Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 55 (1980)

Artikel: Free actions of finite groups on finite CW complexes.

Autor: Wojtkowiak, Zdzislaw

DOI: https://doi.org/10.5169/seals-42372

Comment. Math. Helvetici 55 (1980) 225–232 Birkhäuser Verlag, Basel

Free actions of finite groups on finite CW complexes

ZDZISŁAW WOJTKOWIAK

1. In this paper we examine a nilpotent free action of a finite group G on finite CW complexes. We restrict our attention to CW complexes which are "relatively prime to the order of G" (see definition below).

DEFINITION 1. Let n be a natural number. We say that $(n; X) = 1$ if X is simply connected, $X_{(n)} \sim \bigvee_{i=1}^{k} S^{n_i}_{(n)}$ and $\pi_i(X) \otimes Z_{(n)} \to \tilde{H}_i(X) \otimes Z_{(n)}$ is an isomorphism for $i \leq \dim X$.

PROPOSITION 1. *If X is a finite complex then $(p; S^N X) = 1$ for N sufficiently large and all but a finite number of primes.*

Proof. SX has a decomposition into Moore spaces $M(Z/q^n, k)$ and $S^k = M(Z, k)$ (see [1]). After inverting a finite number of primes torsion parts vanish. Moreover the Hurewicz homomorphism $\pi_i(S^N X) \otimes \mathbf{Q} \to \tilde{H}_i(S^N X) \otimes \mathbf{Q}$ is an isomorphism for $i \leq \dim S^N X$ and N sufficiently large. As both modules are finitely generated Z-modules the Hurewicz will be an isomorphism after inverting a finite number of primes. If we represent the homology classes by a bouquet of spheres, we shall obtain a required homotopy equivalence.

Now we prove an existence theorem. For the definition of an admissible chain complex and of a O-admissible chain map, which are necessary to understand the theorem, see [6] p. 131 and 132.

THEOREM 2. *Let A_* be an admissible finitely generated chain complex of free $Z[G]$-modules with a basis chosen. Let X be a finite* CW *complex with one O-cell and without 1-cells. Suppose that $h: A_* \to C_*(X)$ is a O-admissible Z-chain homotopy equivalence, G acts trivially on $H_*(A_*)$, $A_i = 0$ for $i > \dim X$ and $(n, X) = 1$, where n is the order of G. Then there exists a simply connected* CW *complex Y with a free cellular action of G such that $C_*(Y) = A_*$ as $Z[G]$-modules and there exists a cellular homotopy equivalence $f: Y \to X$ such that $C_*(f) = h$.*

Proof. The admissibility of A_* implies that there is a 2-dimensional complex Y^2 with a free action of G. It follows from our assumption on X that there exists a cellular map $f^2: Y^2 \to X$. Now, according to [3] Prop. 4.3, an obstruction σ to extendibility of our construction lies in the group G^{n+1} $(A_*;\ \ker(\pi_n(X^n) \to H_n(X^n))$. σ is an n-torsion element since G acts trivially on $H_*(A_*)$ (see [3] p. 390). On the other hand, since $(n, X) = 1$ the group $\ker(\pi_n(X^n) \to H_n(X^n))$ consists only of the torsion relatively prime to n. Therefore the obstruction vanishes.

An example constructed in [7] shows that Theorem 2 does not hold without the assumption $(n, X) = 1$.

2. In this rather long section we investigate the classification problem. These investigations were inspired by Theorem 5.2 of [5] proved by C. B. Thomas.

DEFINITION 2. Let Y be a CW complex with a finite fundamental group and let $\theta: \pi_1(Y) \to G$ be an isomorphism. We denote such a pair by (Y, θ). We say that two such pairs (Y_1, θ_1) and (Y_2, θ_2) are equivalent if there is a homotopy equivalence $h: Y_1 \to Y_2$ such that $\theta_2 \circ \pi_1(h) = \theta_1$.

Now we prove our fundamental lemma.

