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Crossed n-fold extensions of groups and cohomology

JOHANNES HUEBSCHMANN

1. Introduction

Crossed modules (§2 below) were introduced by J. H. C. Whitehead [22], [25], and also by Peiffer [19] and Reidemeister [20]. Whitehead was lead to the definition of a crossed module when he investigated the structure of a second relative homotopy group (cf. [8 p. 39]).

The concept of a crossed module admits a natural generalisation to that of a crossed complex (§5). Complexes of this kind were considered in [1], [2], [3], [6], [9], [23], [25] and [26].

An exact crossed complex involving only finitely many non-zero groups and modules may be thought of as a crossed n-fold extension

$$0 \rightarrow A \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow G \rightarrow Q \rightarrow 1, \quad n \ge 1,$$

with Q a group and A a Q-module (see §3). The purpose of this paper is to show that under a suitable similarity relation the classes of crossed n-fold extensions of A by Q constitute an Abelian group $\operatorname{Opext}^n(Q, A)$ naturally isomorphic to the cohomology group $H^{n+1}(Q, A)$ (main Theorem in §7). Thereby the group composition is given by a "Baer sum". This generalises MacLane's interpretation of $H^2(Q, A)$ as group of operator extensions of A by Q [16].

Our major tools are the concepts of free crossed modules (§4), of free (projective) crossed resolutions of groups (§5), and that of homotopy between morphisms of crossed complexes (§6). The main Theorem is proved in §§7 and 8. In §9 we introduce the crossed standard resolution which will be used in [13], [14] and [15]. In §10 we give an illustrative application which will be needed in [13].

As crossed n-fold extensions do occur in mathematics, our interpretation seems to cast new light on group cohomology. We (hope to) demonstrate the significance of our theory in [11], [12], [13], [14] and [15].

Similar results as ours were obtained by other people; we refer to MacLane's Historical Note [17].

The contents of this paper are part of my doctoral dissertation [10] written with the help and encouragement of Professor B. Eckmann to whom I would like

to express my deepest gratitude. I also offer my warmest thanks to R. Beyl, Prof. R. Brown, Prof. S. MacLane, Prof. D. Puppe, Prof. U. Stammbach and R. Strebel.

2. Crossed modules

A crossed module (C, G, ∂) [25] consists of groups C and G, an operation of G on the left of C, written $(g, c) \mapsto {}^{g}c$, and a homomorphism $\partial: C \to G$ of G-groups, where G acts on the left of itself by conjugation. The map ∂ must satisfy the rule

$$bcb^{-1} = {}^{\partial(b)}c, b, c \in C.$$

A morphism $(\alpha, \beta): (C, G, \partial) \to (C', G', \partial')$ of crossed modules consists of homomorphisms $\alpha: C \to C', \beta: G \to G'$ of groups such that $\beta \partial = \partial' \alpha$ and $\alpha({}^{g}c) = {}^{\beta(g)}\alpha(c), c \in C, g \in G$. If (C, G, ∂) is a crossed module, then C is called a *crossed G-module*.

A crossed module generalises the concepts of both an ordinary module and that of a normal subgroup. For if Q is a group and A a Q-(left-) module, then (A, Q, 0) is a crossed module with 0 the trivial map $0(a) = 1 \in Q$, $a \in A$. If G is a group and N a normal subgroup, then (N, G, i) is a crossed module, with i the inclusion and G acting on N by conjugation.

We note at once certain consequences of the definition of a crossed module:

- (a) The image ∂C is a normal subgroup of G.
- (b) The kernel ker (∂) lies in the center Z of C.
- (c) The operation of G on C induces a natural $(G/\partial C)$ -module structure on Z, and ker (∂) is a submodule of Z.
- (d) The action of G on C induces a natural $(G/\partial C)$ -module structure on the commutator factor group $C^{Ab} = C/[C, C]$.

It is clear that the crossed modules constitute a category **XMod**: if G is a fixed group, the crossed G-modules constitute a (full) subcategory G-**XMod**.

