

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 55 (1980)

Artikel: Crossed n-folds extensions of groups and cohomology.
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DOI: <https://doi.org/10.5169/seals-42377>

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Crossed n -fold extensions of groups and cohomology

JOHANNES HUEBSCHMANN

1. Introduction

Crossed modules (§2 below) were introduced by J. H. C. Whitehead [22], [25], and also by Peiffer [19] and Reidemeister [20]. Whitehead was lead to the definition of a crossed module when he investigated the structure of a second relative homotopy group (cf. [8 p. 39]).

The concept of a crossed module admits a natural generalisation to that of a *crossed complex* (§5). Complexes of this kind were considered in [1], [2], [3], [6], [9], [23], [25] and [26].

An exact crossed complex involving only finitely many non-zero groups and modules may be thought of as a *crossed n -fold extension*

$$0 \rightarrow A \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow G \rightarrow Q \rightarrow 1, \quad n \geq 1,$$

with Q a group and A a Q -module (see §3). The purpose of this paper is to show that under a suitable similarity relation the classes of crossed n -fold extensions of A by Q constitute an Abelian group $\text{Opext}^n(Q, A)$ naturally isomorphic to the cohomology group $H^{n+1}(Q, A)$ (main Theorem in §7). Thereby the group composition is given by a “Baer sum”. This generalises MacLane’s interpretation of $H^2(Q, A)$ as group of operator extensions of A by Q [16].

Our major tools are the concepts of *free crossed modules* (§4), of *free (projective) crossed resolutions of groups* (§5), and that of *homotopy between morphisms of crossed complexes* (§6). The main Theorem is proved in §§7 and 8. In §9 we introduce the *crossed standard resolution* which will be used in [13], [14] and [15]. In §10 we give an illustrative application which will be needed in [13].

As crossed n -fold extensions do occur in mathematics, our interpretation seems to cast new light on group cohomology. We (hope to) demonstrate the significance of our theory in [11], [12], [13], [14] and [15].

Similar results as ours were obtained by other people; we refer to MacLane’s Historical Note [17].

The contents of this paper are part of my doctoral dissertation [10] written with the help and encouragement of Professor B. Eckmann to whom I would like

to express my deepest gratitude. I also offer my warmest thanks to R. Beyl, Prof. R. Brown, Prof. S. MacLane, Prof. D. Puppe, Prof. U. Stambach and R. Strebel.

2. Crossed modules

A *crossed module* (C, G, ∂) [25] consists of groups C and G , an operation of G on the left of C , written $(g, c) \mapsto {}^g c$, and a homomorphism $\partial: C \rightarrow G$ of G -groups, where G acts on the left of itself by conjugation. The map ∂ must satisfy the rule

$$bcb^{-1} = {}^{\partial(b)}c, \quad b, c \in C.$$

A *morphism* $(\alpha, \beta): (C, G, \partial) \rightarrow (C', G', \partial')$ of crossed modules consists of homomorphisms $\alpha: C \rightarrow C'$, $\beta: G \rightarrow G'$ of groups such that $\beta\partial = \partial'\alpha$ and $\alpha({}^g c) = {}^{\beta(g)}\alpha(c)$, $c \in C$, $g \in G$. If (C, G, ∂) is a crossed module, then C is called a *crossed G -module*.

A crossed module generalises the concepts of both an ordinary module and that of a normal subgroup. For if Q is a group and A a Q -(left-) module, then $(A, Q, 0)$ is a crossed module with 0 the trivial map $0(a) = 1 \in Q$, $a \in A$. If G is a group and N a normal subgroup, then (N, G, i) is a crossed module, with i the inclusion and G acting on N by conjugation.

We note at once certain consequences of the definition of a crossed module:

- (a) The image ∂C is a normal subgroup of G .
- (b) The kernel $\ker(\partial)$ lies in the center Z of C .
- (c) The operation of G on C induces a natural $(G/\partial C)$ -module structure on Z , and $\ker(\partial)$ is a submodule of Z .
- (d) The action of G on C induces a natural $(G/\partial C)$ -module structure on the commutator factor group $C^{Ab} = C/[C, C]$.

It is clear that the crossed modules constitute a category **XMod**: if G is a fixed group, the crossed G -modules constitute a (full) subcategory **G-XMod**.