LEMMA 3. *Suppose that A_* and X are such as in Theorem 2 and $A_0 = Z[G]$. Let (Y_1, θ_1) and (Y_2, θ_2) be two pairs such that $C_*(\tilde{Y}_i) = A_*$ as complexes of $Z[G]$-modules. Suppose that there are homotopy equivalences $f_1: \tilde{Y}_1 \to X$ and $f_2: \tilde{Y}_2 \to X$ (where $\tilde{Y}_i$ is the universal cover) such that $H_*(f_1) = H_*(f_2)$. Then there exists a homotopy equivalence $h: Y_1 \to Y_2$ such that*

(i) $\theta_2 \circ \pi_1(h) = \theta_1$.

(ii) *$f_2 \circ \tilde{h} \sim f_1$ ($\tilde{h}$ is an equivariant map between the universal covers induced by h).*

Proof. It follows from the assumptions on A_* that Y_1 and Y_2 are nilpotent spaces. Let $p_i: \tilde{Y}_i \to Y_i$, $i = 1, 2$, be the natural projections. Set $f = f_2^{-1} \circ f_1$. We define localizations $h_{[1/n]} = (p_2)_{[1/n]} \circ f_{[1/n]} \circ (p_1)^{-1}_{[1/n]}$ and $h_0 = (p_2)_0 \circ f_0 \circ (p_1)_0^{-1}$. If $|\pi_1(Y_i)| = n$ and Y_i^0 is one point then the localized space $(Y_i)_{(n)}$ can be obtained by successive localizations of cells. Since

$$Y_i^2 = (\bigvee S^1) \cup C(f_i^1) \cup \cdots \cup C(f_i^k)$$

we set

$$(Y_i^2)_{(n)} = (\bigvee S^1_{(n)}) \cup C(f_i^1)_{(n)} \cup \cdots \cup C(f_i^k)_{(n)}.$$

Suppose that we have built $(Y_i^j)_{(n)}$. The condition $(n;\ \tilde{Y}_i) = 1$ implies that

$\pi_r((Y_i^r)_{(n)}) = \pi_r((\tilde{Y}_i^r)_{(n)}) = H_r((\tilde{Y}_i^r)_{(n)})$. Therefore the attaching map α of an $r+1$-local cell is determined by $H_*(\tilde{\alpha})$, where $\tilde{\alpha}$ is lifting of α. $H_r((\tilde{Y}_i^r)_{(n)}) = \ker(A_r \otimes Z_{(n)} \to A_{r-1} \otimes Z_{(n)})$ for $r \geq 2$ since cells of Y_i^r correspond to local cells of $(Y_i^r)_{(n)}$ and, consequently, cells of $\tilde{Y}_i^r$ correspond to local cells of $(\tilde{Y}_i^r)_{(n)}$. Hence the complex A_* determines $(Y_1)_{(n)}$ and $(Y_2)_{(n)}$. Therefore there is a map $h_{(n)}: (Y_1)_{(n)} \to (Y_2)_{(n)}$ such that the lifting $\tilde{h}_{(n)}$ induces an identity on $H_*(A_* \otimes Z_{(n)})$. The maps $(h_{(n)})_0$ and h_0 induces the same map on homology and therefore they are homotopic. It follows from [4] Cor. 5.13 that there is a map $h: Y_1 \to Y_2$. It is easy to see that h satisfies (i) and (ii).

Now we formulate our classification problem. Let A_* and X be fixed and such as in Theorem 2, and $A_0 = Z[G]$. We investigate the equivalence classes of pairs (Y, θ) such that:

(i) $\tilde{Y} \sim X$.

(ii) $C_*(\tilde{Y})$ is $Z[G]$-homotopy equivalent to A_* and the homotopy equivalence is O-admissible.

(iii) Y is finite.

