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**Autor:** Hausmann, Jean-Claude  
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## Open books and $h$ -cobordisms

JEAN-CLAUDE HAUSMANN

Let  $(W^{n+1}, M, N)$  be an  $h$ -cobordism between the closed manifolds  $M$  and  $N$ . Suppose that  $M$  is given with an open book decomposition with page  $V$  and monodromy  $\alpha$  (see section 2 for the definition). If  $n \geq 5$  and if the Whitehead torsion  $\tau^0(W, M) \in \text{Wh}(M)$  is in the image of  $\text{Wh}(\partial V) \rightarrow \text{Wh}(M)$ , our main result (Theorem 3.1) provides an open book decomposition for the manifold  $N$ , whose page  $T$  and monodromy  $\beta$  are closely related to  $V$  and  $\alpha$ . (In fact, one has  $\text{int } V \approx \text{int } T$ ). Although this resemblance between the two open book decomposition, the manifold  $N$  has, in general, not the simple homotopy type of an open book with page homeomorphic to  $V$  (Theorem 6.3).

It has been proven that any closed odd dimensional manifold  $M^n$  ( $n \geq 5$ ) has an open book decomposition with page  $V$  satisfying  $\pi_1(V) \cong \pi_1(M)$ . For such an open book decomposition, any  $h$ -cobordism will satisfy the hypothesis of our theorem, which can thus be applied, in theory, to any  $h$ -cobordism over any closed odd dimensional manifold.

Our other main results concern the set  $I_{\text{CAT}}(M)$  of torsions of CAT-inertial  $h$ -cobordisms from  $M$ , when the manifold  $M$  is an open book. (An  $h$ -cobordism  $(W, M, N)$  is CAT-inertial if  $N$  is CAT-homeomorphic to  $M$ ; CAT = DIFF or PL). This is done in Section 5. The results are the most complete when  $M$  is a stable open book (definition in Section 2). In this case,  $I_{\text{CAT}}(M)$  is strongly related to the set  $E_{\text{CAT}}(V)$  of torsion of CAT-tangential self-homotopy equivalences of  $V$  ( $V$  is the page of the stable open book  $M$ ). Some aspects of this relationship have already been noticed in [La 1 and 2], [H-L], and [Ha 1].

We apply these results in Section 6 to perform concrete computations on  $I_{\text{CAT}}(L_{p,q}^3 \times S^{2k})$ , where  $L_{p,q}^3$  is the 3-dimensional lens space and  $S^{2k}$  is the standard sphere. Among the interesting examples are two manifolds  $M$  and  $N$  which are  $h$ -cobordant but such that  $I_{\text{CAT}}(M) \neq \{0\}$  and  $I_{\text{CAT}}(N) = \{0\}$ . We also construct examples where  $\sigma \in I_{\text{CAT}}(M)$  but  $-\sigma \notin I_{\text{CAT}}(M)$ .

Section 1 contains the general definitions used throughout this paper. Section 2 is devoted to defining open books and proving the basic facts we need about them. Although this paper is practically self-contained with regard to techniques involving open books, we would like to mention References [Wi], [Qu] and [La 3].

In these papers, the problem of finding an open book decomposition on a given manifold of high dimension is solved. In addition the reader will find there several historical remarks about open books and the role they play in the theory of manifolds. Concerning the Whitehead torsion and  $h$ -cobordism theories, we assume that the reader has some familiarity with them. References are [Co], [Mi 2] and [Ke].

The abbreviation CAT stands for PL or DIFF (=smooth  $C^\infty$ ). Strictly speaking, the arguments we give work for  $CAT = PL$ . We leave to the reader the standard adjustments needed to obtain valid proofs in the smooth category. Most of our results are likely to be true for topological manifolds as well.

## 1. Definitions and notations

(1.1) As was said in the introduction, the abbreviation CAT means “PL or DIFF” where “DIFF” is used for “smooth  $C^\infty$ ”. The notation  $M \simeq_{CAT} N$  stands for:  $M$  is CAT-homeomorphic to  $N$ .

(1.2) BCAT is the classifying space BO if  $CAT = DIFF$  or BPL if  $CAT = PL$ .

(1.3) If  $V$  is a manifold,  $\partial V$  denotes its boundary. If a CAT-manifold  $M$  bounds CAT-manifolds  $A$  and  $B$ , we denote by  $A \cup_{\partial} B$  the manifold constructed by gluing  $A$  to  $B$  along their common boundary  $M$ , with the identity of  $M$  as gluing map.

(1.4) If  $K$  is a connected finite complex, we denote by  $Wh(K)$  the Whitehead group  $Wh(\pi_1(K))$  ([Co, Chapter III] or [Mi 2]). This coincides with the geometric definition of  $Wh(K)$  [Co, Chapter II, §6]. The abelian group  $Wh(K)$  has a  $\mathbf{Z}[\mathbf{Z}_2]$ -module structure generated by the involution  $\sigma \rightarrow \bar{\sigma}$  [Mi 2, p. 373].

(1.5) If  $V$  is a compact CAT-manifold,  $Wh(V)$  must be understood as the abelian group  $Wh(\pi_1(V))$  endowed with the  $\mathbf{Z}[\mathbf{Z}_2]$ -module structure for which we take into account the orientation character  $\omega_V: \pi_1(V) \rightarrow \mathbf{Z}_2$  in the definition of the involution  $\sigma \rightarrow \bar{\sigma}$  [Mi 2, p. 398].

(1.6) As in [Mi 2], the torsion  $\tau(f)$  of a homotopy equivalence  $f: X \rightarrow Y$  is measured in  $Wh(Y)$ . We also use the “source torsion”  $f_*^{-1}(\tau(f)) \in Wh(X)$  which simplifies the exposition in places and is denoted by  $\tau^0(f)$ . A homotopy equivalence is simple if  $\tau(f) = 0$  (or equivalently  $\tau^0(f) = 0$ ). Recall the composition formulas: for  $X \xrightarrow{f} Y \xrightarrow{g} Z$ :

$$\tau(g \circ f) = g_*(\tau(f)) + \tau(g)$$

$$\tau^0(g \circ f) = \tau^0(f) + f_*^{-1}(\tau^0(g)).$$

(1.7) Let  $X$  be a subcomplex of a finite complex  $Y$ . We denote by  $i_{X,Y} : \text{Wh}(X) \rightarrow \text{Wh}(Y)$  the homomorphism induced by the inclusion  $X \subset Y$ . If this inclusion is a homotopy equivalence, we denote by  $\tau(Y; X) \in \text{Wh}(Y)$  its torsion, and  $\tau^0(Y, X) = i_{X,Y}^{-1}(\tau(Y; X)) \in \text{Wh}(X)$ .

(1.8) A cobordism  $(W^{n+1}, M, N)$  (i.e. a compact CAT-manifold with  $\partial W = M \cup_{\partial} N$ ) is called an  $h$ -cobordism if both inclusions  $M \subset W$  and  $N \subset W$  are homotopy equivalence. An  $s$ -cobordism is an  $h$ -cobordism with  $\tau(W, M) = 0$  (and consequently with  $\tau(W, N) = 0$  by the duality formula [Mi 2, p. 395]). If  $n \geq 5$ , the  $s$ -cobordism theorem ([Ke] for  $\text{CAT} = \text{DIFF}$ , [Hn] for  $\text{CAT} = \text{PL}$ ) asserts that any CAT-homeomorphism  $h : M \times 0 \cup \partial M \times I \rightarrow M \cup \partial M \times I \subset \partial W$  extends to a CAT-homeomorphism  $H : M \times I \rightarrow W$ .