3. Crossed n -fold extensions

Let Q be a group and A a Q -module. A *crossed n -fold extension of A by Q* ($n \geq 1$) is an exact sequence

$$e: 0 \longrightarrow A \xrightarrow{\gamma} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G \longrightarrow Q \longrightarrow 1$$

of groups with the following properties:

- (i) (C_1, G, ∂_1) is a crossed module,
- (ii) for $1 < k < n$, C_k is a Q -module, and ∂_k and γ are Q -linear.

Note that it makes sense to require ∂_2 to be Q -linear, since the kernel of ∂_1 is naturally a Q -module. Now a *morphism* $(\sigma, \alpha, \varphi) : e \rightarrow e'$ of crossed n -fold extensions consists of group homomorphisms $\varphi : Q \rightarrow Q'$, $\alpha_0 : G \rightarrow G'$, $\alpha_k : C_k \rightarrow C'_k$, $0 < k < n$, and $\sigma : A \rightarrow A'$ such that $(\sigma, \alpha_{n-1}, \dots, \alpha_1, \alpha_0, \varphi)$ provides a commutative diagram of groups which preserves all the structure. So we have a category of crossed n -fold extensions of A by Q . For completeness, by a *crossed 0-fold extension* of A by Q we mean a derivation $d : Q \rightarrow A$.

Given a group K with center Z and automorphism group $\text{Aut}(K)$, we have the crossed 2-fold extension

$$0 \rightarrow Z \rightarrow K \xrightarrow{\partial_K} \text{Aut}(K) \rightarrow \text{Out}(K) \rightarrow 1,$$

where ∂_K sends $k \in K$ to the corresponding inner automorphism; here $\text{Out}(K)$ is the group of outer automorphisms. Now any abstract Q -kernel $\psi : Q \rightarrow \text{Out}(K)$ (see [7]) provides a crossed 2-fold extension

$$e^\psi : 0 \rightarrow Z \rightarrow K \xrightarrow{\partial^\psi} G^\psi \rightarrow Q \rightarrow 1$$

with G^ψ the fibre product $\text{Aut}(K) \times_{\text{Out}(K)} Q$ and ∂^ψ the obvious map. Crossed 2-fold extensions of this kind with G^ψ a free group were studied in [16], see also [18]. An example of a crossed n -fold extension for $n > 2$ will be given in [13].

4. Free crossed modules

Let $\mathbf{Grp}(2)$ denote the category whose objects are group homomorphisms and whose morphisms are commutative squares in the category of groups. The forgetful functor $V : \mathbf{XMod} \rightarrow \mathbf{Grp}(2)$ which forgets the group action has a left adjoint $(\lambda : H \rightarrow G) \mapsto U(\lambda) = (C, G, \partial)$, the *free crossed module on λ* , see [4, p. 207]. If H is (as group) free on a set S , then C coincides with Whitehead's *free crossed G -module* [25, p. 455]; in this case S is called a *basis* for C . Thereby the use of the word "basis" is justified by the fact that the induced map $S \rightarrow C$ is injective. This follows from

LEMMA 1. *If C is the free crossed G -module with basis S , then C^{Ab} is an ordinary $(G/\partial C)$ -module free on the elements $s[C, C]$, $s \in S$.*

Note, however, that the induced map $H \rightarrow C$ need not be injective (where still H is free on S).

Let now $(X; R)$ be a presentation of a group Q . Let N_0 be free on a set \hat{R} in one-one correspondence with R (via $\hat{r} \mapsto r$), and let $\lambda : N_0 \rightarrow F$ be the map that is induced by the relators, where F is free on X ; we denote by N the normal closure of R in F .

PROPOSITION 1. *Any presentation $(X; R)$ of a group Q determines a crossed module (C, F, ∂) which is unique up to isomorphism; thereby F is (as group) free on X and C is the free crossed F -module with basis R (resp. \hat{R}). Moreover, the following holds:*

(a) *If F has at least two free generators, then the center of C coincides with the kernel of ∂ .*

(b) *The elements $\hat{r}[C, C]$, $r \in R$, constitute a Q -basis of C^{Ab} .*

(c) *The induced map $\ker(\partial) \rightarrow C^{Ab}$ is injective, and*

$$0 \rightarrow \ker(\partial) \rightarrow C^{Ab} \rightarrow N^{Ab} \rightarrow 0$$

is a Q -free presentation of N^{Ab} .

