^2$ . Elements of  $\mathcal{L}$  are called lines.

DEFINITION.  $L_1, L_2 \in \mathcal{L}$  are independent if  $L_1 + L_2 = L_1 \oplus L_2 = R^2$ .

Let  $\mathcal{U}(R)$  be the simplicial complex whose vertices are the elements of  $\mathcal{L}$  and whose  $q$ -simplices are determined by a set  $\{L_0, \dots, L_q\}$ ,  $L_i \in \mathcal{L}$  where  $L_i, L_j$  are independent for  $0 \leq i \neq j \leq q$ .

Let  $R(a, b)$  be a vertex of  $\mathcal{U}(R)$  and suppose  $R(c, d)$  is independent of  $R(a, b)$ .

LEMMA 1. Any line in  $R^2$  independent of  $R(a, b)$  is of the form

$$L = R(ra + c, rb + d)$$

for some  $r \in R$ .

*Proof.* Suppose  $L = R(c', d')$  is independent of  $R(a, b)$ . Put  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,

$B = \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}$ ;  $A, B$  are in  $GL_2(R)$ . Let  $A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  Then

$$BA^{-1} = \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}$$

for  $s \in R$ ,  $t \in R^*$  (units of  $R$ ). Thus  $c' = sa + tc$ ,  $d' = sb + td$ ; hence  $R(c', d') = R(ra + c, rb + d)$  with  $r = t^{-1}s$ .

**LEMMA 2.** *The lines  $L_1 = R(r_1a + c, r_1b + d)$   $L_2 = R(r_2a + c, r_2b + d)$  are independent iff  $r_1 - r_2 \in R^*$ .*

*Proof.* Let  $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ ; then

$$\begin{pmatrix} r_1a + c & r_1b + d \\ r_2a + c & r_2b + d \end{pmatrix} C = \begin{pmatrix} r_1 & r_2 \\ 1 & 1 \end{pmatrix}.$$

Hence  $L_1, L_2$  are independent iff  $r_1 - r_2 \in R^*$ .

Consider the link of a vertex  $R(a, b)$  in  $\mathcal{U}(R)$ ,  $\text{Link } R(a, b)$ ; this is the full subcomplex of  $\mathcal{U}(R)$  containing all lines which are independent of  $R(a, b)$ . Let  $\mathcal{R}$  be the simplicial complex whose vertices are given by the elements of  $R$  and in which a  $q$ -simplex is given by a set  $\{r_0, \dots, r_q\}$ ,  $r_i \in R$  with  $r_i - r_j \in R^*$  for  $0 \leq i \neq j \leq q$ . The next lemma follows easily from the previous discussion.

**LEMMA 3.** *Link  $R(a, b) \cong \mathcal{R}$ .*

Put  $M_R = \sup \{m \mid \exists r_1, \dots, r_m \in R \ni \forall i, j, 1 \leq i < j \leq m, r_i - r_j \in R^*\}$ .

**LEMMA 4.** (Lenstra [2])  *$M_R$  is finite if  $R$  has an ideal ( $\neq R$ ) of finite index.*

**COROLLARY.**  *$\mathcal{U}(R)$  is finite dimensional if  $R$  is the ring of integers in a number field and  $\dim \mathcal{U}(R) = M_R$ .*

*Proof.* The ring of integers in a number field has an ideal of finite index, for example (2). It follows easily that the dimension of a simplex in  $\mathcal{U}(R)$  is  $\leq 1 + \dim \mathcal{R} = M_R$ .

When  $R$  is the ring of integers in an algebraic field, and  $R$  has a unit of infinite order then according to a result of Vaserstein [2],  $SL_2(R)$  is generated by

elementary matrices. If  $R$  is a Euclidean ring then  $SL_2(R)$  is generated by elementary matrices. It follows then in case  $R$  is Euclidean or  $R$  has a unit of infinite order that  $\mathcal{U}(R)$  is connected. The cases excluded by this are the non-Euclidean rings of integers in imaginary quadratic number fields.

## §2

We suppose now that  $R$  is a Euclidean ring with respect to the function  $|\cdot|: R \rightarrow N$ . Suppose also that  $|\cdot|$  is multiplicative and thus gives rise to a function on the quotients field of  $R$ ,  $K$ . Define

$$|\cdot|: \mathcal{L} \rightarrow N \quad \text{via}$$

$|R(a, b)| = |b|$ . This is independent of the particular representation of the line since units have value one under the Euclidean function. Let  $\mathcal{U}(n, R)$  be the full subcomplex of  $\mathcal{U}(R)$  containing all vertices  $L$  of  $\mathcal{L}$  with  $|L| \leq n$ .