3. Crossed n-fold extensions

Let Q be a group and A a Q-module. A crossed n-fold extension of A by Q $(n \ge 1)$ is an exact sequence

$$e: 0 \longrightarrow A \xrightarrow{\gamma} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G \longrightarrow Q \longrightarrow 1$$

of groups with the following properties:

- (i) (C_1, G, ∂_1) is a crossed module,
- (ii) for 1 < k < n, C_k is a Q-module, and ∂_k and γ are Q-linear.

Note that it makes sense to require ∂_2 to be Q-linear, since the kernel of ∂_1 is naturally a Q-module. Now a morphism $(\sigma, \alpha, \varphi) : e \rightarrow e'$ of crossed n-fold extensions consists of group homomorphisms $\varphi : Q \rightarrow Q', \ \alpha_0 : G \rightarrow G', \ \alpha_k : C_k \rightarrow C'_k, \ 0 < k < n, \ \text{and} \ \sigma : A \rightarrow A'$ such that $(\sigma, \alpha_{n-1}, \ldots, \alpha_1, \alpha_0, \varphi)$ provides a commutative diagram of groups which preserves all the structure. So we have a category of crossed n-fold extensions of A by Q. For completeness, by a crossed 0-fold extension of A by Q we mean a derivation $d: Q \rightarrow A$.

Given a group K with center Z and automorphism group Aut(K), we have the crossed 2-fold extension

$$0 \rightarrow Z \rightarrow K \xrightarrow{\partial_K} \operatorname{Aut}(K) \rightarrow \operatorname{Out}(K) \rightarrow 1,$$

where ∂_K sends $k \in K$ to the corresponding inner automorphism; here $\mathrm{Out}(K)$ is the group of outer automorphisms. Now any abstract Q-kernel $\psi : Q \to \mathrm{Out}(K)$ (see [7]) provides a crossed 2-fold extension

$$e^{\psi}: 0 \rightarrow Z \rightarrow K \xrightarrow{\partial^{\psi}} G^{\psi} \rightarrow Q \rightarrow 1$$

with G^{ψ} the fibre product $\operatorname{Aut}(K) \underset{\operatorname{Out}(K)}{\times} Q$ and ∂^{ψ} the obvious map. Crossed 2-fold extensions of this kind with G^{ψ} a free group were studied in [16], see also [18]. An example of a crossed *n*-fold extension for n > 2 will be given in [13].

4. Free crossed modules

Let Grp(2) denote the category whose objects are group homomorphisms and whose morphisms are commutative squares in the category of groups. The forgetful functor $V: XMod \rightarrow Grp(2)$ which forgets the group action has a left adjoint $(\lambda: H \rightarrow G) \mapsto U(\lambda) = (C, G, \partial)$, the free crossed module on λ , see [4, p. 207]. If H is (as group) free on a set S, then C coincides with Whitehead's free crossed G-module [25, p. 455]; in this case S is called a basis for C. Thereby the use of the word "basis" is justified by the fact that the induced map $S \rightarrow C$ is injective. This follows from

LEMMA 1. If C is the free crossed G-module with basis S, then C^{Ab} is as ordinary $(G/\partial C)$ -module free on the elements s[C, C], $s \in S$.

Note, however, that the induced map $H \rightarrow C$ need not be injective (where still H is free on S).

Let now (X; R) be a presentation of a group Q. Let N_0 be free on a set \hat{R} in one-one correspondence with R (via $\hat{r} \mapsto r$), and let $\lambda : N_0 \to F$ be the map that is induced by the relators, where F is free on X; we denote by N the normal closure of R in F.

PROPOSITION 1. Any presentation (X; R) of a group Q determines a crossed module (C, F, ∂) which is unique up to isomorphism; thereby F is (as group) free on X and C is the free crossed F-module with basis R (resp. \hat{R}). Moreover, the following holds:

- (a) If F has at least two free generators, then the center of C coincides with the kernel of ∂ .
 - (b) The elements $\hat{r}[C, C]$, $r \in R$, constitute a Q-basis of C^{Ab} .
 - (c) The induced map $ker(\partial) \rightarrow C^{Ab}$ is injective, and

$$0 \rightarrow \ker(\partial) \rightarrow C^{Ab} \rightarrow N^{Ab} \rightarrow 0$$

is a Q-free presentation of N^{Ab} .