Proof of (c). Since $\ker(\partial)$ is central in C , and since N is a free group, C is a direct product $\ker(\partial) \times \bar{N}$, where $C \rightarrow N$ induces an isomorphism $\bar{N} \rightarrow N$; hence $\ker(\partial) \rightarrow C^{Ab}$ is injective. Q.E.D.

For a group G , the notion of a free crossed G -module may be generalised: A *projective crossed G -module* is a projective object in **G-XMod**.

5. Crossed complexes and free (projective) crossed resolutions of groups

A *crossed complex \mathbf{C}* (over a group) is a sequence

$$\mathbf{C}: \cdots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G$$

of groups with the following properties:

(C1) The triple (C_1, G, ∂_1) is a crossed module;

(C2) for $k \geq 2$ each C_k is a Q -module, where $Q = G/(\partial_1 C_1)$, and each ∂_k is a Q -map (for $k = 2$ this shall mean that ∂_2 commutes with the action of G ; note, however, that the image $\partial_2(C_2) \subset C_1$ is a Q -module);

(C3) $\partial\partial = 0$.

A crossed complex \mathbf{C} is called *free (projective)* if G is a free group, if C_1 is a free (projective) crossed G -module, and if each $C_k, k \geq 2$, is a free (projective)

Q -module ($Q = G/(\partial_1 C_1)$). If a crossed complex \mathbf{C} is exact, and if a group Q , given in advance, is isomorphic to the quotient $G/(\partial_1 C_1)$, then \mathbf{C} is called a *crossed resolution* of Q (a *free* resp. a *projective* crossed resolution, if \mathbf{C} is free resp. projective). Now a *morphism* $\alpha : \mathbf{C} \rightarrow \mathbf{C}'$ of *crossed complexes* consists of group homomorphisms $\alpha_0 : G \rightarrow G'$, $\alpha_k : C_k \rightarrow C'_k$, $k \geq 1$, such that $(\dots, \alpha_k, \alpha_{k-1}, \dots, \alpha_1, \alpha_0)$ provides a commutative diagram of groups which preserves all the structure.

Clearly, crossed n -fold extensions yield special examples of (exact) crossed complexes with $C_k = 0$, $k > n$. The standard example of a crossed complex is given by the sequence of relative homotopy groups of a filtered space [3], [6], [25] (“homotopy system”).

As for a given group Q any Q -module has a free (projective) resolution, from Proposition 1 we infer

PROPOSITION 2. *Any group has a free (projective) crossed resolution.*

The following is clear:

PROPOSITION 3. *Let \mathbf{C} be a free (projective) crossed complex with $Q = \text{coker}(\partial_1)$, and let \mathbf{C}' be a crossed resolution of a group Q' . Then any homomorphism $\varphi : Q \rightarrow Q'$ may be lifted to a morphism $\alpha : \mathbf{C} \rightarrow \mathbf{C}'$ of crossed complexes.*

If \mathbf{C} is a free (projective) crossed resolution of Q , denote by \mathbf{C}^n the crossed complex (for $n \geq 2$ it is a crossed n -fold extension)

$$\mathbf{C}^n : 0 \rightarrow J_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow F \rightarrow Q \rightarrow 1,$$

where $J_n = \ker(C_{n-1} \rightarrow C_{n-2})$ (with $C_0 = F$ and $C_{-1} = Q$). We shall refer to \mathbf{C}^n as a *free (projective) crossed n -fold extension* (as to \mathbf{C}^1 see Note added in proof).

PROPOSITION 3'. *Let e' be a crossed n -fold extension with $Q' = \text{coker}(\partial_1)$. Then any homomorphism $\varphi : Q \rightarrow Q'$ may be lifted to a morphism $(\sigma, \alpha, \varphi) : \mathbf{C}^n \rightarrow e'$ of crossed n -fold extensions.*

6. Homotopy

Let there be given two crossed complexes \mathbf{C}, \mathbf{C}' with $Q = \text{coker}(\partial_1)$ and $Q' = \text{coker}(\partial'_1)$; let further α and β be morphisms $\mathbf{C} \rightarrow \mathbf{C}'$ of crossed complexes.