(1.9) Let  $V$  be a compact CAT-manifold. Define  $\text{Aut}_{\text{CAT}}(V \text{ rel } \partial V)$  as the group of CAT-homeomorphisms  $\alpha : V \rightarrow V$  such that the restriction of  $\alpha$  to a neighborhood of  $\partial V$  is the identity. The group structure is given by composition of automorphisms. We denote by  $\text{Aut}_{\text{CAT}}^c(V \text{ rel } \partial V)$  the group of concordance classes of elements of  $\text{Aut}_{\text{CAT}}(V \text{ rel } \partial V)$  (the concordances are relative to  $\partial V$ ). If  $\alpha \in \text{Aut}_{\text{CAT}}(V \text{ rel } \partial V)$ , we denote by  $\alpha^c$  its class in  $\text{Aut}_{\text{CAT}}^c(V \text{ rel } \partial V)$ .

## 2. Open books: definitions and basic facts

(2.1) Let  $V$  be a compact CAT-manifold of dimension  $n - 1$  with  $\partial V \neq \emptyset$ . Let  $\alpha \in \text{Aut}_{\text{CAT}}(V, \text{ rel } \partial V)$  (Definition (1.9)). Let  $\mathcal{M}_\alpha$  be the mapping torus of  $\alpha$ , i.e.  $\mathcal{M}_\alpha = V \times [-1, 1] / (x, -1) = (\alpha(x), 1)$ . The boundary  $\partial \mathcal{M}_\alpha$  admits a natural identification with  $\partial V \times S^1$ . The open book  $\text{OB}(V, \alpha)$  with page  $V$  and monodromy  $\alpha$  is the closed CAT-manifold of dimension  $n$  defined by:

$$\text{OB}(V, \alpha) = \mathcal{M}_\alpha \cup_{\partial} \partial V \times D^2.$$

To complete the terminology,  $\partial V$  is called the binding and  $\text{int } V$  the detached page of the open book. Observe that  $\alpha | \text{int } V$  is a CAT-automorphism of  $\text{int } V$  with compact support.

(2.2) Let  $I = [0, 1]$ . One has a canonical inclusion  $V \times I \subset \text{OB}(V, \alpha)$  which gives a decomposition  $\text{OB}(V, \alpha) = V \times I \cup_{\partial} W$  where  $W$  is the image of  $V \times [0, -1] \cup \partial V \times D^2$  in  $\text{OB}(V, \alpha)$ . Clearly,  $(W, V \times 0, V \times 1)$  is an  $s$ -cobordism.

(2.3) LEMMA. *Let  $\alpha$  and  $\beta \in \text{Aut}_{\text{CAT}}(V^{n-1} \text{ rel } \partial V)$ ,  $(n \geq 5)$ . Then  $\text{OB}(V, \alpha)$  and  $\text{OB}(V, \beta)$  are  $s$ -cobordant relative  $V \times I$  if and only if  $\alpha^c = \beta^c$  in  $\text{Aut}_{\text{CAT}}^c(V \text{ rel } \partial V)$ .*

*Proof.* Let  $\gamma: V \times I \rightarrow V \times I$  be a CAT-concordance relative  $\partial V$  from  $\alpha$  to  $\beta$ . Then the manifold  $R = \mathcal{M}_\gamma \cup \partial V \times I \times D^2$  is an  $s$ -cobordism rel  $V \times I$  from  $\text{OB}(V, \alpha)$  to  $\text{OB}(V, \beta)$ . Conversely, any such  $s$ -cobordism can be realized in this way, using the  $s$ -cobordism theorem.

(2.4) For a closed CAT-manifold  $M^n$ , a CAT-homeomorphism  $h: \text{OB}(V, \alpha) \rightarrow M$  is called an open book decomposition of  $M$ . Two open books decomposition  $h_i: \text{OB}(V, \alpha_i) \rightarrow M$  ( $i = 0, 1$ ) with page  $V$  are called concordant if there is an  $s$ -cobordism  $(R, \text{OB}(V, \alpha_0), \text{OB}(V, \alpha_1))$  relative to  $V \times I$  and a CAT-homeomorphism  $H: (R, \text{OB}(V, \alpha_0), \text{OB}(V, \alpha_1)) \rightarrow (M \times I, M \times 0, M \times 1)$  such that  $H|_{\text{OB}(V, \alpha_i)} = h_i$ . By (2.3), this implies that  $\alpha_0^c = \alpha_1^c$  in  $\text{Aut}_{\text{CAT}}^c(V, \text{rel } \partial V)$ .

(2.5) Let  $h: \text{OB}(V, \alpha) \rightarrow M$  be an open book decomposition. The restriction of  $h$  to  $V \times I$  defines a CAT-embedding  $\mu: V \times I \rightarrow M$  such that  $(M - \text{int } \mu(V \times I), \mu(V \times 0), \mu(V \times 1))$  is an  $s$ -cobordism. On the other hand, if  $\mu: V \times I \rightarrow M$  is such an embedding, then  $\mu|_{V \times 0}: V \times 0 \rightarrow M$  extends to a CAT-homeomorphism  $\bar{\mu}: V \times [0, -1] \rightarrow M - \text{int } \mu(V \times I)$  and  $\mu \cup \bar{\mu}$  essentially provides an open book decomposition  $h: \text{OB}(V, \alpha) \rightarrow M$  where  $\alpha$  is defined by  $(\alpha(x), 1) = \mu^{-1}(\bar{\mu}(x; -1))$ . Concordant open book decompositions give rise to concordant embeddings and conversely. This proves the following lemma.

**LEMMA.** *There is a bijection (given above) between concordance classes of open book decompositions with page  $V$  on a CAT-manifold  $M^n$  ( $n \geq 6$ ) and concordance classes of CAT-embeddings  $\mu: V \times I \rightarrow M$  such that  $(M - \text{int } \mu(V \times I), \mu(V \times 0), \mu(V \times 1))$  is an  $s$ -cobordism.*

In order to simplify our notations we usually identify  $V \times I$  with  $\mu(V \times I)$ . Thus, an open book decomposition of  $M^n$  is viewed as a decomposition  $M = V \times I \cup W$  where  $(W, V \times 0, V \times 1)$  is an  $s$ -cobordism. The use of this point of view can be seen at once in the proof of the following proposition.

(2.6) **PROPOSITION.** *Let  $M^n$  ( $n \geq 7$ ) be a closed CAT-manifold which has an open book decomposition with page  $V$ . Suppose that the homomorphism  $\pi_1(\partial V) \rightarrow \pi_1(V)$  induced by the inclusion is an isomorphism. Then, any manifold  $N^n$  is the same simple homotopy type as  $M$  has an open book decomposition with page  $T$ , where  $T$  (respectively  $\partial T$ ) has the same simple homotopy type as  $V$  (respectively  $\partial V$ ).*

*Proof.* Let  $f: N \rightarrow M$  be a simple homotopy equivalence. Using (2.5), write  $M = V \times I \cup_{\partial} W$ , where  $(W, V \times 0, V \times 1)$  is an  $s$ -cobordism. By Van Kampen Theorem, the condition  $\pi_1(\partial V) \simeq \pi_1(V)$  implies that the homomorphisms  $\pi_1(\partial(V \times I)) \rightarrow \pi_1(V \times I)$  and  $\pi_1(\partial(V \times I)) \rightarrow \pi_1(W)$  are isomorphisms. By [Wa 2,

Theorem 12.1], there is a decomposition  $N = U_0 \cup U_1$  and  $f$  is homotopic to  $g$ , where  $g$  is a simple homotopy equivalence of 4-ads:

$$g : (N; U_0, U_1, \partial U_1) \rightarrow (M; V \times I, W, \partial W).$$

Now  $\partial W = \partial(V \times I) \cong V \cup_{\partial} V$ . Using [Wa 2, Theorem 12.1] again we get a decomposition  $\partial U_0 = T_0 \cup_{\partial} T_1$  and  $g|_{\partial U_0}$  is homotopic to a simple equivalence  $g'$  of 4-ads:

$$g' : (\partial U_0; T_0, T_1, \partial T_0) \rightarrow (\partial(V \times I), V \times 0, V \times 1, \partial(V \times 0)).$$

By the homotopy extension property,  $g'$  can be extended to a map from  $N$  to  $M$  having the same properties as  $g$ . The cobordisms  $(U_0, T_0, T_1)$  and  $(U_1, T_0, T_1)$  are then  $s$ -cobordism so we can identify one of them with  $T_0 \times I$ . The result now follows from (2.5).