Proof of (c). Since ker (∂) is central in C, and since N is a free group, C is a direct product ker (∂) $\times \bar{N}$, where $C \to N$ induces an isomorphism $\bar{N} \to N$; hence ker (∂) $\to C^{Ab}$ is injective. Q.E.D.

For a group G, the notion of a free crossed G-module may be generalised: A projective crossed G-module is a projective object in G-XMod.

5. Crossed complexes and free (projective) crossed resolutions of groups

A crossed complex C (over a group) is a sequence

$$\mathbf{C}: \cdots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G$$

of groups with the following properties:

- (C1) The triple (C_1, G, ∂_1) is a crossed module;
- (C2) for $k \ge 2$ each C_k is a Q-module, where $Q = G/(\partial_1 C_1)$, and each ∂_k is a Q-map (for k = 2 this shall mean that ∂_2 commutes with the action of G; note, however, that the image $\partial_2(C_2) \subset C_1$ is a Q-module);

(C3) $\partial \partial = 0$.

A crossed complex \mathbb{C} is called *free* (projective) if G is a free group, if C_1 is a free (projective) crossed G-module, and if each C_k , $k \ge 2$, is a free (projective)

Q-module $(Q = G/(\partial_1 C_1))$. If a crossed complex \mathbb{C} is exact, and if a group Q, given in advance, is isomorphic to the quotient $G/(\partial_1 C_1)$, then \mathbb{C} is called a crossed resolution of Q (a free resp. a projective crossed resolution, if \mathbb{C} is free resp. projective). Now a morphism $\alpha: \mathbb{C} \to \mathbb{C}'$ of crossed complexes consists of group homomorphisms $\alpha_0: G \to G'$, $\alpha_k: C_k \to C'_k$, $k \ge 1$, such that $(\ldots, \alpha_k, \alpha_{k-1}, \ldots, \alpha_1, \alpha_0)$ provides a commutative diagram of groups which preserves all the structure.

Clearly, crossed n-fold extensions yield special examples of (exact) crossed complexes with $C_k = 0$, k > n. The standard example of a crossed complex is given by the sequence of relative homotopy groups of a filtered space [3], [6], [25] ("homotopy system").

As for a given group Q any Q-module has a free (projective) resolution, from Proposition 1 we infer

PROPOSITION 2. Any group has a free (projective) crossed resolution.

The following is clear:

PROPOSITION 3. Let \mathbb{C} be a free (projective) crossed complex with $Q = \operatorname{coker}(\partial_1)$, and let \mathbb{C}' be a crossed resolution of a group Q'. Then any homomorphism $\varphi: Q \to Q'$ may be lifted to a morphism $\alpha: \mathbb{C} \to \mathbb{C}'$ of crossed complexes.

If **C** is a free (projective) crossed resolution of Q, denote by \mathbb{C}^n the crossed complex (for $n \ge 2$ it is a crossed n-fold extension)

$$\mathbb{C}^n: 0 \to J_n \to C_{n-1} \to \cdots \to C_1 \to F \to Q \to 1,$$

where $J_n = \ker(C_{n-1} \to C_{n-2})$ (with $C_0 = F$ and $C_{-1} = Q$). We shall refer to \mathbb{C}^n as a free (projective) crossed n-fold extension (as to \mathbb{C}^1 see Note added in proof).

PROPOSITION 3'. Let e' be a crossed n-fold extension with $Q' = \operatorname{coker}(\partial_1)$. Then any homomorphism $\varphi : Q \to Q'$ may be lifted to a morphism $(\sigma, \alpha, \varphi) : \mathbb{C}^n \to e'$ of crossed n-fold extensions.

6. Homotopy

Let there be given two crossed complexes C, C' with $Q = \operatorname{coker}(\partial_1)$ and $Q' = \operatorname{coker}(\partial_1')$; let further α and β be morphisms $C \to C'$ of crossed complexes.