Remark. If Y is a finite complex with a fundamental group G such that $\tilde{Y} \sim X$ then Y is homotopy equivalent to a complex Z such that $Z^0 = *$ and $\dim Z = \dim X$. Hence our assumptions on A_* are not restrictive.

Let $M(A_*, X)$ be the set of equivalence classes of such (Y, θ). Let $\mathscr{A}$ be the set of O-admissible chain homotopy equivalences of A_* and let $\operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)$ be the set of automorphisms of $\tilde{H}_*(X)$ which preserve gradation. Let us fix a O-admissible chain homotopy equivalence $h: A_* \to C_*(X)$. Define a map $\beta: \mathscr{A} \to \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)$ by $\beta(r) = \tilde{H}_*(h) \circ \tilde{H}_*(r) \circ \tilde{H}_*(h)^{-1}$. Let $\varepsilon(X)$ be the set of all homotopy equivalences of X and $\alpha: \varepsilon(X) \to \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)$ a natural map. In the set $\operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)$ we define the following relation. We say that f_1 and f_2 are equivalent iff there exist $r \in \mathscr{A}$ and $e \in \varepsilon(X)$ such that $f_2 = \alpha(e) \circ f_1 \circ \beta(r^{-1})$. It is an equivalence relation and we denote the set of equivalence classes by $\operatorname{im} \alpha \backslash \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)/\operatorname{im} \beta$. Now we shall formulate our classification theorem.

THEOREM 4. *Let A_* and X be such as in Theorem 2, $A_0 = Z[G]$ and let $h: A_* \to C_*(X)$ be a O-admissible Z-chain homotopy equivalence. Then we have a bijection:*

$$\varphi: \operatorname{im} \alpha \backslash \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)/\operatorname{im} \beta \to M(A_*; X).$$

Proof. If $s \in \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)$ then there is a O-admissible chain map $s_1: C_*(X) \to C_*(X)$ such that $\tilde{H}_*(s_1) = s$. It follows from Theorem 2 of [6] that there exists a CW complex X_1 and a cellular map $\bar{s}_1: X_1 \to X$ such that $C_*(X_1) = C_*(X)$ and $C_*(\bar{s}_1) = s_1$. We can consider a map $h: A_* \to C_*(X)$ as a map $h_1: A_* \to$

$C_*(X_1)$. By Theorem 2 there exists a finite CW complex $\tilde{Y}_1$ with a free action of G and a map $i_1: \tilde{Y}_1 \to X_1$ such that $C_*(i_1) = C_*(h_1)$. The action of G on $\tilde{Y}_1$ determines an isomorphism $\theta_1: \pi_1(\tilde{Y}_1/G) \to G$. Hence we obtain a pair $(Y_1 = \tilde{Y}_1/G; \theta_1)$. Let $s_2: C_*(X) \to C_*(X)$ also satisfy $\tilde{H}_*(s_2) = s$ and let $\tilde{Y}_2$ be obtained in the same way as $\tilde{Y}_1$ but using s_2. Applying Lemma 3 to $i_1: \tilde{Y}_1 \to X_1$ and $k = \bar{s}_1^{-1} \circ \bar{s}_2 \circ i_2: \tilde{Y}_2 \to X_1$ we see that the pairs (Y_1, θ_1) and (Y_2, θ_2) are equivalent. Thus we have a well defined map $\Phi: \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X) \to M(A_*, X)$.

Let $(Y_1, \theta_1) = \Phi(f_1)$ and $(Y_2, \theta_2) = \Phi(f_2)$. This means that there are CW complex X_1 and X_2 such that $C_*(X_i) = C_*(X)$ $i = 1, 2$. There are also maps $F_i: X_i \to X$ and $i_i: \tilde{Y}_i \to X_i$ such that $\tilde{H}_*(F_i) = f_i$ and $H_*(i_i) = h$ for $i = 1, 2$.