(2.7) If  $h : V \rightarrow V'$  is a CAT-homeomorphism, then  $h \times id : V \times I \rightarrow V' \times I$  extends to a CAT-homeomorphism  $H : OB(V, \alpha) \rightarrow OB(V', h \cdot \alpha \cdot h^{-1})$ .

(2.8) An open book  $L = OB(V, \alpha)$  is called a *stable open book* if  $V^{n-1}$  has a handle decomposition with handles of index  $\leq (n/2) - 1$ . We write  $M = SOB(V, \alpha)$ .

If  $M = SOB(V, \alpha)$ , then  $\pi_1(\partial V) = \pi_1(V)$  and so, by Proposition 2.6, a manifold of the same simple homotopy type as a stable open book is a stable open book. This is not true if we replace “simple homotopy type” by “homotopy type” [Ha 2], so the two words “simple” cannot be crossed out in Proposition 2.6. Stable open books play an important role in the understanding of manifolds without middle dimensional handles [Ha 2]. The problem of the existence and classification of stable open book structures on a manifold will be solved in this forthcoming paper [Ha 2].

### 3. Open book decomposition for manifolds $h$ -cobordant to $OB(V, \alpha)$

This section is devoted to the proof of Theorem 3.1 below and to some of its applications. We first need some definitions:

Let  $V^n$  be a CAT-manifold, with  $\partial V \neq \emptyset$ . Let  $\sigma \in Wh(\partial V)$ . A  $\sigma$ -enlargement of  $V$  is a manifold  $V_\sigma$  such that  $V \subset \text{int } V_\sigma$  and  $U = V_\sigma - \text{int } V$  is an  $h$ -cobordism from  $\partial V$  to  $\partial V_\sigma$  with  $\tau^0(U, \partial V) = \sigma$ .

For any  $\sigma$ -enlargement  $V_\sigma$  of  $V$ , one has a map  $e(V_\sigma) : \text{Aut}_{\text{CAT}}(V \text{ rel } \partial V) \rightarrow \text{Aut}_{\text{CAT}}(V_\sigma \text{ rel } \partial V_\sigma)$  defined by

$$e(V_\sigma)(\alpha) = \begin{cases} \alpha & \text{on } V \\ id & \text{elsewhere.} \end{cases}$$

Clearly  $e(V_\sigma)$  is a monomorphism of groups. It induces a homomorphism  $e^c(V_\sigma): \text{Aut}_{\text{CAT}}^c(V \text{ rel } \partial V) \rightarrow \text{Aut}_{\text{CAT}}^c(V_\sigma \text{ rel } \partial V_\sigma)$ . As  $V_\sigma - V$  is contained in a collar along  $\partial V_\sigma$ , the homomorphism  $e^c(V_\sigma)$  is an isomorphism.

(3.1) **THEOREM.** *Let  $M^n = \text{OB}(V, \alpha)$  with  $n \geq 5$ . Let  $(R, M, M')$  be an  $h$ -cobordism with  $\tau(R, M) = i_{\partial V, M}(\sigma)$  for  $\sigma \in \text{Wh}(\partial V)$ . Then, for any  $\sigma$ -enlargement  $V_\sigma$  of  $V$ , the manifold  $M'$  has an open book decomposition*

$$\text{OB}(V_\sigma, e(V_\sigma)(\alpha)) \xrightarrow{\cong} M'.$$

*Proof.* Let  $M = \mathcal{M}_\alpha \cup_{\partial} \partial V \times D^2$  be our open book decomposition. Let  $V_\sigma$  be a  $\sigma$ -enlargement of  $V$  and write  $W = V_\sigma - \text{int } V$ . The  $h$ -cobordism  $(R, M, M')$  with  $\tau^0(R, M) = i_{\partial V, M}(\sigma)$  is homeomorphic relative to  $M$  to  $M \times I \cup_{\partial V \times D^2} W \times D$ . Indeed, one checks by excision [Co, (20.3)] that the torsion of the latter is  $i_{\partial V, M}(\sigma)$ . Thus  $M' \simeq [\mathcal{M}_\alpha \cup_{\partial V \times S^1} W \times S^1] \cup_{\partial V_\sigma \times S^1} \partial V_\sigma \times D^2$ . Clearly  $\mathcal{M}_\alpha \cup_{\partial V \times S^1} W \times S^1 = \mathcal{M}_{e(V_\sigma)(\alpha)}$ ; thus  $M' \simeq \text{OB}(V_\sigma, e(V_\sigma)(\alpha))$ .

(3.2) **COROLLARY.** *Let  $M^n \simeq \text{OB}(V, \alpha)$  with  $n \geq 7$  and  $i_{\partial V, M}: \text{Wh}(\partial V) \rightarrow \text{Wh}(M)$  surjective. Then any manifold  $M'$  which is  $h$ -cobordant to  $M$  admits an open book decomposition  $M' \simeq \text{OB}(T, \beta)$  with  $\text{int } T$  CAT-homeomorphic to  $\text{int } V$ .*

*Proof.* This is a direct consequence of (3.1) together with the fact that  $h$ -cobordisms are characterized by their torsions [Mi 2, Theorem 11.3] and that  $V$  and  $V_\sigma$  have CAT-homeomorphic interior (an  $h$ -cobordism  $(U, P, Q)$  satisfies  $U - Q \simeq_{\text{CAT}} P \times [0, \infty)$ , [Ke, p. 41]).

(3.3) **Remarks.** The homomorphism  $i_{\partial V, M}$  is surjective in the following situations:

(1)  $i_{\partial V, V}: \text{Wh}(\partial V) \rightarrow \text{Wh}(V)$  is surjective and  $\pi_1(\alpha) = \text{id}$ . Indeed, one has  $\pi_1(M) = \pi_1(V) \times_{\pi_1(\alpha)} \langle t \rangle / \{t = 1\}$ . So, if  $\pi_1(\alpha) = \text{id}$ , one has  $\pi_1(V) \simeq \pi_1(M)$ .

(2)  $\pi_1(\partial V) \rightarrow \pi_1(V)$  is surjective and  $i_{\partial V, V}$  is surjective. Indeed, as  $\alpha \mid \partial V = \text{id}$ , the surjectivity of  $\pi_1(\partial V) \rightarrow \pi_1(V)$  implies  $\pi_1(\alpha) = \text{id}$  and we are in Case 1).

(3)  $\pi_1(\partial V) \rightarrow \pi_1(V)$  is split surjective. So we are in Case 2).

(4)  $V = V_0 \times I$ . This is a particular case of Case 3), since the inclusion  $V_0 \subset \partial(V_0 \times I) \subset V_0 \times I$  makes  $\pi_1(\partial V) \rightarrow \pi_1(V)$  split surjective.

(5) Another interesting situation for which  $i_{\partial V, V}$  is surjective is when  $\pi_1(V)$  is a finite group and  $\text{Im}(\pi_1(\partial V) \rightarrow \pi_1(V))$  contains a conjugate of any maximal hyperelementary subgroup  $H$  of  $\pi_1(V)$  such that  $\text{Wh}(H) \neq 0$  (see [St]). Example:  $\pi_1(V)$  is the alternating group  $A_5$  and  $\pi_1(\partial V)$  is the dihedral group  $D_{10}$ .

