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# The first k-invariant, Quillen's space $BG^+$ and the construction of Kan and Thurston

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## 1. Outline

In this paper we shall use the interpretation of group cohomology in terms of crossed $n$-fold extensions [7].

Let $Y$ be a connected $CW$-complex with a single 0-cell; we assume $\pi_k(Y)=0$ for $1<k<n$, $n\geqq 2$ (nothing is assumed for $n=2$). Then we have an exact sequence

$$e_Y : 0 \longrightarrow \pi_n(Y) \xrightarrow{\partial_n} \pi_n(Y, Y^{n-1}) \xrightarrow{\partial_{n-1}} \pi_{n-1}(Y^{n-1}, Y^{n-2}) \xrightarrow{\partial_{n-2}}$$

$$\cdots \xrightarrow{\partial_2} \pi_2(Y^2, Y^1) \xrightarrow{\partial_1} \pi_1(Y^1) \longrightarrow \pi_1(Y) \longrightarrow 1$$

where each $\partial_j$ is obtained from the exact homotopy sequences of the corresponding pairs of spaces in the obvious way. Now the action of $\pi_1(Y^1)$ on $\pi_2(Y^2, Y^1)$ (on $\pi_2(Y, Y^1)$ in case $n=2$) and that of $\pi_1(Y)$ on the remaining groups turn $e_Y$ into a crossed $n$-fold extension [7§3] (that the relative $\pi_2$ is a crossed $\pi_1(Y^1)$-module is due to J. H. C. Whitehead, see [6 p. 39]). In view of the main Theorem of [7§7], $e_Y$ represents a class $[e_Y]\in H^{n+1}(\pi_1(Y), \pi_n(Y))$.

THEOREM 1. *The class $[e_Y]$ is the first (non-trivial) k-invariant of $Y$.*

This offers an answer to a question in [2 p. 301]. Furthermore, we obtain, as a consequence, a description of the n'th cohomology group which recovers the description in [5 p. 75] (Theorem 14.1′):

COROLLARY. *Let $A$ be a $\pi_1(Y)$-module, and let the crossed $n$-fold extension*

$$0\to\pi_n(Y)\to C_{n-1}\to\cdots\to C_1\to F\to\pi_1(Y)\to 1$$

*represent $[e_Y]$, with $C_{n-2},\ldots, C_1$, $F$ free ($e_Y$ itself is such a representative). Then* *the group* $H^n(Y, A)$ (*local coefficients*) *is the cokernel of the induced map* $\mathrm{Hom}_G(C_{n-2}, A) \to \mathrm{Hom}_G(C_{n-1}, A)$ $(C_0 = F)$; *here* $G = \pi_1(Y)$ *resp.* $G = F$ (*in low dimensions*), *and* $\mathrm{Hom}_G(F, A) = \mathrm{Der}(F, A)$.

We shall use Theorem 1 to determine the first $k$-invariant of Quillen's $(\ )^+$-construction: Let $G$ be a group, and let $E$ be a perfect normal subgroup. Now $E$ has a *universal central extension* [11 p. 43]

$$e: 0 \to H_2(E) \to X \to E \to 1.$$

Using the universal property of $e$, we extend the action of $G$ on $E$ to a unique action of $G$ on $X$, turning $X$ into a crossed $G$-module, whence we obtain a crossed 2-fold extension

$$e_{(G,E)}: 0 \longrightarrow H_2(E) \longrightarrow X \xrightarrow{\partial} G \longrightarrow Q \longrightarrow 1$$

where $Q = G/E$. In view of the main Theorem in [7§7], $e_{(G,E)}$ represents a class $[e_{(G,E)}] \in H^3(Q, H_2(E)) = H^3(\pi_1(BG^+), \pi_2(BG^+))$. Here $BG^+$ is Quillen's space [10].

THEOREM 2. *The class* $[e_{(G,E)}]$ *is the first k-invariant of* $BG^+$.

By a recent result of Kan and Thurston [8], to any connected space $Y$ there may be associated a group $G$ together with a map $\chi: BG \to Y$ such that (i) the kernel $E$ of the induced map $G \to \pi_1(Y)$ is perfect, and (ii) the map $\chi$ extends to a (possibly weak) $h$-equivalence $BG^+ \to Y$. Theorem 2 shows how the first $k$-invariant of $Y$ is determined by $G$ and $E$. Note that $[e_{(G,E)}]$ may be non-zero. As to the vanishing of $[e_{(G,E)}]$ we have

THEOREM 3. *The class* $[e_{(G,E)}]$ *is zero, and* $Q$ (*resp.* $G$) *acts trivially on* $H_2(E)$ *if and only if the induced map* $H_2(E) \to H_2(G)$ *is a split injection* (*i.e. admits a left inverse*).

A particular example arises in algebraic $K$-theory: If $\Lambda$ is a ring with unit, in view of the above, the class of

$$e_\Lambda: 0 \to K_2(\Lambda) \to St(\Lambda) \to GL(\Lambda) \to K_1(\Lambda) \to 1$$

is the first $k$-invariant of $\mathrm{BGL}(\Lambda)^+$. For commutative rings, this was also announced in [3].

