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# The first $k$ -invariant, Quillen’s space $BG^+$ and the construction of Kan and Thurston

JOHANNES HUEBSCHMANN

## 1. Outline

In this paper we shall use the interpretation of group cohomology in terms of crossed  $n$ -fold extensions [7].

Let  $Y$  be a connected CW-complex with a single 0-cell; we assume  $\pi_k(Y) = 0$  for  $1 < k < n$ ,  $n \geq 2$  (nothing is assumed for  $n = 2$ ). Then we have an exact sequence

$$e_Y : 0 \longrightarrow \pi_n(Y) \xrightarrow{\partial_n} \pi_n(Y, Y^{n-1}) \xrightarrow{\partial_{n-1}} \pi_{n-1}(Y^{n-1}, Y^{n-2}) \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} \pi_2(Y^2, Y^1) \xrightarrow{\partial_1} \pi_1(Y^1) \longrightarrow \pi_1(Y) \longrightarrow 1$$

where each  $\partial_j$  is obtained from the exact homotopy sequences of the corresponding pairs of spaces in the obvious way. Now the action of  $\pi_1(Y^1)$  on  $\pi_2(Y^2, Y^1)$  (on  $\pi_2(Y, Y^1)$  in case  $n = 2$ ) and that of  $\pi_1(Y)$  on the remaining groups turn  $e_Y$  into a crossed  $n$ -fold extension [7§3] (that the relative  $\pi_2$  is a crossed  $\pi_1(Y^1)$ -module is due to J. H. C. Whitehead, see [6 p. 39]). In view of the main Theorem of [7§7],  $e_Y$  represents a class  $[e_Y] \in H^{n+1}(\pi_1(Y), \pi_n(Y))$ .

**THEOREM 1.** *The class  $[e_Y]$  is the first (non-trivial)  $k$ -invariant of  $Y$ .*

This offers an answer to a question in [2 p. 301]. Furthermore, we obtain, as a consequence, a description of the  $n$ 'th cohomology group which recovers the description in [5 p. 75] (Theorem 14.1'):

**COROLLARY.** *Let  $A$  be a  $\pi_1(Y)$ -module, and let the crossed  $n$ -fold extension*

$$0 \rightarrow \pi_n(Y) \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow F \rightarrow \pi_1(Y) \rightarrow 1$$

*represent  $[e_Y]$ , with  $C_{n-2}, \dots, C_1, F$  free ( $e_Y$  itself is such a representative). Then*

the group  $H^n(Y, A)$  (local coefficients) is the cokernel of the induced map  $\text{Hom}_G(C_{n-2}, A) \rightarrow \text{Hom}_G(C_{n-1}, A)$  ( $C_0 = F$ ); here  $G = \pi_1(Y)$  resp.  $G = F$  (in low dimensions), and  $\text{Hom}_G(F, A) = \text{Der}(F, A)$ .

We shall use Theorem 1 to determine the first  $k$ -invariant of Quillen's  $( )^+$ -construction: Let  $G$  be a group, and let  $E$  be a perfect normal subgroup. Now  $E$  has a universal central extension [11 p. 43]

$$e : 0 \rightarrow H_2(E) \rightarrow X \rightarrow E \rightarrow 1.$$

Using the universal property of  $e$ , we extend the action of  $G$  on  $E$  to a unique action of  $G$  on  $X$ , turning  $X$  into a crossed  $G$ -module, whence we obtain a crossed 2-fold extension

$$e_{(G, E)} : 0 \longrightarrow H_2(E) \longrightarrow X \xrightarrow{\partial} G \longrightarrow Q \longrightarrow 1$$

where  $Q = G/E$ . In view of the main Theorem in [7§7],  $e_{(G, E)}$  represents a class  $[e_{(G, E)}] \in H^3(Q, H_2(E)) = H^3(\pi_1(BG^+), \pi_2(BG^+))$ . Here  $BG^+$  is Quillen's space [10].

**THEOREM 2.** *The class  $[e_{(G, E)}]$  is the first  $k$ -invariant of  $BG^+$ .*

By a recent result of Kan and Thurston [8], to any connected space  $Y$  there may be associated a group  $G$  together with a map  $\chi : BG \rightarrow Y$  such that (i) the kernel  $E$  of the induced map  $G \rightarrow \pi_1(Y)$  is perfect, and (ii) the map  $\chi$  extends to a (possibly weak)  $h$ -equivalence  $BG^+ \rightarrow Y$ . Theorem 2 shows how the first  $k$ -invariant of  $Y$  is determined by  $G$  and  $E$ . Note that  $[e_{(G, E)}]$  may be non-zero. As to the vanishing of  $[e_{(G, E)}]$  we have

**THEOREM 3.** *The class  $[e_{(G, E)}]$  is zero, and  $Q$  (resp.  $G$ ) acts trivially on  $H_2(E)$  if and only if the induced map  $H_2(E) \rightarrow H_2(G)$  is a split injection (i.e. admits a left inverse).*

A particular example arises in algebraic  $K$ -theory: If  $\Lambda$  is a ring with unit, in view of the above, the class of

$$e_\Lambda : 0 \rightarrow K_2(\Lambda) \rightarrow \text{St}(\Lambda) \rightarrow GL(\Lambda) \rightarrow K_1(\Lambda) \rightarrow 1$$

is the first  $k$ -invariant of  $BGL(\Lambda)^+$ . For commutative rings, this was also announced in [3].