(3.4) COROLLARY. Let  $V^{n-1}$  be a compact CAT-manifold with  $n \geq 5$  and  $\partial V \neq \emptyset$ . Let  $\sigma_1$  and  $\sigma_2 \in \text{Wh}(\partial V)$  be such that  $i_{\partial V, V}(\sigma_1) = i_{\partial V, V}(\sigma_2)$  in  $\text{Wh}(V)$ . Then, for any  $\sigma_i$ -enlargement  $V_{\sigma_i}$  of  $V$  and any  $\alpha \in \text{Aut}_{\text{CAT}}(V \text{ rel } \partial V)$  one has a CAT-homeomorphism:

$$\text{OB}(V_{\sigma_1}, e(V_{\sigma_1})(\alpha)) \simeq_{\text{CAT}} \text{OB}(V_{\sigma_2}, e(V_{\sigma_2})(\alpha)).$$

*Proof.* Let  $M = \text{OB}(V, \alpha)$  and let  $(R, M, M')$  be an  $h$ -cobordism with  $\tau^0(R, M) = i_{\partial V, M}(\sigma_1) = i_{\partial V, M}(\sigma_2)$ . Then, by (4.1),  $\text{OB}(V_{\sigma_i}; e(V_{\sigma_i})(\alpha)) \simeq_{\text{CAT}} M$  for  $i = 1, 2$ .

(3.5) *Remark.* The condition  $i_{\partial V, V}(\sigma_1) = i_{\partial V, V}(\sigma_2)$  in (4.6) does not imply in general that  $V_{\sigma_1} \simeq_{\text{CAT}} V_{\sigma_2}$  ( $\partial V_{\sigma_1}$  could be not homeomorphic to  $\partial V_{\sigma_2}$ ).

#### 4. Pre-open book decompositions

The material of this section will be used in Section 5. It plays also an important role in the classification of manifolds without middle dimensional handles [Ha 2].

Let  $V^{n-1}$  be a compact CAT-manifold with  $\partial V \neq \emptyset$ . Let  $M^n$  be a closed manifold.

(4.1) DEFINITION. A decomposition  $M = V \times I \cup_{\partial} W$  is a pre-openbook decomposition of  $M$  if  $(W, V \times 0, V \times 1)$  is an  $h$ -cobordism.

By (2.5), an open book decomposition of  $M$  is a pre-open book decomposition.

Let  $M = V \times I \cup_{\partial} W$  be a pre-open book decomposition. Let  $\sigma \in \text{Wh}(\partial V)$  and let  $V_{\sigma}$  be a  $\sigma$ -enlargement (Definition in §3). i.e.  $V_{\sigma} = V \cup U$  where  $(U, \partial V, \partial V_{\sigma})$  is an  $h$ -cobordism with  $\tau^0(U, \partial V) = \sigma$ . Choose an embedding of  $U \times I$  into a collar of  $\partial V \times I$  in  $W$ . This gives a decomposition  $M = V_{\sigma} \times I \cup W_{\sigma}$  where  $W_{\sigma} = W - \text{int}(V_{\sigma} \times I)$ .

(4.2) LEMMA.  $M = V_{\sigma} \times I \cup W_{\sigma}$  is a pre-open book decomposition.

*Proof.* For  $i = 0, 1$ , one has the following diagram of inclusions:

$$\begin{array}{ccc} V \times \{i\} & \xrightarrow{j_1} & W \\ \downarrow j_2 & & \uparrow j_4 \\ V_{\sigma} \times \{i\} & \xrightarrow{j_3} & W_{\sigma} \end{array}$$

As  $j_1$ ,  $j_2$  and  $j_4$  are homotopy equivalences, so is  $j_3$ , which implies the lemma.

The torsions  $\tau(W, V \times 0)$  and  $\tau(W_\sigma, V_\sigma \times 0)$  are related by the following *main formula*:

(4.3) PROPOSITION.

$$i_{W_\sigma, W}(\tau(W_\sigma, V_\sigma \times 0)) = \tau(W, V \times 0) - i_{\partial V, W}(\sigma) + (-1)^n \overline{i_{\partial V, W}(\sigma)}.$$

*Proof.* First fix some notation. Let  $X \subset Y \subset Z$  be inclusions between finite subcomplexes of a complex  $N$ . If  $X \subset Y$  and  $Y \subset Z$  are homotopy equivalences, we define

$$\tau^N(Y, X) = i_{Y, N}(\tau(Y; X)) \in \text{Wh}(N).$$

The composition formula implies that

$$\tau^N(Z, X) = \tau^N(Y; X) + \tau^N(Z, Y)$$

so  $\tau^N$  behaves as a homomorphism.

The main formula is now equivalent to

$$\tau^W(W_\sigma, V_\sigma \times 0) = \tau(W, V \times 0) - \tau^W(U, \partial V) + \tau^W(U, \partial V_\sigma)$$

since  $(-1)^n \overline{\tau(U, \partial V)} = \tau(U, \partial V_\sigma)$  by the duality formula [Mi 2, p. 394]. The commutativity of the diagram in the proof of (4.1) and the fact that  $\tau^W$  is a homomorphism for the composition of inclusions between subspaces of  $W$  give:

$$\tau^W(W_\sigma, V_\sigma \times 0) = \tau(W, V \times 0) - \tau^W(V_\sigma, V) - \tau^W(W, W_\sigma).$$

By the excision principle for Whitehead torsion [Co, (20.3)], one has  $\tau^W(V_\sigma, V) = \tau^W(U, \partial V)$  and  $\tau^W(W, W_\sigma) = \tau^W(U \times I, Z)$  for  $Z = \partial(U \times I) - \text{int}(\partial V \times I)$ . The commutativity of the following diagram

$$\begin{array}{ccc} & U \times 1 \cup \partial V_\sigma \times I & \\ \swarrow & & \searrow \\ U \times I & \xleftarrow{\quad} & Z \end{array}$$

implies that  $\tau^W(U \times I, Z) = -\tau^W(Z, U \times 1 \cup \partial V_\sigma \times I)$ . By excision again,  $\tau^W(Z, U \times 1 \cup \partial V_\sigma \times I) = \tau^W(U \times 0, \partial V_\sigma \times 0)$ . This proves our formula.

## 5. Inertial $h$ -cobordisms for openbooks

An  $h$ -cobordism  $(W, M, N)$  is called CAT-inertial (or just “inertial”) if  $M$  is CAT-homeomorphic to  $N$ . Define  $I_{\text{CAT}}(M) \subset \text{Wh}(M)$  as the set of torsions  $\tau^0(W, M)$  of CAT-inertial  $h$ -cobordisms  $(W, M, N)$ . In this section we study the set  $I_{\text{CAT}}(M)$  when  $M$  is an open book. We need some definitions:

A homotopy equivalence  $f: A^n \rightarrow B^n$  between CAT-manifolds is called CAT-tangential (or, by abuse of language, just “tangential”) if there is an isomorphism of stable vector bundles over  $A$  between  $T^s A$  and  $f^*(T^s B)$ , where  $T^s$  denotes the stable tangent bundle. It is equivalent to say that  $f$  commutes up to homotopy with the classifying maps for  $T^s A$  and  $T^s B$ :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ T^s B \searrow & & \downarrow T^s A \\ & & \text{BCAT} \end{array}$$

Thus, a homotopy inverse of a tangential homotopy equivalence is a tangential homotopy equivalence.