Now a family $\Sigma = {\Sigma_k, k \ge 0}$ of maps $\Sigma_0 : G \to C'_1, \Sigma_k : C'_k \to C_{k+1}, k \ge 1$, is called a homotopy between α and β , denoted $\Sigma : \alpha = \beta$, if

(i) $\Sigma_0: G \to C_1$ is a (left-) derivation (crossed homomorphism) associated with β_0 , i.e. $\Sigma_0(xy) = \Sigma_0(x)(^{\beta_0(x)}\Sigma_0(y)), x, y \in G$, such that

$$\partial_1 \Sigma_0(x) = \alpha_0(x) \beta_0(x)^{-1}, \qquad x \in G,$$

(ii) $\Sigma_1: C_1 \to C_2'$ is a G-homomorphism, with G acting on C_2' via α_0 (or β_0 , which yields the same action in view of (i)), such that

$$\partial_2 \Sigma_1(x) = \beta_1(x)^{-1} (\Sigma_0 \partial_1(x))^{-1} \alpha_1(x), \qquad x \in C_1,$$

(iii) for $k \ge 2$, Σ_k is a Q-homomorphism, with Q acting on the C'_k via the induced map $Q \to Q'$ (note that α and β induce the same map $Q \to Q'$ in view of (i)), such that

$$\partial_{k+1} \Sigma_k + \Sigma_{k-1} \partial_k = \alpha_k - \beta_k$$
.

LEMMA 2. Homotopy is an equivalence relation.

PROPOSITION 4. Let \mathbb{C} be a free (projective) crossed complex with $Q = \operatorname{coker}(\partial_1)$, and let \mathbb{C}' be a crossed resolution of Q'; let further $\alpha, \beta : \mathbb{C} \to \mathbb{C}'$ be morphisms of crossed complexes. If α and β induce the same homomorphism $\varphi : Q \to Q'$, there is a homotopy $\Sigma : \alpha \simeq \beta$.

It is clear that we also have the notion of a homotopy $\Sigma : (\sigma, \alpha, \varphi) \simeq (\tau, \beta, \varphi)$ of morphisms $e \to e'$ of crossed *n*-fold extensions with the same *right end* $\varphi : Q \to Q'$: it is a family $(\Sigma_{n-1}, \ldots, \Sigma_0)$ of maps satisfying (i), (ii) and (iii) above; thereby $\partial_n = \gamma$, $\Sigma_n = 0 = \partial_{n+1}$, $\alpha_n = \sigma$, $\beta_n = \tau$, $C_n = A$.

PROPOSITION 4'. Let \mathbb{C}^n be a free (projective) crossed n-fold extension with $Q = \operatorname{coker}(\partial_1)$, and let e' be a crossed n-fold extension with $Q' = \operatorname{coker}(\partial_1)$. If $(\sigma, \alpha, \varphi)$ and (τ, β, φ) are morphisms $\mathbb{C}^n \to e'$ of crossed n-fold extension with the same right end φ , then there is a homotopy $\Sigma : (\sigma, \alpha, \varphi) \cong (\tau, \beta, \varphi)$.

Proofs are routine and left to the reader. If we combine the above with Proposition 3 resp. Proposition 3', we obtain

PROPOSITION 5. The set Hom(Q, Q') classifies the homotopy classes of morphisms $\mathbb{C} \to \mathbb{C}'$ resp. of morphisms $\mathbb{C}^n \to e'$ with the same right end.

It is now clear how to introduce the notion of homotopy equivalence of crossed complexes, and we have the

COROLLARY. Any two free (projective) crossed resolutions of a group are homotopy equivalent.