Suppose that the pairs (Y_1, θ_1) and (Y_2, θ_2) are equivalent. Let $\rho: \tilde{Y}_1 \to \tilde{Y}_2$ be G-equivariant cellular homotopy equivalence. Set $C_*(\rho) = r$ and $e = F_2 \circ i_2 \circ \rho \circ i_1^{-1} \circ F_1^{-1}$. Then we have

$$
\begin{aligned}
&\alpha(e) \circ f_1 \circ \beta(r^{-1}) \\
&= H_*(F_2) \circ H_*(i_2) \circ H_*(\rho) \circ H_*(i_1)^{-1} \circ H_*(F_1)^{-1} \circ f_1 \circ H_*(h) \circ H_*(r)^{-1} \circ H_*(h)^{-1} \\
&= f_2 \circ H_*(h) \circ H_*(r) \circ H_*(h)^{-1} \circ f_1^{-1} \circ f_1 \circ H_*(h) \circ H_*(r)^{-1} \circ H_*(h)^{-1} = f_2.
\end{aligned}
$$

Now we show that $\Phi(f) = \Phi(\alpha(e) \circ f \circ \beta(r^{-1}))$. Let $\Phi(f) = (Y_1, \theta_1)$ and $\Phi(\alpha(e) \circ f \circ \beta(r^{-1})) = (Y_2, \theta_2)$. Let us apply Theorem 2 of [6] to Y_2 and $r: A_* \to A_*$. Then we obtain a chain complex Y_3 and a map $\rho: \tilde{Y}_3 \to \tilde{Y}_2$. Consider two maps $i_1: \tilde{Y}_1 \to X_1$ and $k = F_1^{-1} \circ e^{-1} \circ F_2 \circ i_2 \circ \rho$. It is easy to check that $H_*(k) = H_*(i_1)$. Therefore it follows from Lemma 3 that $(Y_1, \theta_1) = (Y_2, \theta_2)$. Hence the map Φ defines φ.

Now we show that φ is onto. Let $(Y, \theta) \in M(A_*, X)$. Applying Theorem 2 of [6] to the map $h^{-1}: C_*(X) \to C_*(\tilde{Y}) = A_*$ we obtain a CW complex X_1 and a cellular map $t: X_1 \to \tilde{Y}$. Let $g: X_1 \to X$ be a homotopy equivalence. Since $C_*(X_1) = C_*(X)$, the map g determines an element $s = \tilde{H}_*(g) \in \operatorname{Aut} \operatorname{gr} \tilde{H}_*(X)$. It is clear that $\varphi(s) = (Y, \theta)$.

COROLLARY 5. *Let $X = \bigvee_{i \in I} S^{n_i}$ and $(n, X) = 1$. Suppose that (Y_i, θ_i), $i = 1, 2$ are nilpotent spaces such that $\tilde{Y}_i \sim X$ for $i = 1, 2$. Then the pairs (Y_1, θ_1) and (Y_2, θ_2) are equivalent iff there exists a O-admissible $Z[G]$-chain homotopy equivalence $h: C_*(\tilde{Y}_1) \to C_*(\tilde{Y}_2)$.*

Proof. In this case $\operatorname{im} \alpha = \operatorname{Aut} \tilde{H}_*(X)$. Hence our result follows.

DEFINITION 3. Let $r: G \to G$ be an automorphism and let $f: A_* \to B_*$ be a Z-chain map between complexes of free $Z[G]$-modules. We say that f has type (r) if $f(g \cdot x) = r(g) \cdot f(x)$ for all $g \in G$.

Let $\mathscr{A}_G$ denote the set of all O-admissible chain homotopy equivalences of A_* of all types (r). We shall now investigate the homotopy types of CW-complexes Y satisfying

(i) Y is finite, $\tilde{Y} \sim X$, $\pi_1(Y) \approx G$ and $(n, X) = 1$.

(ii) $C_*(\tilde{Y})$ is Z-chain homotopy equivalent to A_*, where A_* is such as in Theorem 4, and the homotopy equivalence is O-admissible and has type (r) for some $r \in \mathrm{Aut}\,(G)$.