Now a family $\Sigma = \{\Sigma_k, k \geq 0\}$ of maps $\Sigma_0 : G \rightarrow C'_1, \Sigma_k : C'_k \rightarrow C'_{k+1}, k \geq 1$, is called a *homotopy* between α and β , denoted $\Sigma : \alpha \simeq \beta$, if

(i) $\Sigma_0 : G \rightarrow C'_1$ is a (left-) derivation (crossed homomorphism) associated with β_0 , i.e. $\Sigma_0(xy) = \Sigma_0(x)(\beta_0(x)\Sigma_0(y)), x, y \in G$, such that

$$\partial_1 \Sigma_0(x) = \alpha_0(x)\beta_0(x)^{-1}, \quad x \in G,$$

(ii) $\Sigma_1 : C'_1 \rightarrow C'_2$ is a G -homomorphism, with G acting on C'_2 via α_0 (or β_0 , which yields the same action in view of (i)), such that

$$\partial_2 \Sigma_1(x) = \beta_1(x)^{-1}(\Sigma_0 \partial_1(x))^{-1} \alpha_1(x), \quad x \in C'_1,$$

(iii) for $k \geq 2$, Σ_k is a Q -homomorphism, with Q acting on the C'_k via the induced map $Q \rightarrow Q'$ (note that α and β induce the same map $Q \rightarrow Q'$ in view of (i)), such that

$$\partial_{k+1} \Sigma_k + \Sigma_{k-1} \partial_k = \alpha_k - \beta_k.$$

LEMMA 2. *Homotopy is an equivalence relation.*

PROPOSITION 4. *Let \mathbf{C} be a free (projective) crossed complex with $Q = \text{coker}(\partial_1)$, and let \mathbf{C}' be a crossed resolution of Q' ; let further $\alpha, \beta : \mathbf{C} \rightarrow \mathbf{C}'$ be morphisms of crossed complexes. If α and β induce the same homomorphism $\varphi : Q \rightarrow Q'$, there is a homotopy $\Sigma : \alpha \simeq \beta$.*

It is clear that we also have the notion of a homotopy $\Sigma : (\sigma, \alpha, \varphi) \simeq (\tau, \beta, \varphi)$ of morphisms $e \rightarrow e'$ of crossed n -fold extensions with the same *right end* $\varphi : Q \rightarrow Q'$: it is a family $(\Sigma_{n-1}, \dots, \Sigma_0)$ of maps satisfying (i), (ii) and (iii) above; thereby $\partial_n = \gamma, \Sigma_n = 0 = \partial_{n+1}, \alpha_n = \sigma, \beta_n = \tau, C_n = A$.

PROPOSITION 4'. *Let \mathbf{C}^n be a free (projective) crossed n -fold extension with $Q = \text{coker}(\partial_1)$, and let e' be a crossed n -fold extension with $Q' = \text{coker}(\partial_1)$. If $(\sigma, \alpha, \varphi)$ and (τ, β, φ) are morphisms $\mathbf{C}^n \rightarrow e'$ of crossed n -fold extension with the same right end φ , then there is a homotopy $\Sigma : (\sigma, \alpha, \varphi) \simeq (\tau, \beta, \varphi)$.*

Proofs are routine and left to the reader. If we combine the above with Proposition 3 resp. Proposition 3', we obtain

PROPOSITION 5. *The set $\text{Hom}(Q, Q')$ classifies the homotopy classes of morphisms $\mathbf{C} \rightarrow \mathbf{C}'$ resp. of morphisms $\mathbf{C}^n \rightarrow e'$ with the same right end.*

It is now clear how to introduce the notion of *homotopy equivalence* of crossed complexes, and we have the

COROLLARY. Any two free (projective) crossed resolutions of a group are homotopy equivalent.

7. Opextⁿ-groups and cohomology; the main Theorem.

Let Q be a given group, and let

$$\mathbf{C}: \cdots \rightarrow C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \rightarrow F \dashrightarrow Q$$

be a free (projective) crossed resolution of Q . For any Q -module A , consider the complex (the arrows are the obvious maps)

$$\text{Hom}(\mathbf{C}, A): \text{Der}(F, A) \rightarrow \text{Hom}_F(C_1, A) \rightarrow \text{Hom}_Q(C_2, A) \rightarrow \cdots$$

(For a group G and a G -module A , “Der(G , A)” denotes the Abelian group of derivations from G to A .) Its cohomology groups are as follows:

PROPOSITION 6. $H^0(\text{Hom}(\mathbf{C}, A)) = \text{Der}(Q, A)$, $H^q(\text{Hom}(\mathbf{C}, A)) = H^{q+1}(Q, A)$, $q \geq 1$.