Let  $K$  be a finite complex and let  $V$  be a compact CAT-manifold. The torsions  $\tau^0(f) \in \text{Wh}(K)$  of self-homotopy equivalences  $f: K \rightarrow K$  form a subset of  $\text{Wh}(K)$  which is denoted by  $E^0(K)$ . The set  $E_{\text{CAT}}^0(V)$  is the set of torsions  $\tau^0(f) \in \text{Wh}(V)$  of CAT-tangential self-homotopy equivalences  $f: V \rightarrow V$ . One has  $E_{\text{CAT}}^0(V) \subset E^0(V)$  and  $E^0(K) \simeq E_{\text{CAT}}^0(V^n(K))$  for  $V^n(K)$  a regular neighborhood of an embedding of  $K$  in  $R^n$ . (If  $V$  is parallelizable, then  $E^0(V) = E_{\text{CAT}}^0(V)$ .) The corresponding sets for  $\tau(f)$  are denoted by  $E(K)$  and  $E_{\text{CAT}}(V)$ .

For a subset  $X$  of  $\text{Wh}(V)$ , we define  $-X = \{\sigma \in \text{Wh}(V) \mid -\sigma \in X\}$  and  $\bar{X} = \{\sigma \in \text{Wh}(V) \mid \bar{\sigma} \in X\}$ . The precise relationship between  $E_{\text{CAT}}^0(V)$  and  $E_{\text{CAT}}(V)$  is as follows:

(5.1) LEMMA. *For any compact CAT-manifold  $V$  or finite complex  $K$ , one has:*

$$E_{\text{CAT}}^0(V) = -E_{\text{CAT}}(V) \quad \text{and} \quad E^0(K) = -E(K).$$

(5.2) Remark. In general,  $E_{\text{CAT}}^0(V) \neq E_{\text{CAT}}(V)$  and  $E^0(K) \neq E(K)$ , see (6.2).

*Proof.* Let  $f: K \rightarrow K$  be a self homotopy equivalence and  $\varphi: K \rightarrow K$  be a

homotopy inverse of  $f$ . One has

$$\begin{aligned} 0 &= \tau(f \circ \varphi) = \tau(f) + f_* \tau(\varphi) = \tau(f) + \varphi_*^{-1} \tau(\varphi) \\ &= \tau(f) + \tau^0(\varphi) \end{aligned}$$

whence  $-E^0(K) \subset E(K)$  and also  $-E(K) \subset E^0(K)$ . Thus  $E^0(K) = -E(K)$ . The same argument holds for  $E_{\text{CAT}}^0(V)$  and  $E_{\text{CAT}}(V)$ , since a homotopy inverse of a tangential homotopy equivalence is a tangential homotopy equivalence.

Let  $S_n(\text{Wh}(V))$  denote the subgroup of  $\text{Wh}(V)$  constituted by the elements  $\sigma$  with  $\sigma = (-1)^n \bar{\sigma}$  ( $n$ -symmetric elements of  $\text{Wh}(V)$ ). For  $\sigma \in \text{Wh}(V)$  we denote by  $[\sigma]_n$  its image in  $\text{Wh}(V)/S_n(\text{Wh}(V))$ . When  $X \subset \text{Wh}(V)$ , we denote by  $[X]_n$  the set  $\{[\sigma]_n \mid \sigma \in X\} \subset \text{Wh}(V)/S_n(\text{Wh}(V))$ . For instance,  $[\text{Wh}(V)]_n = \text{Wh}(V)/S_n(\text{Wh}(V))$ .

(5.3) PROPOSITION. *Let  $M^n = \text{SOB}(V, \alpha)$  with  $n \geq 5$  and let  $(R^{n+1}, M, M')$  be an  $h$ -cobordism. If  $M$  is CAT-homeomorphic to  $\text{SOB}(V, \beta)$  for some  $\beta$ , then*

$$[\tau^0(R, M)]_n \in [i_{V, M}(E_{\text{CAT}}^0(V))]_n \cap [i_{V, M}(\overline{E_{\text{CAT}}^0(V)})]_n.$$

*Proof.* A stable open book satisfies  $\pi_1(\partial V) \simeq \pi_1(V) \simeq \pi_1(M)$ . Let  $\sigma = i_{V, M}^{-1}(\tau^0(R, M)) \in \text{Wh}(V)$ . Let  $M \simeq V \times I \cup_{\partial} W$  be the open book decomposition under consideration viewed as in (2.5).

By the characterization of  $h$ -cobordisms by their torsion [Mi 2, Theorem 11.3] and the sum theorem for Whitehead torsion [Co, 23.1],  $R$  is CAT-homeomorphic to  $V \times I \times I \cup R_0$ , where  $V \times I \times I \cap R_0 = \partial(V \times I) \times I$  and  $(R_0, W, W')$  is an  $h$ -cobordism with  $\tau(R_0, W) = i_{V, R_0}(\sigma)$ . Hence  $M' \simeq V \times I \cup_{\partial} W'$  is a pre-open book decomposition (see §4). Let  $\rho : R_0 \rightarrow W'$  be a deformation retraction.

We want to compute  $\nu = \tau(W', V \times 0 \times 1)$ . Identifying  $V$  with  $V \times 0 \times 0$  or  $V \times 0 \times 1$  (both inclusions are isotopic in  $R_0$ ), one gets the following equation in  $\text{Wh}(R_0)$ :

$$i_{W', R_0}(\nu) + \tau(R_0, W') = \tau(R_0, W)$$

whence:

$$\begin{aligned} i_{W', R_0}(\nu) &= \tau(R_0, W) - \tau(R_0, W') \\ &= i_{V, R_0}(\sigma) + (-1)^{n+1} \overline{i_{V, R_0}(\sigma)}. \end{aligned}$$

As  $\rho_* \circ i_{W', R_0} = id_{\text{Wh}(W')}$ , and  $\rho_* \circ i_{V, R_0} = i_{V, W'}$ , we get

$$\omega = i_{V, W'}(\sigma) + (-1)^{n+1} i_{V, W'}(\bar{\sigma}). \tag{1}$$

Now use the fact that  $M' \simeq \text{SOB}(V, \beta)$  which corresponds to  $M' = (V \times I)_0 \cup W_0$ . Observe that  $W'$  has the homotopy type of  $V$  and  $\pi_1(\partial W') \simeq \pi_1(W')$ . Therefore, the classical procedure for eliminating handles [Ke] implies that  $W'$  has a handlebody decomposition with handles of index  $\leq (n/2) - 1$ . By general position and unicity of regular neighborhoods, one can move the decomposition  $M' = V \times I \cup W'$  by an ambient isotopy so that

$$V \times 0 \subset \text{int}(V \times 0)_0 \quad \text{and} \quad (V \times I)_0 = V \times I \cup U \times I$$

where  $U = (V \times 0)_0 - \text{int}(V \times 0)$ . By connexity of the inclusions  $V \times I \subset M'$  and  $(V \times I)_0 \subset M'$ , the inclusion  $j: V \times 0 \subset (V \times 0)_0$  is a homotopy equivalence, which is CAT-tangential since  $j$  is a CAT-embedding. Thus  $\eta = \tau^0(j) \in E_{\text{CAT}}^0(V)$ . Also, as  $\pi_1(\partial V) \cong \pi_1(V)$ , the region  $U$  is an  $h$ -cobordism. By excision [Co, (23.1)], one has  $i_{\partial V, V}^{-1}(\eta) = \tau^0(U, \partial(V \times 0))$ , so  $(V \times 0)_0$  is an  $i_{\partial V, V}^{-1}(\eta)$ -enlargement of  $V \times 0$ . As  $\tau(W_0, (V \times 0)_0) = 0$ , the main formula (4.3) together with formula (1) gives:

$$0 = i_{V, M'}(\sigma) + (-1)^{n+1} i_{V, M'}(\bar{\sigma}) - i_{V, M'}(\eta) + (-1)^n i_{V, M'}(\bar{\eta}). \quad (1)$$

As  $i_{V, M'}$  is an isomorphism, this leads us to

$$0 = \sigma + (-1)^{n+1} \bar{\sigma} - \eta + (-1)^n \bar{\eta}. \quad (2)$$

From formula (2) one deduces:

$$\sigma - \eta = (-1)^n (\bar{\sigma} - \bar{\eta}) \quad (3)$$

$$(-1)^{n+1} \bar{\sigma} - \eta = -\sigma - (-1)^n \bar{\eta}. \quad (4)$$

Formula (3) implies that  $[\sigma]_n = [\eta]_n$ . One checks in both cases ( $n$  odd and even) that formula (4) makes  $[\bar{\sigma}]_n = [\eta]_n$ . As  $\eta \in E_{\text{CAT}}^0(V)$ , Proposition (5.3) is proved.