7. Opextⁿ-groups and cohomology; the main Theorem.

Let Q be a given group, and let

$$\mathbf{C}: \cdots \to C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \to F \dashrightarrow Q$$

be a free (projective) crossed resolution of Q. For any Q-module A, consider the complex (the arrows are the obvious maps)

$$\operatorname{Hom}(\mathbf{C}, A) : \operatorname{Der}(F, A) \to \operatorname{Hom}_F(C_1, A) \to \operatorname{Hom}_Q(C_2, A) \to \cdots$$

(For a group G and a G-module A, "Der (G, A)" denotes the Abelian group of derivations from G to A.) Its cohomology groups are as follows:

PROPOSITION 6.
$$H^0(\text{Hom }(\mathbb{C},A)) = \text{Der }(Q,A), \quad H^q(\text{HOM}(\mathbb{C},A)) = H^{q+1}(Q,A), \quad q \ge 1.$$

Proof. Assume for convenience that, in case \mathbb{C} is a proper projective crossed resolution, the crossed F-module C_1 is free. The case of a proper projective crossed F-module C_1 is left as an exercise. Now the crossed complex \mathbb{C} may be transformed into the complex

$$\hat{\mathbf{C}}: \cdots \to C_k \to \cdots \to C_2 \to C_1^{Ab} \to \mathbb{Z}Q \otimes_F IF,$$

where $C_2 \rightarrow C_1^{Ab}$ is the obvious map, and where $C_1^{Ab} \rightarrow \mathbb{Z} Q \otimes_F IF$ is given by the rule $x[C_1, C_1] \mapsto 1 \otimes (\partial x - 1)$, $x \in C_1$. (Here "IG" denotes the augmentation ideal of a group G.) By Proposition 2, C_1^{Ab} is a free Q-module, and the cokernel of $C_2 \rightarrow C_1^{Ab}$ is the relation module N^{Ab} , where $N = \ker(F \rightarrow Q)$. Hence $\hat{\mathbf{C}}$ is a free (projective) resolution of IQ. Applying the functor $\operatorname{Hom}_Q(-, A)$ to $\hat{\mathbf{C}}$ yields a complex canonically isomorphic to $\operatorname{Hom}(\mathbf{C}, A)$ whence the cohomology of $\operatorname{Hom}(\mathbf{C}, A)$ is as stated. Q.E.D.

The fact that $H^2(Q, A)$ is $H^1(\text{Hom}(\mathbb{C}, A))$ was already proved by MacLane [16, Theorem A'].

We now divide the crossed n-fold extensions of A by Q ($n \ge 1$) into classes as follows: Two crossed n-fold extensions e, e' of A by Q are related if there is a morphism $(1, \alpha, 1) : e \rightarrow e'$ of crossed n-fold extensions; this relation generates an equivalence relation which shall be denoted by " \equiv ". The equivalence class of e, also called *similarity class*, is to be denoted by [e].

We next consider a crossed n-fold extension e of A by Q. If \mathbb{C} is a projective crossed resolution of Q, it follows from Proposition 3 that the identity map of Q lifts to

$$\mathbf{C}: \cdots \to C_{n} \to C_{n-1} \to \cdots \to C_{1} \to F \to Q \to 1$$

$$\downarrow^{\zeta} \qquad \downarrow^{\alpha_{n-1}} \qquad \qquad \downarrow^{\alpha_{1}} \qquad \downarrow^{\alpha_{0}} \qquad \parallel$$

$$e: \qquad 0 \to A \to A_{n-1} \to \cdots \to A_{1} \to G \to Q \to 1$$

In view of the above, ζ represents a class $[\zeta] \in H^{n+1}(Q, A)$. If \mathbb{C} is replaced by \mathbb{C}^n (introduced in §5), the above induces a morphism $(\nu, \alpha, 1) : \mathbb{C}^n \to e$ of crossed n-fold extensions. Now, for $n \ge 2$, the coequaliser $C_{n-1,\nu}$, say, of $J_n \xrightarrow{\nu} A \times C_{n-1}$, where i denotes the inclusion $J_n \to C_{n-1}$, yields the crossed n-fold extension

$$\nu \mathbb{C}^n: 0 \to A \to C_{n-1,\nu} \to C_{n-2} \to \cdots \to C_1 \to F \to Q \to 1$$

with $C_{n-1,\nu} \to C_{n-2}$ the obvious map. If n = 1, the coequaliser $C_{0,\nu}$ of $J_1 \Longrightarrow A$] F $(J_1 = N = \ker(F \to Q))$ yields the ordinary group extension

$$\nu$$
C¹: $0 \rightarrow A \rightarrow C_{0,\nu} \rightarrow Q \rightarrow 1$;

here "]" denotes the semi-direct product. Clearly, there is a morphism $(1, \beta, 1) : \mathbb{C}^n \to e$ of crossed *n*-fold extensions; hence

PROPOSITION 7. Each equivalence class of crossed n-fold extensions of A by Q has a representative of the form $\nu \mathbb{C}^n$.