THEOREM 4. *The class* $[e_\Lambda] \in H^3(K_1(\Lambda), K_2(\Lambda))$ *is zero.*

I am indebted to K. Dennis who provided me with the crucial argument for the proof of Theorem 4. In fact, he has shown that the induced map $H_2(E(\Lambda)) \to H_2(GL(\Lambda))$ is a split injection [4 Cor. 8]. Hence Theorem 4 is a consequence of Theorem 3. I am also grateful to K. Brown; he read an earlier version of the paper and conjectured Theorem 3.

## 2. Proofs

*Proof of Theorem* 1. It is known [1] that the first (non-trivial) $k$-invariant of $Y$ is the Eilenberg- Mac Lane invariant $l_Y \in H^{n+1}(\pi_1(Y), \pi_n(Y))$ [5]. If $\mathbf{C}$ is the crossed standard resolution of $\pi_1(Y)$ [7§9], we may lift the identity map of $\pi_1(Y)$ to

$$\begin{array}{ccccccccccc}
\mathbf{C}: \cdots \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow \cdots \longrightarrow & F & \longrightarrow & \pi_1(Y) & \longrightarrow 1 \\
& \downarrow{\scriptstyle \xi} & & \downarrow & & \downarrow & & \| & \\
e_Y: 0 \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(Y, Y^{n-1}) & \longrightarrow \cdots \longrightarrow & \pi_1(Y^1) & \longrightarrow & \pi_1(Y) & \longrightarrow 1.
\end{array}$$

Now $\xi$ is Eilenberg–Mac Lane's cocycle. This, together with the main Theorem in [7 §7] proves Theorem 1.

*Proof of Theorem* 2. In view of Theorem 1, the Eilenberg–Mac Lane class resp. the first $k$-invariant $l_{BG^+}$ is represented by

$$\hat{e}: O \to \pi_2(BG^+) \to \pi_2(BG^+, (BG^+)^1) \to \pi_1((BG^+)^1) \to \pi_1(BG^+) \to 1.$$

But $\hat{e}$ is clearly equivalent to

$$e^+: 0 \to \pi_2(BG^+) \to \pi_2(BG^+, BG) \to \pi_1(BG) \to \pi_1(BG^+) \to 1,$$

since there is an obvious morphism $(1, \ldots, 1): \hat{e} \to e^+$ of crossed 2-fold extensions. We complete the proof by showing that $\hat{e}$ and $e_{(G,E)}$ are essentially the same crossed 2-fold extension:

The group $\pi_2(BG^+, BG)$ is $\pi_1(Z)$, $Z$ the homotopy fibre of $BG \to BG^+$. Since $Z$ is acyclic (this is just the universal property of the $(\ )^+$-construction), $H_1(\pi_1(Z)) = 0 = H_2(\pi_1(Z))$. It follows from Lemma 2 in [9] (p. 215) that

$$0 \to \pi_2(BG^+) \to \pi_2(BG^+, BG) \to E \to 1$$

is the universal central extension of $E$.

*Proof of Theorem* 3. By the Theorem in [7 §10], $[e_{(G, E)}]$ is zero if and only if there is a group extension $1 \to X \xrightarrow{i} D \to Q \to 1$ together with a morphism $(1, \alpha) : (X, D, j) \to (X, G, \partial)$ of crossed modules inducing the identity map of $Q$. It follows that $[e_{(G, E)}]$ is zero if and only if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(E) & \longrightarrow & X & \longrightarrow & E & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & H_2(E) & \longrightarrow & D & \longrightarrow & G & \longrightarrow & 1, \end{array} \qquad (*)$$

such that conjugation in $D$ induces the crossed $G$-structure on $X$; here $i$ denotes the inclusion $E \subset G$.

It is clear that $Q$ (resp. $G$) acts trivially on $H_2(E)$ if $H_2(E) \to H_2(G)$ is a split injection. Hence we may assume that $Q$ acts trivially on $H_2(E)$. Consider now the commutative diagram

$$\begin{array}{ccc} H^2(G, H_2(E)) & \longrightarrow & \mathrm{Hom}\,(H_2(G), H_2(E)) \\ \downarrow & & \downarrow \\ H^2(E, H_2(E)) & \longrightarrow & \mathrm{Hom}\,(H_2(E), H_2(E)) \end{array} \qquad (**)$$

with the obvious maps. Inspection of (**) shows that we have a diagram (*) if and only if $H_2(E) \to H_2(G)$ is a split injection. Hence the condition of the Theorem is necessary. Now, if we have a diagram (*), conjugation in $D$ induces an action of $G$ on $X$; since this one extends the action of $G$ on $E$ given by conjugation, it agrees with that one we used to define the crossed $G$-structure on $X$, as there is only one such $G$-action on $X$. This last fact is a consequence of the universal property of the universal central extension. It follows that the condition of the Theorem is also sufficient.

*Remark.* The same arguments, applied to the Kan–Thurston construction, may be used to prove the following classical result (which seems to be folk-lore):

THEOREM. *For a given connected space $Y$, the Hurewicz map $\pi_2(Y) \to H_2(Y)$ is a split injection if and only if $\pi_1(Y)$ acts trivially on $\pi_2(Y)$ and if the first $k$-invariant of $Y$ is zero.*

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