Let  $\text{Em}_{\text{CAT}}(V)$  be the set of torsions  $\tau(F) \in \text{Wh}(V)$  for which  $F: V \rightarrow V$  is a homotopy equivalence which is a CAT-embedding of  $V$  into  $\text{int } V$ . Denote by  $\text{Em}_{\text{CAT}}^0(V)$  the set of torsions  $\tau^0(F) \in \text{Wh}(V)$  for such maps  $F$ . If  $F$  is such an embedding, then  $V = F(V) \cup U$  where  $(U, F(\partial V), \partial V)$  is an  $h$ -cobordism. Gluing to  $V$  an  $h$ -cobordism  $(U', \partial V, Z)$  inverse to  $U$ , one checks that  $F$  has a homotopy inverse  $\Phi$  which is also a CAT-embedding of  $V \rightarrow \text{int } V$ . Therefore, as in Lemma (5.1), one proves that  $\text{Em}_{\text{CAT}}^0(V) = -\text{Em}_{\text{CAT}}(V)$ .

(5.4) PROPOSITION. *Let  $M^n = \text{OB}(V, \text{id})$  with  $n \geq 5$ . Then*

$$i_{V, M}[\text{Em}_{\text{CAT}}^0(V \times D^2) \cup (-1)^{n+1} \overline{\text{Em}}_{\text{CAT}}^0(V \times D^2)] \subset I_{\text{CAT}}(M).$$

*Proof.* Let  $\sigma \in \text{Em}_{\text{CAT}}^0(V \times D^2)$  and let  $F: V \times D^2 \rightarrow \text{int}(V \times D^2)$  be a CAT-embedding which is a homotopy equivalence with  $\tau(F) = -\sigma$ . Recall that  $M = \partial(V \times D^2)$  and that the Van-Kampen Theorem applied to the decomposition  $M = V \times I \cup_{\partial} V \times I$  implies that  $\pi_1(M) \rightarrow \pi_1(V \times D^2)$  is an isomorphism. Therefore, the fact that  $\sigma \in I_{\text{CAT}}(M)$  and  $(-1)^{n+1}\bar{\sigma} \in I_{\text{CAT}}(M)$  comes from Lemma (5.5) and (5.6) below.

(5.5) LEMMA. *Let  $T^m$  be a compact CAT-manifold with  $m \geq 6$  and  $\pi_1(\partial T) \simeq \pi_1(T)$  an isomorphism. Let  $F: T \rightarrow \text{int } T$  be a CAT-embedding which is a homotopy equivalence with  $\tau(F) = i_{\partial T, T}(\sigma)$ , for  $\sigma \in \text{Wh}(\partial T)$ . Then any  $(-\sigma)$ -enlargement of  $T$  is CAT-homeomorphic to  $T$ . (In consequence,  $-\sigma \in I_{\text{CAT}}(\partial T)$ .)*

*Proof.* Write  $Z = F(\partial T)$  and  $U = T - \text{int } F(T)$ . As  $\pi_1(\partial T) \simeq \pi_1(T)$ , the cobordism  $(U, \partial T, Z)$  is an  $h$ -cobordism. Let  $(U', \partial T, Z')$  be an inverse  $h$ -cobordism for  $U$ , i.e.  $(U \cup U', Z) \simeq_{\text{CAT}} (Z \times I, Z \times 0)$ . Define  $N = T \cup_{\partial T} U'$ .

The excision principle [Co, (20.3)] gives

$$\tau^N(U', \partial T) = \tau^N(N, T)$$

(see proof of (4.3) for the definition of  $\tau^N$ ), and

$$\tau^N(U, Z) = \tau^N(T, F(T)) = i_{T, N}(\tau(F)).$$

One also has

$$\tau^N(U', \partial T) + \tau^N(U, Z) = \tau^N(U \cup U', Z) = 0$$

whence  $\tau^N(U', \partial T) = -i_{T, N}(\tau(F))$ . Thus  $\tau^0(U', \partial T) = -\sigma$ . By construction,  $N$  is a  $(-\sigma)$ -enlargement of  $T$  and is CAT-homeomorphic to  $T$ . As any two  $(-\sigma)$ -enlargements of  $T$  are CAT-homeomorphic, this proves Lemma (5.5).

(5.6) LEMMA. *Let  $M^n$  be a compact manifold with  $n \geq 5$ . Then  $(-1)^{n+1}\overline{I_{\text{CAT}}(M)} = I_{\text{CAT}}(M)$ .*

*Proof.* Let  $\eta \in I_{\text{CAT}}(M)$ . Let  $(W, M, N)$  and  $(\bar{W}, M, \bar{N})$  be  $h$ -cobordisms with  $\tau^0(W, M) = \sigma$  and  $\tau^0(\bar{W}, M) = (-1)^{n+1}\bar{\sigma}$ . One checks that  $(W \cup_M \bar{W}, N, \bar{N})$  is an  $s$ -cobordism (use excision [Co, (23.1)] and the duality formula [Mi, p. 394]). Therefore  $N = \bar{N}$  and  $N \simeq M$  by hypothesis. Thus  $(-1)^{n+1}\overline{I_{\text{CAT}}(M)} \subset I_{\text{CAT}}(M)$ . The reverse inclusion is obtained by symmetry.

Observe that  $\text{Em}_{\text{CAT}}^0(V \times D^2) \subset E_{\text{CAT}}^0(V \times D^2) = E_{\text{CAT}}^0(V)$ . If  $V$  has a handle decomposition with handles of index  $\leq n/2$ , then  $E_{\text{CAT}}^0(V) \subset \text{Em}_{\text{CAT}}^0(V \times D^2)$ .

Indeed,  $V \times D^2$  is then a stable thickening [Wa 1, §5] and the proof of Proposition 5.1 of [Wa 1] shows that a CAT-tangential homotopy equivalence  $f: V \times D^2 \rightarrow V \times D^2$  is homotopic to a CAT-embedding. Therefore, Propositions (5.3) and (5.4) imply the following theorem:

(5.7) THEOREM. *Let  $M^n = \text{SOB}(V, \text{id})$  with  $n \geq 5$ . Then*

$$[I_{\text{CAT}}(M)]_n = [i_{V,M} E_{\text{CAT}}^0(V)]_n.$$

Moreover, if  $n = 2k$ , then

$$[I_{\text{CAT}}(M)]_n = -[I_{\text{CAT}}(M)]_n.$$

(5.8) *Remark.* Theorem (5.7) is of special interest when  $S_n(\text{Wh}(M))$  is small. For instance, if  $n = 2k + 1$ ,  $\pi_1(M)$  is finite abelian and  $\omega_1(M) = 0$ ,  $S_n(\text{Wh}(M))$  is the subgroup of  $\text{Wh}(M)$  of the elements of order 2. Also, if  $x = \bar{x}$  and  $\text{Wh}(M)$  has no 2-torsion, then Theorem (5.7) says that  $I_{\text{CAT}}(M) = i_{V,M}(E_{\text{CAT}}^0(V))$  ( $n = 2k + 1$ ). This happens in the following cases:

- (1)  $M$  is orientable,  $\pi_1(M)$  is finite abelian with cyclic 2-torsion.
- (2)  $M$  orientable,  $\pi_1(M)$  is a dihedral group, or an alternating group  $A_r$  ( $r \leq 6$ ) or the binary icosahedral group [St].