Consider now the link of a vertex  $R(a, b)$ ,  $|R(a, b)| = n$ , in  $\mathcal{U}(n, R)$ , denoted  $\text{Link}_n R(a, b)$ . This link is the full subcomplex of  $\mathcal{U}(n, R)$  containing lines  $L$  independent of  $R(a, b)$  with  $|L| \leq n$ . Let  $R(c, d)$  be independent of  $R(a, b)$ . It follows from Lemma 1 that this link contains only vertices  $R(ra + c, rb + d)$  with  $|rb + d| \leq n$ . Using the Euclidean algorithm we write  $d = qb + d_0$  with  $|d_0| < |b| = n$ ; let  $c_0 = c - qa$ . Thus the link contains only those lines  $R(c_0 + ra, d_0 + rb)$  with  $|d_0 + rb| \leq n$ .

Now if  $x \in K$ , put  $R_x = \{r \in R \mid |x - r| \leq 1\}$ . Let  $\mathcal{R}_x$  be the simplicial complex in which a  $q$ -simplex is determined by a set  $\{r_0, \dots, r_q\}$ ,  $r_i \in R_x$ ,  $r_1 - r_j \in R^*$ ,  $0 \leq i \neq j \leq q$ ; this is the full subcomplex of  $\mathcal{R}$  containing the vertices  $R_x$ .

LEMMA 5.  $\text{Link}_n R(a, b) \cong \mathcal{R}_x$ ,  $x = d_0/b$ .

*Proof.* The vertices in  $\text{Link}_n R(a, b)$  are  $R(c_0 + ra, d_0 + rb)$ ,  $|d_0 + rb| \leq |b|$  or equivalently  $|d_0/b + r| \leq 1$ . Thus there is a 1-1 correspondence between the simplices of the link and the simplices of  $\mathcal{R}_x$ ,  $x = d_0/b$ . The incidence relation on  $\mathcal{R}_x$  is designed so as to agree with that for the link.

We make the observations below which will be of use later.

LEMMA 6.  $\mathcal{R}_x \cong \mathcal{R}_{x+a}$   $x \in K$ ,  $a \in R$ .

LEMMA 7.  $\mathcal{R}_x \cong \mathcal{R}_{ux}$   $x \in K$ ,  $u \in R^*$ .



There are two types of elements of  $K$  which we need to distinguish. If  $x \in K$  and  $R_x = \{r \mid |x - r| < 1\}$  then  $x$  will be called of type I; otherwise  $x$  is of type II.

### §3

In this section we analyze the structure of the complexes  $\mathcal{R}$  and  $\mathcal{R}_x$  for the ring  $R = Z[\omega]$ ,  $\omega^2 = \omega - 1$ . The simplices of  $\mathcal{R}$  are given by sets  $\{r_0, \dots, r_q\}$ ; we shall at times denote this by  $r + u\{s_0, \dots, s_q\}$ ,  $r \in R$ ,  $u \in R^*$ ,  $r_i = r + us_i$ ,  $0 \leq i \leq q$ .

LEMMA 8. (a) *Every 1-simplex of  $\mathcal{R}$  is uniquely of the form  $r + \omega^i\{0, 1\}$ ,  $0 \leq i < 3$ .*

(b) *Every 2-simplex of  $\mathcal{R}$  is uniquely of the form  $r + \omega^i\{0, 1, \omega\}$ ,  $0 \leq i < 2$ .*

*Proof.* Given a 1-simplex,  $\{r_0, r_1\}$ , we may write this as  $r_0 + (r_1 - r_0)\{0, 1\}$ ,  $r_1 - r_0 \in R^*$ . Notice that  $\{0, -1\} = -1 + \{0, 1\}$  and this provides the required form. For a 2-simplex we may suppose that it has the form  $r + u\{0, 1, \eta\}$  with  $\eta, \eta - 1 \in R^*$ . Then it follows easily that  $\eta = \omega$  or  $\eta = \omega^5$ . Notice that  $\{0, 1, \omega^5\} = \omega^{-1}\{0, 1, \omega\}$ . Thus every 2-simplex has the form  $r + u\{0, 1, \omega\}$ ; in order to get the proper restriction on  $u$  notice the relations:

$$-\{0, 1, \omega\} = -\omega + \omega\{0, 1, \omega\}. \quad \omega^2\{0, 1, \omega\} = -1 + \{0, 1, \omega\}.$$