It is now clear that the Abelian group $\operatorname{Hom}_F(J_n, A)$ (= $\operatorname{Hom}_Q(J_n, A)$, if $n \ge 2$) maps onto the classes of crossed *n*-fold extensions of A by Q by rule $\nu \mapsto \nu \mathbb{C}^n$. Consequently, these classes constitute a set, denoted henceforth by $\operatorname{Opext}^n(Q, A)$.

Given two crossed *n*-fold extensions e, e' of A by Q, it is routine to construct their "Baer- sum" e+e'. We refrain from writing down details. Moreover, the Baer- sum induces a sum on similarity classes, and the surjection $\operatorname{Hom}_F(J_n, A) \to \operatorname{Opext}^n(Q, A)$ is a homomorphism with respect to the Baer- sum, i.e. $(\mu + \nu)\mathbb{C}^n \equiv \mu \mathbb{C}^n + \nu \mathbb{C}^n$, $\mu, \nu: J_n \to A$ operator maps. Consequently, under the Baer- sum, $\operatorname{Opext}^n(Q, A)$ is an Abelian group, with zero element OC^n , i.e. the image of the zero map $J_n \to A$, and $\operatorname{Hom}_F(J_n, A) \to \operatorname{Opext}^n(Q, A)$ is an epimorphism of Abelian groups.

LEMMA 3. Let $\nu: J_n \to A$, $n \ge 1$, be an operator map which may be extended over C_{n-1} to

- (i) a derivation $F \rightarrow A$, if n = 1,
- (ii) an F-map $C_1 \rightarrow A$, if n = 2, and
- (iii) a Q-map $C_{n-1} \rightarrow A$, if $n \ge 3$.

Then the extension

$$E: 0 \rightarrow A \rightarrow C_{n-1,\nu} \rightarrow J_{n-1} \rightarrow 1$$

 $(J_1 = N, J_0 = Q)$ splits, i.e. there is a section $J_{n-1} \rightarrow C_{n-1,\nu}$ which is a group homomorphism, if n = 1, an F-homomorphism, if n = 2, and a Q-homomorphism, if $n \ge 3$.

The proof is straightforward.

If, given an operator map $\nu: J_n \to A$, $n \ge 2$, the extension E splits (as in Lemma 3), there is a morphism $(1, \alpha, 1): \nu \mathbb{C}^n \to \mathbf{0}$ of crossed n-fold extensions, where $\mathbf{0}$ denotes

$$0: 0 \to A \xrightarrow{=} A \to 0 \to \cdots \to 0 \to Q \xrightarrow{=} Q \to 1,$$

whence $\nu \mathbb{C}^n$ and $\mathbf{0}$ are equivalent; since $\mathbf{0}$ represents $0 \in \operatorname{Opext}^n(Q, A)$, so does $\nu \mathbb{C}^n$. By Proposition 6, the cokernel of $\operatorname{Hom}_F(C_{n-1}, A) \to \operatorname{Hom}_Q(J_n, A)$ ($\operatorname{Hom}_F(C_{n-1}, A) = \operatorname{Hom}_Q(C_{n-1}, A)$ if $n \ge 3$) is the cohomology group $H^{n+1}(Q, A)$. It follows from Lemma 3 that for $n \ge 2$ the rule $\nu \mapsto \nu \mathbb{C}^n$, $\nu : J_n \to A$ an operator map, induces an epimorphism $\Phi : H^{n+1}(Q, A) \to \operatorname{Opext}^n(Q, A)$ of Abelian groups; this also follows for n = 1, as $H^2(Q, A)$ is the cokernel of $\operatorname{Der}(F, A) \to \operatorname{Hom}_F(N, A)$.