Proof. Assume for convenience that, in case \mathbf{C} is a proper projective crossed resolution, the crossed F -module C_1 is free. The case of a proper projective crossed F -module C_1 is left as an exercise. Now the crossed complex \mathbf{C} may be transformed into the complex

$$\hat{\mathbf{C}}: \cdots \rightarrow C_k \rightarrow \cdots \rightarrow C_2 \rightarrow C_1^{Ab} \rightarrow \mathbb{Z}Q \otimes_F IF,$$

where $C_2 \rightarrow C_1^{Ab}$ is the obvious map, and where $C_1^{Ab} \rightarrow \mathbb{Z}Q \otimes_F IF$ is given by the rule $x[C_1, C_1] \mapsto 1 \otimes (\partial x - 1)$, $x \in C_1$. (Here “ IG ” denotes the augmentation ideal of a group G .) By Proposition 2, C_1^{Ab} is a free Q -module, and the cokernel of $C_2 \rightarrow C_1^{Ab}$ is the relation module N^{Ab} , where $N = \ker(F \rightarrow Q)$. Hence $\hat{\mathbf{C}}$ is a free (projective) resolution of IQ . Applying the functor $\text{Hom}_Q(-, A)$ to $\hat{\mathbf{C}}$ yields a complex canonically isomorphic to $\text{Hom}(\mathbf{C}, A)$ whence the cohomology of $\text{Hom}(\mathbf{C}, A)$ is as stated. Q.E.D.

The fact that $H^2(Q, A)$ is $H^1(\text{Hom}(\mathbf{C}, A))$ was already proved by MacLane [16, Theorem A’].

We now divide the crossed n -fold extensions of A by Q ($n \geq 1$) into classes as follows: Two crossed n -fold extensions e, e' of A by Q are related if there is a morphism $(1, \alpha, 1): e \rightarrow e'$ of crossed n -fold extensions; this relation generates an equivalence relation which shall be denoted by “ \equiv ”. The equivalence class of e , also called *similarity class*, is to be denoted by $[e]$.

We next consider a crossed n -fold extension e of A by Q . If \mathbf{C} is a projective crossed resolution of Q , it follows from Proposition 3 that the identity map of Q lifts to

$$\begin{array}{ccccccccccc} \mathbf{C}: & \cdots & \rightarrow & C_n & \rightarrow & C_{n-1} & \rightarrow & \cdots & \rightarrow & C_1 & \rightarrow & F & \rightarrow & Q & \rightarrow & 1 \\ & & & \downarrow \zeta & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \parallel & & \\ e: & 0 & \rightarrow & A & \rightarrow & A_{n-1} & \rightarrow & \cdots & \rightarrow & A_1 & \rightarrow & G & \rightarrow & Q & \rightarrow & 1 \end{array}$$

In view of the above, ζ represents a class $[\zeta] \in H^{n+1}(Q, A)$. If \mathbf{C} is replaced by \mathbf{C}^n (introduced in §5), the above induces a morphism $(\nu, \alpha, 1) : \mathbf{C}^n \rightarrow e$ of crossed n -fold extensions. Now, for $n \geq 2$, the coequaliser $C_{n-1,\nu}$, say, of $J_n \xrightarrow[\nu]{i} A \times C_{n-1}$, where i denotes the inclusion $J_n \rightarrow C_{n-1}$, yields the crossed n -fold extension

$$\nu\mathbf{C}^n : 0 \rightarrow A \rightarrow C_{n-1,\nu} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_1 \rightarrow F \rightarrow Q \rightarrow 1,$$

with $C_{n-1,\nu} \rightarrow C_{n-2}$ the obvious map. If $n = 1$, the coequaliser $C_{0,\nu}$ of $J_1 \rightrightarrows A]F$ ($J_1 = N = \ker(F \rightarrow Q)$) yields the ordinary group extension

$$\nu\mathbf{C}^1 : 0 \rightarrow A \rightarrow C_{0,\nu} \rightarrow Q \rightarrow 1;$$

here “ $]F$ ” denotes the semi-direct product. Clearly, there is a morphism $(1, \beta, 1) : \mathbf{C}^n \rightarrow e$ of crossed n -fold extensions; hence