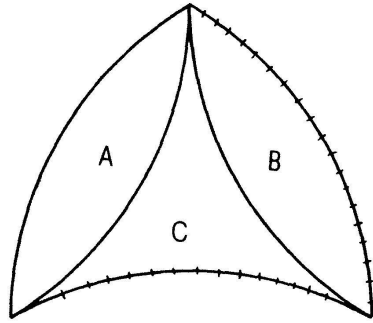
For the uniqueness part suppose that  $r + u\sigma = s + v\sigma$  where  $\sigma$  is  $\{0, 1\}$  in part (a) or  $\sigma = \{0, 1, \omega\}$  in part (b) and  $u, v$  are restricted suitably. We obtain then a relation  $\sigma = v^{-1}(r - s) + v^{-1}u \cdot \sigma$ . Thus we may suppose that there is a relation  $\sigma = \rho + \tau\sigma$  and show that  $\tau = 1$  and  $\rho = 0$ . This is quite simple in case (a). In case (b) we observe that  $\rho$  must be one of  $0, 1, \omega$ . If  $\rho = 1$  then either  $\tau = -1$  or  $\tau = \omega^2$ ; both of these are excluded by the form. If  $\rho = \omega$  then either  $\omega + \tau = 0$  or  $\omega + \tau\omega = 0$ ; one checks that this is impossible.

COROLLARY. *If  $R = Z[\omega]$  then  $\mathcal{R}$  is contractible.*

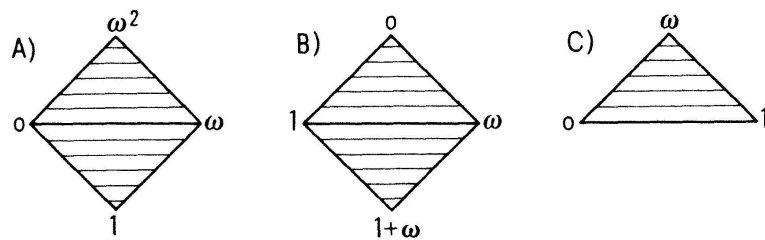
*Proof.* View  $R$  embedded in  $\mathbb{C}$  as a lattice, then the simplices  $r + \omega\{0, 1, \omega\}$ ,  $r + \{0, 1, \omega\}$  provide  $\mathbb{C}$  with a simplicial structure tessellated by these two types of simplices.

Now for the structure of  $\mathcal{R}_x$  we may using Lemma 6 assume that  $0 \in R_x$ ; our only concern is with  $x \in K - R$ . View  $R$  embedded in  $\mathbb{C}$  and hence also  $K$ . The norm  $N: K \rightarrow \mathbb{Q}$ , which is the square of the usual absolute value on  $\mathbb{C}$ , provides

the multiplicative Euclidean function on  $R$ . Using Lemma 7 we may assume that  $x$  belongs to the region below.



If  $x \in K - R$  belongs to this region and is not on one of three solid arcs then it is of type I. Following the labeling of the three regions we describe  $R_x$ . For region  $A$  which includes the two solid arcs we have  $R_x = \{0, 1, \omega, \omega^2\}$ ; for region  $B$  which includes the third solid arc  $R_x = \{0, 1, \omega, 1 + \omega\}$ ; for region  $C$ ,  $R_x = \{0, 1, \omega\}$ . The complexes  $\mathcal{R}_x$  have the following structure:



Notice that for  $x \in K - R$  of type II,  $R_x$  is a union of two 2-simplices.

#### §4

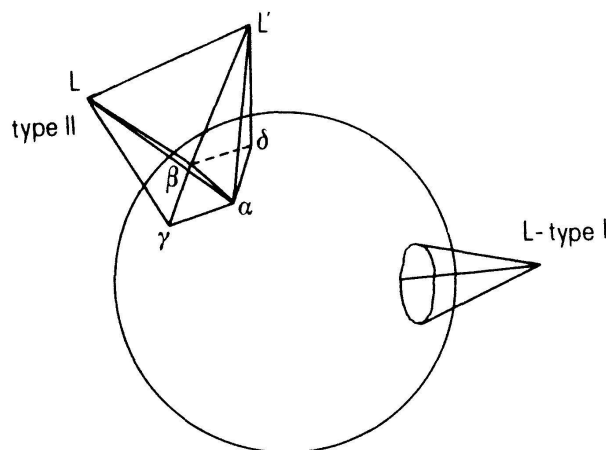
**THEOREM.**  $\mathcal{U}(Z[\omega])$  is contractible.