**THEOREM 4.** *The class  $[e_\Lambda] \in H^3(K_1(\Lambda), K_2(\Lambda))$  is zero.*

I am indebted to K. Dennis who provided me with the crucial argument for the proof of Theorem 4. In fact, he has shown that the induced map  $H_2(E(\Lambda)) \rightarrow H_2(GL(\Lambda))$  is a split injection [4 Cor. 8]. Hence Theorem 4 is a consequence of Theorem 3. I am also grateful to K. Brown; he read an earlier version of the paper and conjectured Theorem 3.

**2. Proofs**

*Proof of Theorem 1.* It is known [1] that the first (non-trivial)  $k$ -invariant of  $Y$  is the Eilenberg- Mac Lane invariant  $l_Y \in H^{n+1}(\pi_1(Y), \pi_n(Y))$  [5]. If  $\mathbf{C}$  is the crossed standard resolution of  $\pi_1(Y)$  [7§9], we may lift the identity map of  $\pi_1(Y)$  to

$$\begin{array}{ccccccccccc}
 \mathbf{C} : \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow & \pi_1(Y) & \longrightarrow & 1 \\
 & & \downarrow \xi & & \downarrow & & & & \downarrow & & \parallel & & \\
 e_Y : 0 & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(Y, Y^{n-1}) & \longrightarrow & \cdots & \longrightarrow & \pi_1(Y^1) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1.
 \end{array}$$

Now  $\xi$  is Eilenberg–Mac Lane’s cocycle. This, together with the main Theorem in [7 §7] proves Theorem 1.

*Proof of Theorem 2.* In view of Theorem 1, the Eilenberg–Mac Lane class resp. the first  $k$ -invariant  $l_{BG^+}$  is represented by

$$\hat{e} : O \rightarrow \pi_2(BG^+) \rightarrow \pi_2(BG^+, (BG^+)^1) \rightarrow \pi_1((BG^+)^1) \rightarrow \pi_1(BG^+) \rightarrow 1.$$

But  $\hat{e}$  is clearly equivalent to

$$e^+ : 0 \rightarrow \pi_2(BG^+) \rightarrow \pi_2(BG^+, BG) \rightarrow \pi_1(BG) \rightarrow \pi_1(BG^+) \rightarrow 1,$$

since there is an obvious morphism  $(1, \dots, 1) : \hat{e} \rightarrow e^+$  of crossed 2-fold extensions. We complete the proof by showing that  $\hat{e}$  and  $e_{(G, E)}$  are essentially the same crossed 2-fold extension:

The group  $\pi_2(BG^+, BG)$  is  $\pi_1(Z)$ ,  $Z$  the homotopy fibre of  $BG \rightarrow BG^+$ . Since  $Z$  is acyclic (this is just the universal property of the  $( )^+$ -construction),  $H_1(\pi_1(Z)) = 0 = H_2(\pi_1(Z))$ . It follows from Lemma 2 in [9] (p. 215) that

$$0 \rightarrow \pi_2(BG^+) \rightarrow \pi_2(BG^+, BG) \rightarrow E \rightarrow 1$$

is the universal central extension of  $E$ .

*Proof of Theorem 3.* By the Theorem in [7 §10],  $[e_{(G, E)}]$  is zero if and only if there is a group extension  $1 \rightarrow X \xrightarrow{i} D \rightarrow Q \rightarrow 1$  together with a morphism  $(1, \alpha) : (X, D, j) \rightarrow (X, G, \partial)$  of crossed modules inducing the identity map of  $Q$ . It follows that  $[e_{(G, E)}]$  is zero if and only if there is a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_2(E) & \longrightarrow & X & \longrightarrow & E \longrightarrow 1 \\
 & & \parallel & & \downarrow & & \downarrow i \\
 0 & \longrightarrow & H_2(E) & \longrightarrow & D & \longrightarrow & G \longrightarrow 1,
 \end{array} \tag{*}$$

such that conjugation in  $D$  induces the crossed  $G$ -structure on  $X$ ; here  $i$  denotes the inclusion  $E \subset G$ .

It is clear that  $Q$  (resp.  $G$ ) acts trivially on  $H_2(E)$  if  $H_2(E) \rightarrow H_2(G)$  is a split injection. Hence we may assume that  $Q$  acts trivially on  $H_2(E)$ . Consider now the commutative diagram

$$\begin{array}{ccc}
 H^2(G, H_2(E)) & \longrightarrow & \text{Hom}(H_2(G), H_2(E)) \\
 \downarrow & & \downarrow \\
 H^2(E, H_2(E)) & \longrightarrow & \text{Hom}(H_2(E), H_2(E))
 \end{array} \tag{**}$$

with the obvious maps. Inspection of (\*\*) shows that we have a diagram (\*) if and only if  $H_2(E) \rightarrow H_2(G)$  is a split injection. Hence the condition of the Theorem is necessary. Now, if we have a diagram (\*), conjugation in  $D$  induces an action of  $G$  on  $X$ ; since this one extends the action of  $G$  on  $E$  given by conjugation, it agrees with that one we used to define the crossed  $G$ -structure on  $X$ , as there is only one such  $G$ -action on  $X$ . This last fact is a consequence of the universal property of the universal central extension. It follows that the condition of the Theorem is also sufficient.

*Remark.* The same arguments, applied to the Kan–Thurston construction, may be used to prove the following classical result (which seems to be folk-lore):

**THEOREM.** *For a given connected space  $Y$ , the Hurewicz map  $\pi_2(Y) \rightarrow H_2(Y)$  is a split injection if and only if  $\pi_1(Y)$  acts trivially on  $\pi_2(Y)$  and if the first  $k$ -invariant of  $Y$  is zero.*

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