PROPOSITION 7. *Each equivalence class of crossed n -fold extensions of A by Q has a representative of the form $\nu\mathbf{C}^n$.*

It is now clear that the Abelian group $\text{Hom}_F(J_n, A)$ ($= \text{Hom}_Q(J_n, A)$, if $n \geq 2$) maps onto the classes of crossed n -fold extensions of A by Q by rule $\nu \mapsto \nu\mathbf{C}^n$. Consequently, these classes constitute a set, denoted henceforth by $\text{Opext}^n(Q, A)$.

Given two crossed n -fold extensions e, e' of A by Q , it is routine to construct their “Baer- sum” $e + e'$. We refrain from writing down details. Moreover, the Baer- sum induces a sum on similarity classes, and the surjection $\text{Hom}_F(J_n, A) \rightarrow \text{Opext}^n(Q, A)$ is a homomorphism with respect to the Baer- sum, i.e. $(\mu + \nu)\mathbf{C}^n \equiv \mu\mathbf{C}^n + \nu\mathbf{C}^n$, $\mu, \nu : J_n \rightarrow A$ operator maps. Consequently, under the Baer- sum, $\text{Opext}^n(Q, A)$ is an Abelian group, with zero element $0\mathbf{C}^n$, i.e. the image of the zero map $J_n \rightarrow A$, and $\text{Hom}_F(J_n, A) \rightarrow \text{Opext}^n(Q, A)$ is an epimorphism of Abelian groups.

LEMMA 3. Let $\nu : J_n \rightarrow A$, $n \geq 1$, be an operator map which may be extended over C_{n-1} to

- (i) a derivation $F \rightarrow A$, if $n = 1$,
- (ii) an F -map $C_1 \rightarrow A$, if $n = 2$, and
- (iii) a Q -map $C_{n-1} \rightarrow A$, if $n \geq 3$.

Then the extension

$$E: 0 \rightarrow A \rightarrow C_{n-1, \nu} \rightarrow J_{n-1} \rightarrow 1$$

($J_1 = N$, $J_0 = Q$) splits, i.e. there is a section $J_{n-1} \rightarrow C_{n-1, \nu}$ which is a group homomorphism, if $n = 1$, an F -homomorphism, if $n = 2$, and a Q -homomorphism, if $n \geq 3$.

The proof is straightforward.

If, given an operator map $\nu : J_n \rightarrow A$, $n \geq 2$, the extension E splits (as in Lemma 3), there is a morphism $(1, \alpha, 1) : \nu \mathbf{C}^n \rightarrow \mathbf{0}$ of crossed n -fold extensions, where $\mathbf{0}$ denotes

$$\mathbf{0}: 0 \rightarrow A \xrightarrow{=} A \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow Q \xrightarrow{=} Q \rightarrow 1,$$

whence $\nu \mathbf{C}^n$ and $\mathbf{0}$ are equivalent; since $\mathbf{0}$ represents $0 \in \text{Opext}^n(Q, A)$, so does $\nu \mathbf{C}^n$. By Proposition 6, the cokernel of $\text{Hom}_F(C_{n-1}, A) \rightarrow \text{Hom}_Q(J_n, A)$ ($\text{Hom}_F(C_{n-1}, A) = \text{Hom}_Q(C_{n-1}, A)$ if $n \geq 3$) is the cohomology group $H^{n+1}(Q, A)$. It follows from Lemma 3 that for $n \geq 2$ the rule $\nu \mapsto \nu \mathbf{C}^n$, $\nu : J_n \rightarrow A$ an operator map, induces an epimorphism $\Phi : H^{n+1}(Q, A) \rightarrow \text{Opext}^n(Q, A)$ of Abelian groups; this also follows for $n = 1$, as $H^2(Q, A)$ is the cokernel of $\text{Der}(F, A) \rightarrow \text{Hom}_F(N, A)$.