*Proof.* We filter  $\mathcal{U}(Z[\omega])$  by the subcomplexes  $\mathcal{U}(n, Z[\omega])$  according the norm. Let  $\mathcal{L}_n = \{L \in \mathcal{L} \mid |L| = n\}$ . We shall establish that  $\mathcal{U}(n, Z[\omega])$  is contractible to  $Z[\omega](1, 0)$  by induction on  $n$ . Notice first that  $\mathcal{U}(0) = Z[\omega](1, 0)$  and that  $\mathcal{U}(1)$  has  $Z[\omega](1, 0)$  as a cone point; suppose inductively then that  $\mathcal{U}(n-1, Z[\omega])$  is contractible for  $n > 1$ . We have

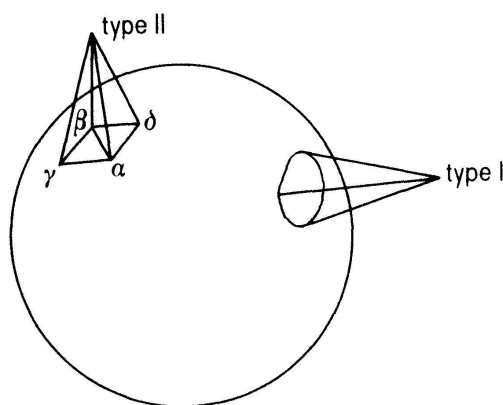
$$\mathcal{U}(n-1, Z[\omega]) \cong \mathcal{U}(n, Z[\omega]) - \bigcup_{L \in \mathcal{L}_n} \text{st}(L).$$

where  $\text{st}(L)$  is the open star of  $L$  in  $\mathcal{U}(n, Z[\omega])$ . Thus  $\mathcal{U}(n, Z[\omega])$  is obtained from  $\mathcal{U}(n-1)$  by attaching the  $\text{Cone}(\text{Link}_n L)$ ,  $L \in \mathcal{L}_n$  to  $\mathcal{U}(n-1)$  along the

$\text{Link}_n L$ . Now the  $\text{Link}_n L$  corresponds to one of the complexes  $\mathcal{R}_x$ ,  $x \in K - R$ , which if  $x$  is of type I has all of its vertices in  $\mathcal{U}(n-1)$ ; however if  $x$  is of type II there is a unique vertex  $L'$  in  $\text{Link}_n L$  which belongs to  $\mathcal{L}_n$ . We diagram this by the picture:



We have noticed here that if  $L$  is of type II then for the vertex  $L'$  in  $\text{Link}_n L$  there must be exactly two 1-simplices meeting at  $L'$  in the link. Now to complete the picture we examine  $\text{Link}_n L'$ ; the link of  $L'$  contains  $\alpha, \beta, L$  and another vertex  $\delta \in \mathcal{U}(n-1)$ ,  $\delta \neq \gamma$ , arranged as in the diagram. Now to contract  $\mathcal{U}(n)$  we first contract  $L$  to  $L'$  along the edge joining them for every pair  $L, L' \in \mathcal{L}_n$  which are in each other's links. We obtain then a complex with the same homotopy type as  $\mathcal{U}(n)$ ,  $\mathcal{V}(n)$ . Now



$$\mathcal{V}(n) \cong \mathcal{U}(n-1) \bigcup_{L \in \mathcal{L}'_n} \text{Cone}(\text{Link}_n(L))$$

where  $\mathcal{L}'_n$  is the subset of  $\mathcal{L}_n$  containing all type I vertices and one from each pair of type II vertices as above. The  $\text{Link}_n(L)$  is unchanged for type I and for the type II the link has the same homotopy type. Now  $\mathcal{U}(n-1)$  is contractible so we

obtain

$$\mathcal{U}(n) \cong \mathcal{V}(n) \cong \bigvee_{L \in \mathcal{L}'_n} \text{Susp}(\text{Cone}(\text{Link}_n(L))).$$

Hence, since each link is contractible we have that  $\mathcal{U}(n)$  is contractible. It follows then that  $\mathcal{U}$  is contractible.

We denote by  $\mathcal{U}'(Z[\omega])$  the first barycentric subdivision of  $\mathcal{U}(Z[\omega])$ .

**COROLLARY.**  $\mathcal{U}(Z[\omega]) \cong \mathcal{U}'(Z[\omega]) - \bigcup_{L \in \mathcal{L}} \text{st}(L)$

*Proof.* According to the corollary of Lemma 8, the simplicial complex  $\mathcal{R}$  is contractible. We have  $\mathcal{R} \cong \text{Link}(L)$  for any  $L \in \mathcal{L}$ . Notice:

$$\mathcal{U} \cong \mathcal{U}' \cong \left( \mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}(L) \right) \cup \bigcup_{L \in \mathcal{L}} \text{Cone}(\text{Link}(L))$$

Now the  $\text{Link}(L)$  above is computed in  $\mathcal{U}'$  but its homotopy type is unchanged, i.e., it's contractible. Thus

$$\mathcal{U} \cong \mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}(L)$$

is contractible.

The complex  $\mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}(L)$  may be described as follows: Consider the partially ordered set (by inclusion) of subsets of  $\mathcal{L}$  of the type

$$\{L_0, \dots, L_q\}, \quad q \geq 1$$

for which  $L_i, L_j$  are independent for  $0 \leq i \neq j \leq q$ ; then  $\mathcal{U}' - \bigcup_{L \in \mathcal{L}} \text{st}(L)$  has the homotopy type of the realization of this poset, say  $\mathcal{Y}(Z[\omega])$ .