The main Theorem. The map Φ is an isomorphism of Abelian groups. In other words, the classes of crossed n-fold extensions of A by Q constitute an Abelian group $\operatorname{Opext}^n(Q, A)$ naturally isomorphic to the cohomology group $H^{n+1}(Q, A)$. The group composition is given by the Baer-sum. The zero element of this group is the class of the crossed n-fold extension $\mathbf{0}$, whereas the inverse of the class of

$$e: 0 \to A \xrightarrow{\gamma} C_{n-1} \to \cdots \to C_1 \to G \to Q \to 1$$

is the class of

$$-e: 0 \to A \xrightarrow{(-\gamma)} C_{n-1} \to \cdots \to C_1 \to G \to Q \to 1.$$

8. The proof of the main Theorem

We have to prove that $\Phi: H^{n+1}(Q, A) \to \operatorname{Opext}^n(Q, A)$, $n \ge 2$, is injective (the case n = 1 is classical). This amounts to show that if $\mu \mathbb{C}^n \equiv \nu \mathbb{C}^n$, with \mathbb{C}^n a free (projective) crossed n-fold extension and μ, ν operator maps $J_n \to A$, then $\mu - \nu$ extends over C_{n-1} as in (ii) resp. (iii) of Lemma 3. We argue as follows:

Since $\mu \mathbb{C}^n \equiv \nu \mathbb{C}^n$, there are crossed n-fold extensions e_1, e_2, \ldots, e_m of A by Q, with $e_m = \nu \mathbb{C}^n$, and morphisms $(1, \alpha^1, 1) : \mu \mathbb{C}^n \to e_1$, $(1, \alpha^2, 1) : e_2 \to e_1$, $(1, \alpha^3, 1) : e_2 \to e_3$, $(1, \alpha^4, 1) : e_4 \to e_3$, and so forth. By construction, there are morphisms $(\mu, \beta^0, 1) : \mathbb{C}^n \to \mu \mathbb{C}^n$ and $(\nu, \beta^m, 1) : \mathbb{C}^n \to \nu \mathbb{C}^n$. Moreover, it follows from Proposition 3' that for $1 \le k < m$ the identity map of Q lifts to a morphism $(\nu^k, \beta^k, 1) : \mathbb{C}^n \to e_k$ of crossed n-fold extensions. We may assume that m is even (otherwise we add the identity morphism $e_{m+1} \to e_m$). It follows from Proposition 4' that the morphisms $(\mu, \alpha^1 \beta^0, 1)$ and $(\nu^2, \alpha^2 \beta^2, 1) : \mathbb{C}^n \to e_1$ are homotopic; likewise, $(\nu^2, \alpha^3 \beta^2, 1)$ and $(\nu^4, \alpha^4 \beta^4, 1) : \mathbb{C}^n \to e_3$ are homotopic also, and so forth. We ultimately arrive at $(\nu^{m-2}, \alpha^{m-1} \beta^{m-2}, 1)$ and $(\nu, \alpha^m \beta^m, 1) : \mathbb{C}^n \to e_{m-1}$ which again are homotopic. Now $\mu - \nu = \mu - \nu^2 + \nu^2 - \nu^4 + \cdots + \nu^{m-2} - \nu$ extends over C_{n-1} as desired.

The proofs of naturality of Φ and of the assertion as to the inverse of a class $[e] \in \operatorname{Opext}^n(Q, A)$ are left to the reader. Q.E.D.

9. The (inhomogenous) crossed standard resolution

The following section will be needed in [13], [14] and [15]; it will provide the bridge between our interpretation of group cohomology and the classical description in terms of cocycles.