The main Theorem. *The map Φ is an isomorphism of Abelian groups. In other words, the classes of crossed n -fold extensions of A by Q constitute an Abelian group $\text{Opext}^n(Q, A)$ naturally isomorphic to the cohomology group $H^{n+1}(Q, A)$. The group composition is given by the Baer-sum. The zero element of this group is the class of the crossed n -fold extension $\mathbf{0}$, whereas the inverse of the class of*

$$e: 0 \rightarrow A \xrightarrow{\gamma} C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow G \rightarrow Q \rightarrow 1$$

is the class of

$$-e: 0 \rightarrow A \xrightarrow{(-\gamma)} C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow G \rightarrow Q \rightarrow 1.$$

8. The proof of the main Theorem

We have to prove that $\Phi : H^{n+1}(Q, A) \rightarrow \text{Opext}^n(Q, A)$, $n \geq 2$, is injective (the case $n = 1$ is classical). This amounts to show that if $\mu\mathbf{C}^n \equiv \nu\mathbf{C}^n$, with \mathbf{C}^n a free (projective) crossed n -fold extension and μ, ν operator maps $J_n \rightarrow A$, then $\mu - \nu$ extends over C_{n-1} as in (ii) resp. (iii) of Lemma 3. We argue as follows:

Since $\mu\mathbf{C}^n \equiv \nu\mathbf{C}^n$, there are crossed n -fold extensions e_1, e_2, \dots, e_m of A by Q , with $e_m = \nu\mathbf{C}^n$, and morphisms $(1, \alpha^1, 1) : \mu\mathbf{C}^n \rightarrow e_1$, $(1, \alpha^2, 1) : e_2 \rightarrow e_1$, $(1, \alpha^3, 1) : e_2 \rightarrow e_3$, $(1, \alpha^4, 1) : e_4 \rightarrow e_3$, and so forth. By construction, there are morphisms $(\mu, \beta^0, 1) : \mathbf{C}^n \rightarrow \mu\mathbf{C}^n$ and $(\nu, \beta^m, 1) : \mathbf{C}^n \rightarrow \nu\mathbf{C}^n$. Moreover, it follows from Proposition 3' that for $1 \leq k < m$ the identity map of Q lifts to a morphism $(\nu^k, \beta^k, 1) : \mathbf{C}^n \rightarrow e_k$ of crossed n -fold extensions. We may assume that m is even (otherwise we add the identity morphism $e_{m+1} \rightarrow e_m$). It follows from Proposition 4' that the morphisms $(\mu, \alpha^1\beta^0, 1)$ and $(\nu^2, \alpha^2\beta^2, 1) : \mathbf{C}^n \rightarrow e_1$ are homotopic; likewise, $(\nu^2, \alpha^3\beta^2, 1)$ and $(\nu^4, \alpha^4\beta^4, 1) : \mathbf{C}^n \rightarrow e_3$ are homotopic also, and so forth. We ultimately arrive at $(\nu^{m-2}, \alpha^{m-1}\beta^{m-2}, 1)$ and $(\nu, \alpha^m\beta^m, 1) : \mathbf{C}^n \rightarrow e_{m-1}$ which again are homotopic. Now $\mu - \nu = \mu - \nu^2 + \nu^2 - \nu^4 + \dots + \nu^{m-2} - \nu$ extends over C_{n-1} as desired.

The proofs of naturality of Φ and of the assertion as to the inverse of a class $[e] \in \text{Opext}^n(Q, A)$ are left to the reader. Q.E.D.

9. The (inhomogenous) crossed standard resolution

The following section will be needed in [13], [14] and [15]; it will provide the bridge between our interpretation of group cohomology and the classical description in terms of cocycles.

Let Q be a group, and let $(Q^*; Q^* \times Q^*)$ be its standard presentation ($Q^* = Q \setminus \{1\}$); hence the relator $[q_1, q_2]$, $q_1, q_2 \in Q^*$, corresponds to the word $^{[q_1]}[q_2][q_1q_2]^{-1}$. Next, let F be the free group on Q^* , and let C be the free crossed F -module with basis $Q^* \times Q^*$; it is then clear that the elements

$$(*) \quad ^{[q_1]}[q_2, q_3][q_1, q_2q_3][q_1q_2, q_3]^{-1}[q_1, q_2]^{-1}, \quad q_1, q_2, q_3 \in Q^*,$$

lie in the kernel of $\partial : C \rightarrow F$. By Proposition 1, we have the Q -free presentation

$$0 \rightarrow \ker(\partial) \rightarrow C^{Ab} \rightarrow N^{Ab} \rightarrow 0$$