## §5

**LEMMA 9.** For the complex  $\mathcal{U}(R)$ ,  $R = Z[\omega]$ ,

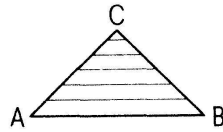
- (a) every vertex is  $GL_2(R)$  equivalent to  $\{R(1, 0)\}$ ;
- (b) every 1-simplex is  $GL_2(R)$  equivalent to  $\{R(1, 0), R(0, 1)\}$ ;
- (c) every 2-simplex is  $GL_2(R)$  equivalent to  $\{R(1, 0), R(0, 1), R(1, 1)\}$ ;
- (d) every 3-simplex is  $GL_2(R)$  equivalent to  $\{R(1, 0), R(0, 1), R(1, 1), R(1, \omega)\}$ .

*Proof.* The first two parts of the lemma are easy. For part (c), any 2-simplex is equivalent via  $GL_2$  to a simplex which must have the form  $\{R(0, 1), R(1, 0), R(1, \alpha)\}$ ,  $\alpha \in R^*$ . Multiplication by the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$  converts this 2-simplex to the required form. For 3-simplices we may by the action of  $GL_2$  bring this to the simplex

$$\{R(0, 1), R(1, 0), R(1, 1), R(1, \alpha)\}.$$

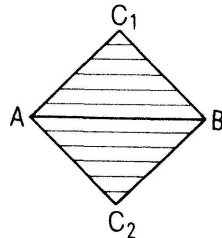
with  $\alpha, \alpha - 1 \in R^*$ . According to the proof of lemma 8,  $\alpha = \omega$  or  $\omega^5$ . If  $\alpha = \omega^5$  then multiplication by the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  on this simplex converts it to the required form.

**COROLLARY.** *The fundamental domain for the action of  $GL_2(Z[\omega])$  on  $\mathcal{Y}(Z[\omega])$  is a single 2-simplex.*



*Proof.* Recall the description of  $\mathcal{Y}(Z[\omega])$  at the end of the previous section. Using the previous lemma now, the fundamental domain for  $GL_2$  on  $\mathcal{Y}(Z[\omega])$  has vertices  $A = \{(0, 1), (1, 0)\}$ ,  $B = \{(0, 1), (1, 0), (1, 1)\}$  and  $C = \{(1, 0), (0, 1), (1, 1), (1, \omega)\}$ . (We have given only the generators for the lines in  $A, B, C$ .)

**COROLLARY.** *The fundamental domain for the action of  $SL_2(Z[\omega])$  on  $\mathcal{Y}(Z[\omega])$  is the 2-complex*



*Proof.* Given a vertex  $\{R(a, b), R(c, d)\}$  we may multiply  $a, b, c, d$ , by  $u = (ad - bc)^{-1}$  assume that the matrix  $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$  is in  $SL_2(Z[\omega])$ . Hence this vertex is  $SL_2$  equivalent to  $A = \{(0, 1), (1, 0)\}$ . For any vertex containing exactly three

lines there is an  $SL_2$  equivalent having the form  $\{(0, 1), (1, 0), (1, \alpha)\}$ ,  $\alpha \in Z[\omega]^*$ . Multiplication by  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$  or  $\begin{pmatrix} -\omega & 0 \\ 0 & \omega^2 \end{pmatrix}$  converts  $(1, -1)$ ,  $(1, \omega)$  or  $(1, \omega^2)$  to  $(1, 1)$  and preserves  $A$ . Thus any vertex containing three lines is  $SL_2$  equivalent to  $B = \{(0, 1), (1, 0), (1, 1)\}$ . Finally any vertex containing four independent lines is  $SL_2$  equivalent to  $\{(0, 1), (1, 0), (1, 1), (1, \alpha)\}$ ,  $\alpha, \alpha - 1 \in Z[\omega]^*$ . Hence  $\alpha = \omega$  or  $-\omega^2$ ; these two vertices  $C_1, C_2$  corresponding to  $\alpha = \omega$  and  $\alpha = -\omega^2$  respectively are easily seen to be inequivalent.