In our applications we shall use the following corollary of Theorem (5.7):

(5.9) COROLLARY. *Let  $N^n$  be a closed CAT-manifold. If  $k \geq n$  and  $n + k \geq 5$ , one has*

$$[I_{\text{CAT}}(N \times S^k)]_{n+k} = [E_{\text{CAT}}^0(N)]_{n+k}$$

In particular, if  $N$  is parallelizable, one has  $[I_{\text{CAT}}(N \times S^k)]_{n+k} = [E^0(N)]_{n+k}$ .

*Proof.* As  $N \times S^k = \text{SOB}(N \times D^{k-1}, \text{id})$ , this follows from Theorem (5.7).

Finally, as a consequence of the techniques of this section, we have the following result:

(5.10) PROPOSITION. *Let  $M_0^n$  and  $M_1^n$  be two closed CAT-manifolds, and let  $k$  be an integer  $\geq n + 2$ . Suppose that  $S_{n+k}(\text{Wh}(M)) = 0$ . Then  $M_0 \times S^k$  is CAT-homeomorphic to  $M_1 \times S^k$  if and only if  $M_0$  and  $M_1$  have the same CAT-tangential simple homotopy type.*

*Proof.* Write  $V_i = M_i \times D^{k-1}$ , so  $M \times S^k = \text{SOB}(V_i, \text{id})$  if  $k \geq n + 2$ . Let

$h: M_0 \times S^k \rightarrow M_1 \times S^k$  be a CAT-homeomorphism. By the general position and connectivity arguments of the proof of (5.3), one may assume that  $h(V_0 \times I) \subset \text{int}(V_1 \times I)$  and that  $h_0 = h|_{V_0 \times I}$  is a tangential homotopy equivalence from  $V_0 \times I$  to  $V_1 \times I$ .

Let  $F = h_0 \times id: V_0 \times I \times I \rightarrow (V_1 \times I \times I)$ . As in the proof of Lemma (5.5), this embedding  $F$  gives rise to an  $h$ -cobordism  $(W, M_1 \times S^k, Z)$  with  $\tau^0(W, M) = -i_{M_1, M_2 \times S^k}(\tau(F))$  and  $Z \simeq_{\text{CAT}} F(M_0 \times S^k) \simeq_{\text{CAT}} M_0 \times S^k$ . Thus  $W$  is an inertial cobordism and by Proposition (6.3),  $-\tau(F) \in E_{\text{CAT}}^0(M_1)$ . Let  $\varphi: M_2 \rightarrow M_1$  be a CAT-tangential homotopy equivalence such that  $\tau^0(\varphi) = -\tau(F)$ . Then:

$$\tau(\varphi \cdot F) = \tau(\varphi) + \varphi_*(\tau(F)) = \tau(\varphi) - \varphi_*(\tau^0(\varphi)) = 0.$$

Hence  $\varphi \circ F: M_0 \rightarrow M_1$  is a CAT-tangential simple homotopy equivalence.

To prove the converse, observe that a CAT-tangential homotopy equivalence  $f: M_0 \rightarrow M_1$  produces a CAT-embedding  $F: M_0 \times D^{k+1} \rightarrow M_1 \times D^{k+1}$  for  $k+1 \geq n+1$ . If  $F$  is a simple homotopy equivalence, the manifold  $M_1 \times D^{k+1} - \text{int } F(M_0 \times D^{k+1})$  is an  $s$ -cobordism from  $F(M_0 \times S^k)$  to  $M_1 \times S^k$ . The result follows from the  $s$ -cobordism theorem.

(5.11) *Remark.* As we have just seen,  $M_0 \times S^k$  and  $M_1 \times S^k$  are  $h$ -cobordant if  $M_0$  and  $M_1$  have the same tangential homotopy type and  $k \geq n$ . But, under the assumptions of Proposition (5.10),  $M_0$  and  $M_1$  need to have the same *simple* tangential homotopy type in order for  $M_0 \times S^k$  and  $M_1 \times S^k$  to be homeomorphic. This generalizes the technique of [Mi 1] to get non-inertial  $h$ -cobordisms.

## 6. Applications and examples

Let  $L_{p,q}$  be the 3-dimensional lens space for  $p$  prime  $\neq 2$  as it is defined classically [Co, §27]. Its fundamental group is  $C_p$ , the cyclic of order  $p$  with generator  $t$ . Let  $S^n$  denote the standard sphere.

(6.1) **THEOREM.** *For  $2k \geq 6$  and  $\text{CAT} = \text{DIFF}$  or  $\text{PL}$ , one has:*

- (1)  $I_{\text{CAT}}(L_{p,q} \times S^{2k})$  is either  $\{0\}$  or  $\{0, \sigma\}$  for some  $\sigma \in \text{Wh}(L_{p,q})$ .
- (2)  $I_{\text{CAT}}(L_{p,q} \times S^{2k}) = \{0\}$  if  $p \equiv 3 \pmod{4}$
- (3)  $I_{\text{CAT}}(L_{p,1} \times S^{2k}) \neq \{0\}$  if  $p \equiv 1 \pmod{4}$ .

*Proof.* As  $L_{p,q}$  is parallelizable, one has  $[I_{\text{CAT}}(L_{p,q} \times S^{2k})]_{2k+3} = [E^0(L_{p,q})]_{2k+3}$  by Theorem (5.7). But  $S_{\text{odd}}(\text{Wh}(L_{p,q})) = 0$  by [Ba, (4.2)]. Thus  $I_{\text{CAT}}(L_{p,q} \times S^{2k}) = E^0(L_{p,q})$  and it suffices to prove 1), 2) and 3) for  $E^0(L_{p,q})$ .

The set of homotopy classes of self-homotopy equivalence of  $L_{p,q}$  is a finite set which is in bijection with  $\{a \in N \mid 0 < a < p \text{ and } a^2 \equiv \pm 1 \pmod{p}\}$  [Co, (29.6)]. The bijection sends  $f: L_{p,q} \rightarrow L_{p,q}$  to  $a(f)$  such that  $\pi_1(f)(t) = t^{a(f)}$ .

If  $a(f) = \pm 1$ , the homomorphism  $\text{Wh}(f)$  is the identity (it sends  $x \rightarrow x$  or  $x \rightarrow \bar{x}$ , but  $\bar{x} = x$  [Ba, Proposition 4.2]). This implies that the set  $\{\tau(f) \mid a(f) = \pm 1\}$  is a subgroup of  $\text{Wh}(L_{p,q})$ . But any finite subgroup of  $\text{Wh}(L_{p,q})$  is trivial [Ba, Proposition 4.2]. Thus  $\tau(f) = 0$  if  $a(f) = \pm 1$ , which proves Point 2).

If  $-1 \equiv b^2 \pmod{p}$ , for  $0 < b < p$ , write  $f_+, f_-: L_{p,q} \rightarrow L_{p,q}$  the self-homotopy equivalence with  $a(f_{\pm}) = \pm b$ . Then  $a(f_+ \cdot f_-) = 1$  and  $a(f_+ \cdot f_+) = -1$ . Thus:

$$\begin{cases} \tau(f_+) + (f_+)_* \tau(f_-) = 0 \\ \tau(f_+) + (f_+)_* \tau(f_+) = 0 \end{cases}$$

which implies  $\tau(f_+) = \tau(f_-)$ . Point 1) is thus proved.