of N^{Ab} . Since C^{Ab} is the corresponding term of the (ordinary) inhomogenous standard resolution of the integers where it is known that the elements (*) generate the kernel of $C^{Ab} \rightarrow N^{Ab}$ (in the operator sense), it follows that the

elements (*) generate $\ker(\partial)$. If we now “splice” our free crossed module $C \rightarrow F$ with the remaining part of the inhomogenous standard resolution of the integers (this is a resolution of $\ker(\partial)$), we obtain a free crossed resolution of Q , henceforth called the (*inhomogenous*) *crossed standard resolution* of Q . Now, if C_2 is the free Q -module on $Q^* \times Q^* \times Q^*$, and if $\partial_2 : C_2 \rightarrow C_1$ ($C_1 = C$) is given by sending $[q_1, q_2, q_3]$ to (*), the kernel of ∂_2 is generated by the elements (written multiplicatively)

$$(**) \quad q_1[q_2, q_3, q_4][q_1q_2, q_3, q_4]^{-1}[q_1, q_2q_3, q_4][q_1, q_2, q_3q_4]^{-1}[q_1, q_2, q_3],$$

$$q_1, q_2, q_3, q_4 \in Q^*.$$

10. An illustration

If e^ψ is the crossed 2-fold extension obtained from an abstract Q -kernel $\psi : Q \rightarrow \text{Out}(K)$ (§3), we may lift the identity map of Q to

$$\begin{array}{ccccccc} \mathbf{C} : & \cdots & \rightarrow & C_2 & \rightarrow & C_1 & \rightarrow & F & \rightarrow & Q & \rightarrow & 1 \\ & & & \downarrow \zeta & & \downarrow & & \downarrow & & \Downarrow & & \\ e^\psi : & 0 & \rightarrow & Z & \rightarrow & K & \rightarrow & G^\psi & \rightarrow & Q & \rightarrow & 1, \end{array}$$

where \mathbf{C} is the crossed standard resolution. This yields an operator map $\zeta : C_2 \rightarrow Z$, which, in view of (**), is a 3-cocycle; it is the Eilenberg–MacLane cocycle [7]. This, together with our main Theorem shows that $[e^\psi] \in \text{Opext}^2(Q, Z)$ is the Eilenberg–MacLane class of (K, ψ) . Eilenberg–MacLane’s extendibility criterion is now recovered by the following

THEOREM. *Let $e : 0 \rightarrow A \rightarrow K \xrightarrow{\partial} G \rightarrow Q \rightarrow 1$ be a crossed 2-fold extension. There is a group extension $1 \rightarrow K \xrightarrow{j} E \rightarrow Q \rightarrow 1$ together with a morphism $(1, \alpha) : (K, E, j) \rightarrow (K, G, \partial)$ of crossed modules inducing the identity map of Q if and only if $[e] = 0 \in \text{Opext}^2(Q, A)$.*

This generalises Eilenberg–MacLane’s extendibility criterion, since A need not coincide with the center of K .

Proof. We show that the condition suffices. To this end, let $\mathbf{C}^2 : 0 \rightarrow J \rightarrow C \rightarrow F \rightarrow Q \rightarrow 1$ be a free crossed 2-fold extension and let $(\nu, \beta_1, \beta_0, 1) : \mathbf{C}^2 \rightarrow e$ be a lifting of the identity map of Q . Since $[e] = 0 \in \text{Opext}^2(Q, A)$, ν extends over C as in (ii) of Lemma 3; it follows that there is a morphism $(\beta, \beta_0) : (N, F, i) \rightarrow (K, G, \partial)$ of crossed modules, where $N = \ker(F \rightarrow Q)$. Now the coequaliser E of $N \xrightarrow[\beta]{i} K] F$ (where F acts on K via β_0) yields the required extension. Q.E.D.

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Received November 14, 1978/October 16, 1979

Note added in Proof: Strictly speaking, the crossed complex \mathbf{C}^1 in §5 is not a crossed 1-fold extension since $J_1 = N = (\ker (F \rightarrow Q))$ is not a Q -module; however, \mathbf{C}^1 may always be replaced by

$$O \rightarrow N^{\text{Ab}} \rightarrow F/[N, N] \rightarrow Q \rightarrow 1.$$