## §6

Let  $R = Z[\omega]$ ; the vertices in the fundamental domain for  $\mathcal{Y}(R)/SL_2(R)$  are  $A = \{R(0, 1), R(1, 0)\}$ ,  $B = \{R(0, 1), R(1, 0), R(1, 1)\}$ ,  $C_1 = \{R(1, 0), R(0, 1), R(1, 1), R(1, \omega)\}$  and  $C_2 = \{R(1, 0), R(0, 1), R(1, 1), R(1, -\omega^2)\}$ . Put  $\Gamma = SL_2(R)$ ; denote by  $\Gamma_v$  the stabilizer of the vertex  $v$ . Each vertex is determined by a collection of pairwise independent lines  $\mathcal{L}(v)$ . Consequently we have homomorphisms

$$\Gamma_v \rightarrow \Sigma_{\mathcal{L}(v)}$$

( $\Sigma_S$  denotes the symmetric group on the set  $S$ ) with kernel denoted  $K_v$ .

In case  $v = A$  then  $\Gamma_A$  contains  $\begin{pmatrix} \omega & 0 \\ 0 & -\omega^2 \end{pmatrix} = \sigma$  and  $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  which induces the transposition of the elements of  $A$ . Consequently there is an exact sequence

$$0 \rightarrow Z_6 \rightarrow \Gamma_A \rightarrow \Sigma_2 \rightarrow 0.$$

It is easy to see then that  $\Gamma_A$  is the dicyclic group of order 12:

$$\Gamma_A = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = (\sigma\tau)^2 \rangle$$

In case  $v = B$  then an analysis yields the fact that  $\Gamma_B$  contains the matrices  $s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  together with  $t = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$  which induces a 3-cycle on the lines in  $B$ . We have an exact sequence

$$0 \rightarrow Z_2 \rightarrow \Gamma_B \rightarrow Z_3 \rightarrow 0$$

so that  $\Gamma_B$  is a cyclic group generated by  $t$  of order 6.

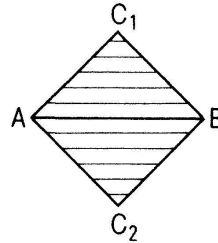
In the last case where  $v = C_1$  or  $C_2$  then it is easy to see that  $K_{C_i}$  is cyclic of order 2 generated by  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ . It is not difficult to see that the image of  $\Gamma_{C_i}$  in  $\Sigma_{\mathcal{L}(C_i)}$  contains no transpositions; however there are 3 cycles and double transpositions. In  $\Gamma_{C_1}$  a 3 cycle is afforded by  $t = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$  and a double transposition by  $r = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$ . We find that  $\Gamma_{C_1}$  is the binary tetrahedral group:

$$\Gamma_{C_1} = \langle t, r \mid t^3 = r^2 = (t^{-1}r)^3 \rangle$$

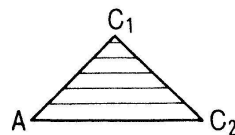
The group  $\Gamma_{C_2}$  is  $\alpha\Gamma_{C_1}\alpha^{-1}$ ,  $\alpha = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$ .

From this information it is then easy to describe the stabilizers of the edges and 2-simplices in the fundamental domain for  $SL_2(R)$ . We summarize this data: ( $\Gamma_{xy} = \Gamma \cap \Gamma_y$ , etc.)  $\Gamma_{AB} = \langle t^3 \rangle$  is cyclic of order 2,  $\Gamma_{AC_1} = \langle r \rangle$  is cyclic of order 4,  $\Gamma_{BC_1} = \langle t \rangle$  is cyclic of order 6,  $\Gamma_{AC_2} = \langle \alpha r \alpha^{-1} \rangle = \langle \tau \sigma^4 \rangle$  is cyclic of order 4,  $\Gamma_{BC_2} = \langle \sigma \tau \alpha^{-1} \rangle = \langle t^{-1} \rangle$  is cyclic of order 6,  $\Gamma_{ABC_1} = \Gamma_{ABC_2} = \langle t^3 \rangle$  is cyclic of order 2. Observe that  $\alpha r \alpha^{-1} = t \sigma^4$ , and  $\alpha \tau \alpha^{-1} = t^{-1}$ .

We have the fundamental domain as below.



so that  $\chi(SL_2(R)) = \frac{1}{12} + \frac{1}{6} + \frac{1}{24} + \frac{1}{24} - \frac{1}{4} - \frac{1}{4} - \frac{1}{6} - \frac{1}{6} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$ . Since  $\Gamma_{C_1 C_2} = \Gamma_B$  we may regard the fundamental domain as a single 2 simplex



Using the presentation of  $\Gamma_A, \Gamma_{C_1}, \Gamma_{C_2}$  we may obtain a presentation for  $SL_2(\text{Soule [5]})$  viz.,

$$SL_2 = \langle \sigma, \tau, t \mid \tau^2 = \sigma^3 = (\sigma\tau)^2 = t^3 = (t^{-1}\sigma\tau)^3 = (t^{-1})^3 = (t\tau^{-1}\sigma)^3 \rangle.$$