Let Q be a group, and let $(Q^*; Q^* \times Q^*)$ be its standard presentation $(Q^* = Q \setminus \{1\})$; hence the relator $[q_1, q_2], q_1, q_2 \in Q^*$, corresponds to the word $[q_1][q_2][q_1q_2]^{-1}$. Next, let F be the free group on Q^* , and let C be the free crossed F-module with basis $Q^* \times Q^*$; it is then clear that the elements

(*)
$$[q_1][q_2, q_3][q_1, q_2q_3][q_1q_2, q_3]^{-1}[q_1, q_2]^{-1}, \qquad q_1, q_2, q_3 \in Q^*,$$

lie in the kernel of $\partial: C \rightarrow F$. By Proposition 1, we have the Q-free presentation

$$0 \rightarrow \ker(\partial) \rightarrow C^{Ab} \rightarrow N^{Ab} \rightarrow 0$$

of N^{Ab} . Since C^{Ab} is the corresponding term of the (ordinary) inhomogenous standard resolution of the integers where it is known that the elements (*) generate the kernel of $C^{Ab} \rightarrow N^{Ab}$ (in the operator sense), it follows that the

elements (*) generate ker (∂). If we now "splice" our free crossed module $C \to F$ with the remaining part of the inhomogenous standard resolution of the integers (this is a resolution of ker (∂)), we obtain a free crossed resolution of Q, henceforth called the (inhomogenous) crossed standard resolution of Q. Now, if C_2 is the free Q-module on $Q^* \times Q^* \times Q^*$, and if $\partial_2 : C_2 \to C_1$ ($C_1 = C$) is given by sending $[q_1, q_2, q_3]$ to (*), the kernel of ∂_2 is generated by the elements (written multiplicatively)

(**)
$$q_1[q_2, q_3, q_4][q_1q_2, q_3, q_4]^{-1}[q_1, q_2q_3, q_4][q_1, q_2, q_3q_4]^{-1}[q_1, q_2, q_3],$$

$$q_1, q_2, q_3, q_4 \in Q^*.$$

10. An illustration

If e^{ψ} is the crossed 2-fold extension obtained from an abstract Q-kernel $\psi: Q \rightarrow \text{Out}(K)$ (§3), we may lift the identity map of Q to

$$\mathbf{C}: \cdots \to C_2 \to C_1 \to F \to Q \to 1$$

$$\downarrow^{\zeta} \qquad \downarrow \qquad \downarrow$$

$$e^{\psi}: \quad 0 \to Z \to K \to G^{\psi} \to Q \to 1,$$

where \mathbb{C} is the crossed standard resolution. This yields an operator map $\zeta: C_2 \to \mathbb{Z}$, which, in view of (**), is a 3-cocycle; it is the Eilenberg-MacLane cocycle [7]. This, together with our main Theorem shows that $[e^{\psi}] \in \operatorname{Opext}^2(Q, \mathbb{Z})$ is the Eilenberg-MacLane class of (K, ψ) . Eilenberg-MacLane's extendibility criterion is now recovered by the following

THEOREM. Let $e: 0 \to A \to K \xrightarrow{\partial} G \to Q \to 1$ be a crossed 2-fold extension. There is a group extension $1 \to K \xrightarrow{j} E \to Q \to 1$ together with a morphism $(1, \alpha): (K, E, j) \to (K, G, \partial)$ of crossed modules inducing the identity map of Q if and only if $[e] = 0 \in \text{Opext}^2(Q, A)$.

This generalises Eilenberg-MacLane's extendibility criterion, since A need not coincide with the center of K.

Proof. We show that the condition suffices. To this end, let $\mathbb{C}^2: 0 \to J \to C \to F \to Q \to 1$ be a free crossed 2-fold extension and let $(\nu, \beta_1, \beta_0, 1): \mathbb{C}^2 \to e$ be a lifting of the identity map of Q. Since $[e] = 0 \in \text{Opext}^2(Q, A)$, ν extends over C as in (ii) of Lemma 3; it follows that there is a morphism $(\beta, \beta_0): (N, F, i) \to (K, G, \partial)$ of crossed modules, where $N = \ker(F \to Q)$. Now the coequaliser E of $N \xrightarrow{i}_{\beta} K F$ (where F acts on K via β_0) yields the required extension. Q.E.D.

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Note added in Proof: Strictly speaking, the crossed complex \mathbb{C}^1 in §5 is not a crossed 1-fold extension since $J_1 = N = (\ker(F \to Q))$ is not a Q-module; however, \mathbb{C}^1 may always be replaced by

$$O \rightarrow N^{Ab} \rightarrow F/[N, N] \rightarrow Q \rightarrow 1.$$