Point (3) is proved using the special torsion  $\Delta(L_{p,1}) \in \mathbb{Q}C_p$  defined in [Mi 2, §13]. One has  $\Delta(L_{p,1}) = (t-1)^2$ . Let  $f: L_{p,1} \rightarrow L_{p,1}$  a self-homotopy equivalence with  $a(f) = b \neq \pm 1$ . One checks easily that there is no integer  $k$  satisfying

$$f_*(\Delta(L_{p,1})) = (t^b - 1)^2 = \pm t^k (t-1)^2 = \pm t^k \Delta(L_{p,1}).$$

This implies that  $\tau(f) \neq 0$  by [Mi 2, Lemma 12.5].

(6.2) *Remark.* As  $\text{Wh}(L_{p,q})$  has no torsion, it follows that  $I_{\text{CAT}}(L_{p,1} \times S^{2k})$  is not a subgroup of  $\text{Wh}(L_{p,q})$ . Also, if  $0 \neq \sigma \in I_{\text{CAT}}(L_{p,q} \times S^{2k})$ , then  $-\sigma \notin I_{\text{CAT}}(L_{p,q} \times S^{2k})$ . Also  $E(L_{p,1}) \neq E^0(L_{p,1})$ .

(6.3) **THEOREM.** *Let  $M = \text{SOB}(V, \alpha)$ , with  $V = L_{p,q} \times D^{2k-1}$ ,  $k \geq 3$ . Then.  $I_{\text{CAT}}(M)$  contains at most 2 elements. If  $(W, M, N)$  is an  $h$ -cobordism with  $\tau^0(W, M) \notin I_{\text{ACT}}(M)$ , then  $N$  has not the same simple homotopy type as an open book with page homeomorphic to  $V$ .*

*Proof.* By Proposition 5.3 and the fact that  $S_{\text{odd}} \text{Wh}(L_{p,q}) = 0$  [Ba, Proposition 4.2], one has  $I_{\text{CAT}}(M) \subset E^0(L_{p,q})$ . The first assertion thus follows from (6.1) and its proof. For the last affirmation, let us suppose that  $N$  has the same simple homotopy type as  $\text{OB}(V, \beta)$ . By Proposition (2.6)  $N$  is CAT-homeomorphic to  $\text{OB}(T, \gamma)$  with  $(T; \partial T)$  of the same simple homotopy type as  $(V; \partial V)$ . Observe that  $[L_{p,q}; \text{BCAT}] = \{\text{constant map}\}$ , by obstruction theory and the fact that  $\pi_i(\text{BCAT}) = \mathbb{Z}_2$  for  $i \leq 2$  and  $\pi_3(\text{BCAT}) = 0$ . Therefore,  $T$  is a trivial thickening of  $L_{p,q}$  [Wa 1, Proposition 5.1] and is homeomorphic to  $V$ . Thus  $M \simeq \text{SOB}(V, \gamma)$  and, by Proposition (5.3) and the discussion above concerning  $S_{\text{odd}}(\text{Wh}(L_{p,q}))$ , one has  $\tau^0(W, M) \in E_{\text{CAT}}^0(L_{p,q}) = I_{\text{CAT}}(M)$ .

(6.4) *Remark.* Theorem (6.3) contrasts with Corollary (3.2), which asserts that  $M'$  has an open book decomposition with detached page  $\text{int } V$ . This shows that the data of the detached page and the restriction of the monodromy to it does not even determine the simple homotopy type of  $\text{OB}(V, \gamma)$ .

The following lemma proves that the condition  $q = 1$  is essential in Theorem (6.1):

(6.5) **LEMMA.** *For  $\text{CAT} = \text{DIFF}$  or  $\text{PL}$  and  $k \geq 3$ , one has  $I_{\text{CAT}}(L_{5,2} \times S^{2k}) = \{0\}$*

*Proof.* (One uses the results and notations of the proof of (6.1)). It suffices to show that  $E^0(L_{5,2}) = \{0\}$ . This can be done by using the Milnor's special torsion  $\Delta(L_{5,2}) = (t-1)(t^2-1) \in \mathbb{Q}C_p$ . The only possible non-zero element of  $E^0(L_{5,2})$  would occur as  $\tau(g)$  for the self-homotopy equivalence  $g$  of  $L_{5,2}$  characterized by  $a(g) = 2$ . But one checks that  $g_*(\Delta(L_{5,2})) = -t^4 \Delta(L_{5,2})$ , which implies that  $\tau(g) = 0$  by [Mi 2, Lemma 12.5].

The situation of Lemma (6.5) gives rise to an interesting phenomenon:

(6.6) **THEOREM.** *Let  $p \equiv 1 \pmod{4}$  and  $q$  such that  $I_{\text{CAT}}(L_{p,q} \times S^{2k}) = \{0\}$  ( $k \geq 3$ ). Then, there exists a manifold  $N$  which is  $h$ -cobordant to  $L_{p,q} \times S^{2k}$  such that  $I_{\text{CAT}}(N) \neq \{0\}$ .*

*Proof.* Again, one uses throughout this proof the results and the notations of the proof of (6.1). The non-zero torsion of  $E^0(L_{p,1})$  is realized by the self-homotopy equivalence  $g$  of  $L_{p,1}$  characterized by  $a(g) = b$ , for  $b$  a square root of  $-1$  in  $\mathbb{Z}/p\mathbb{Z}$ . Thus, the homomorphism  $g_*$  induced on  $\text{Wh}(L_{p,1})$  is different from the identity. Choose  $\sigma \in \text{Wh}(L_{p,q}) = \text{Wh}(L_{p,1})$  such that  $g_*^{-1}(\sigma) \neq \sigma$ .

On the other hand, the self-homotopy equivalence  $f$  of  $L_{p,q}$  characterized by  $a(f) = b$ , although satisfying  $f_* = g_*$ , is simple, since  $E^0(L_{p,q}) = I_{\text{CAT}}(L_{p,q} \times S^{2k}) = 0$ .

By [Co, 20.7 and 8.5], there exists a finite complex  $X$  obtained by attaching 2 and 3 cells to  $L_{p,q}$  such that the inclusion  $i : L_{p,q} \subset X$  is a homotopy equivalence satisfying  $\tau^0(i) = \sigma$ . Let  $r : X \rightarrow L_{p,q}$  be a homotopy inverse to  $i$ . Then  $i \circ f \circ r$  is a self-homotopy equivalence of  $X$ . As  $\tau^0(f) = 0$ , one has:

$$\begin{aligned} \tau^0(i \circ f \circ r) &= \tau^0(r) + r_*^{-1} f_*^{-1}(\tau^0(i)) = -r_*^{-1}(\tau^0(i)) + r_*^{-1} f_*^{-1}(\tau^0(i)) \\ &= r_*^{-1}(-\sigma + f_*^{-1}(\sigma)) \neq 0. \end{aligned}$$

Now let  $V_X$  be a regular neighborhood of  $X$  embedded in  $\mathbb{R}^{2k+2}$ . Then  $V_X$  is an enlargement of  $L_{p,q} \times D^{2k-1}$ . Thus  $L_{p,q} \times S^{2k} = \text{SOB}(L_{p,q} \times D^{2k-1}, \text{id})$  is  $h$ -cobordant to  $\text{SOB}(V_X, \text{id})$  by Theorem (3.1), and  $I_{\text{CAT}}(\text{SOB}(V_X, \text{id})) = E^0(X) \neq 0$ .

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*The Institute for Advanced Study  
Princeton, USA*

*University of Geneva  
Switzerland*

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