## §7

If  $X$  is an acyclic space on which a group  $\Gamma$  acts there is a spectral sequence

$$E_{p,q}^1 = H_q(\Gamma, C_p) \Rightarrow H_{p+q}(\Gamma, Z)$$

where  $C_p$  are  $p$ -chains on  $X$  (Serre [4]). If  $\Gamma$  acts with fundamental domain  $\bar{X}$  then

$$C_p = \bigoplus_i Z\Gamma \otimes_{Z\Gamma_{p_i}} Z$$

where  $p_i$  are the  $p$ -simplices in  $\bar{X}$  and  $\Gamma_{p_i}$  is the stabilizer of  $p_i$  in  $\Gamma$ . Thus

$$E_{p,q}^1 = \bigoplus_i H_q(\Gamma_{p_i}, Z).$$

Now in the case  $X = \mathcal{Y}(Z[\omega])$  with fundamental domain as above,  $\Gamma_A, \Gamma_{C_1}, \Gamma_{C_2}$  are all subgroups of the three sphere  $S^3$  and hence have periodic homology of period 4.

**PROPOSITION.** *If  $G$  is a finite subgroup of  $S^3$  then  $H_{4l+k}(G) = H_k(G)$   $k = 1, 2, 3$   $l \geq 0$ ;  $H_{4l}(G) = 0$   $l \geq 1$ ;  $H_3(G) = Z_{|G|}$ ,  $H_2(G) = 0$ .*

*Proof.* The action of  $G$  on  $S^3$  implies that the homology of  $G$  is periodic of period 4. The determination of  $H_3(G)$  is well known (See Cartan-Eilenberg [1]). If  $G$  is a cyclic or dicyclic then  $H_2(G) = 0$  [1]. Otherwise  $G$  is one of the binary polyhedral groups. In this case  $S^3/G$  is an orientable 3-manifold; using Poincare duality  $H_2(G) \cong \text{torsion } H_0(G) = 0$ .

**PROPOSITION.** *If  $G$  is a cyclic group of order  $n$  then*

$$H_0(G) = Z, \quad H_k(G) = Z_n, \quad H_{k+1}(G) = 0, \quad k \text{ odd.}$$

**COROLLARY.** *The homology of  $SL_2(Z[\omega])$  in dimensions greater than zero is annihilated by 24.*

Recall from §6 that  $\Gamma_A = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = (\sigma\tau)^2 \rangle$ ,

$$\Gamma_{C_1} = \langle t, r \mid t^3 = r^2 = (t^{-1}r)^3 \rangle \quad \text{and} \quad \Gamma_{C_2} = \alpha \Gamma_{C_1} \alpha^{-1}.$$



LEMMA.  $\Gamma_A/\Gamma'_A$  is cyclic of order 4 generated by the image of  $\tau$ , say  $\bar{\tau}$  and  $2\bar{\tau} = \bar{\sigma}$ .  $\Gamma_{C_1}/\Gamma'_{C_1}$  is cyclic of order 3 generated by the image of  $\bar{t}$  and  $2\bar{t} = \bar{t}^{-1}\tau t$ .

The table below indicates the effects on the first homology of the indicated maps

| Inclusion Map                                  | 1st Homology Map          |
|--|---------------------------|
| $\Gamma_{AC_1} \rightarrow \Gamma_A$           | $Z_4 \xrightarrow{3} Z_4$ |
| $\Gamma_{AC_1} \rightarrow \Gamma_{C_1}$       | $Z_4 \xrightarrow{0} Z_3$ |
| $\Gamma_{AC_2} \rightarrow \Gamma_A$           | $Z_4 \xrightarrow{1} Z_4$ |
| $\Gamma_{AC_2} \rightarrow \Gamma_{C_2}$       | $Z_4 \xrightarrow{0} Z_3$ |
| $\Gamma_{C_1C_2} \rightarrow \Gamma_{C_1}$     | $Z_6 \xrightarrow{1} Z_3$ |
| $\Gamma_{C_1C_1} \rightarrow \Gamma_{C_2}$     | $Z_6 \xrightarrow{2} Z_3$ |
| $\Gamma_{AC_1C_2} \rightarrow \Gamma_A$        | $Z_2 \xrightarrow{2} Z_4$ |
| $\Gamma_{AC_1C_2} \rightarrow \Gamma_{C_1}$    | $Z_2 \xrightarrow{0} Z_3$ |
| $\Gamma_{AC_1C_2} \rightarrow \Gamma_{C_2}$    | $Z_2 \xrightarrow{0} Z_3$ |
| $\Gamma_{AC_1C_2} \rightarrow \Gamma_{AC_1}$   | $Z_2 \xrightarrow{2} Z_4$ |
| $\Gamma_{AC_1C_2} \rightarrow \Gamma_{AC_2}$   | $Z_2 \xrightarrow{2} Z_4$ |
| $\Gamma_{AC_1C_2} \rightarrow \Gamma_{C_1C_2}$ | $Z_2 \xrightarrow{3} Z_6$ |

We analyze the spectral sequence in the steps below.

- (1)  $E_{2,p}^1 \xrightarrow{d_1} E_{1,p}^1$ . This map corresponds to  $H_p(Z_2) \rightarrow H_p(Z_4) \oplus H_p(Z_4) \oplus H_p(Z_6)$  from the stabilizer of the 2 simplex to the stabilizers of the edges. This is injective; hence  $E_{2,p}^2 = 0$ .
- (2) Since all the edges have cyclic stabilizers  $E_{1,p}^1 \xrightarrow{d_1} E_{0,p}^1$  is zero for  $p$  even. Thus  $E_{1,p}^2 = 0$  for  $p$  even.
- (3) If  $p = 1$  (4) then  $E_{1,p}^1 \xrightarrow{d_1} E_{0,p}^1$  corresponds to the map

$$H_p(Z_6) \oplus H_p(Z_4) \oplus H_p(Z_4) \rightarrow H_p(\Gamma_{C_1}) \oplus H_p(\Gamma_{C_2}) \oplus H_p(\Gamma_A)$$

or

$$Z_6 \oplus Z_4 \oplus Z_4 \rightarrow Z_3 \oplus Z_3 \oplus Z_4$$

$$(a, b, c) \rightarrow (a, 2a, b - c)$$

so that the kernel is of order 8 generated by  $(3, 0, 0)$  and  $(0, 1, 1)$ . The image of  $d_1: E_{2,p}^1 \rightarrow E_{1,p}^1$  is generated by  $(3, 2, 2)$  so that  $E_{1,p}^2 = Z_2 \oplus Z_4 / (1, 2) \cong Z_4$  generated by  $(0, 1, 1)$ .

- (4) If  $p = 3$  (4) then  $E_{1,p}^1 \xrightarrow{d_1} E_{0,p}^1$  corresponds to the map

$$Z_4 \oplus Z_6 \oplus Z_4 \xrightarrow{d_1} Z_{24} \oplus Z_{24} \oplus Z_{12}$$

$$(a, b, c) \rightarrow (4b + 6c, -6a - 4b, 3a + 3c).$$

If  $(a, b, c)$  is in the kernel then  $4 \mid a + c$ ,  $12 \mid 2b + 3c$ ,  $12 \mid 3a + 2b$ . One finds then that the kernel is generated by  $(2, 3, 2)$  which is precisely the image of  $E_{2,p}^1 \rightarrow E_{1,p}^1$ . Hence  $E_{1,p}^2 = 0$  for  $p = 3$  (4).

(5)  $E_{0,4k}^2 = 0$   $k \geq 1$ .

(6) If  $p = 1$  (4) then  $E_{1,p}^1 \xrightarrow{d_1} E_{0,p}^1$  is the same as in the map in step 3

$$Z_6 \oplus Z_4 \oplus Z_4 \xrightarrow{d_1} Z_3 \oplus Z_3 \oplus Z_4$$

The kernel is of order 8 so that the image is of order 12, hence  $E_{0,p}^2 =$  cokernel of  $d_1 \cong Z_3$ .

(7) If  $p = 3$  (4) then  $E_{0,p}^2$  is the cokernel of

$$E_{1,p}^1 \xrightarrow{d_1} E_{0,p}^1$$

which from step 4 is the cokernel of

$$Z_4 \oplus Z_6 \oplus Z_4 \rightarrow Z_{24} \oplus Z_{24} \oplus Z_{12}$$

$$(a, b, c) \rightarrow (4b + 6c, -6a - 4b, 3a + 3c)$$

If  $x, y, z$  generate  $Z_{24}, Z_{24}, Z_{12}$  respectively then the cokernel has relations  $-6y + 3z$ ,  $4x - 4y$ ,  $6x - 3z$  or equivalently  $6x + 3z$  and  $10x + 2y$ . Consequently,  $E_{0,p}^2 \cong Z_{24} \oplus Z_2 \oplus Z_3$  for  $p = 3$  (4).

(8) If  $p = 2$  (4) then  $E_{1,p}^1 \rightarrow E_{0,p}^1$  is zero. It follows now that  $E^2 = E^\infty$  and the next result is then immediate.

**THEOREM.** For  $\Gamma = SL_2(Z[\omega])$ ,

$$H_n(\Gamma) = \begin{cases} Z & n = 0 \\ Z_3 & n = 1 \text{ (4)} \\ Z_4 & n = 2 \text{ (4)} \\ Z_{24} \oplus Z_6 & n = 3 \text{ (4)} \\ 0 & n = 0 \text{ (4), } n \neq 0 \end{cases}$$

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