Let $H(A_*, X)$ be the set of homotopy types of such Y. Then we have:

THEOREM 6. *If the assumptions of Theorem* 4 *hold then there is a map*

$$\varphi_1 : \mathrm{im}\,\alpha \backslash \mathrm{Aut}\,\mathrm{gr}\,\tilde{H}_*(X)/\mathrm{im}\,\beta' \to H(A_*, X)$$

which is a bijection.

$\beta' : \mathscr{A}_G \to \mathrm{Aut}\,\mathrm{gr}\,\tilde{H}_*(X)$ is defined in a similar way as β. The proof is similar to that of Theorem 4.

3. In the special case when G is a cyclic group of prime order p our classification is much more effective. Any chain complex $C_*(\tilde{Y})$ is chain homotopy equivalent to a "canonical" one. We shall now prove this fact. Consider the following complexes:

$$0 \longrightarrow Z[Z/p] \xrightarrow{e-g^{n_1}} Z[Z/p] \xrightarrow{N} \cdots \xrightarrow{e-g^{n_2}} Z[Z/p] \xrightarrow{N} Z[Z/p] \xrightarrow{e-g^{n_1}} Z[Z/p] \longrightarrow 0,$$

where

$$(n_k ; p) = 1 \quad \text{and} \quad N = e + g + \cdots + g^{p-1}, \tag{1}$$

and

$$0 \longrightarrow Z[Z/p] \xrightarrow{1+g+\cdots+g^k} Z[Z/p] \longrightarrow 0,$$

where

$$(k ; p) = 1. \tag{2}$$

DEFINITION 4. A chain complex A_* of $Z[Z/p]$-modules is called elementary if A_* is a finite direct sum of complexes of the form (1) and (2).

LEMMA 7. *Let C_* be a chain complex of projective $Z[Z/p]$ modules such that:*

(i) *$H_*(C_*)$ is a finitely generated trivial Z/p-module without p-torsion.*
(ii) *$H_*(C_* \otimes_{\mathbf{Z}[\mathbf{Z}/p]} \mathbf{Z})$ is finitely generated.*

Then C_* is chain homotopy equivalent to an elementary complex A_*.

Proof. Let

$$B_* : \cdots \longrightarrow Z[Z/p] \xrightarrow{e-g} Z[Z/p] \xrightarrow{N} Z[Z/p] \xrightarrow{e-g} Z[Z/p] \longrightarrow 0$$

be the standard resolution of Z. Consider a chain complex X_* with

$$X_n = \sum_{p+q=n} \left(C_q \underset{Z[Z/p]}{\otimes} B_p \right)$$

and with a filtration

$$(F_p X)_n = \sum_{i \le p} \left(C_{n-i} \underset{Z[Z/p]}{\otimes} B_i \right).$$

The associated spectral sequence converges to $H_*(C_* \otimes_{\mathbf{Z}[\mathbf{Z}/p]} \mathbf{Z})$ (see [2] XI). In the cohomological spectral sequence the multiplication by a generator of $H^2(Z/p; Z)$ is an isomorphism. Therefore the differentials in the cohomological as well as in the homological spectral sequence are periodic. It follows from (ii) that $E_\infty^{i,q-i} = 0$ for q large enough. Hence any $x \in E_1^{i,q-i}$ is a boundary or $d_r(x) \neq 0$. Let $x = x_1 \otimes e_i \in E_1^{i,q-i} = H_{q-i}(C_*) \otimes_{Z[Z/p]} B_i$, where x_1 is a generator of a cyclic summand of an infinite order. Let i be odd and $d_k x = 0$ for $k < r$ and $d_r x \neq 0$. Then r is even. Let $a = \sum_{i=0}^{i} c_{q-i} \otimes e$ be a representative of x such that $da \in (F_{i-r}X)_{q-1}$. Hence a part of da which belongs to $F_i X$ vanishes i.e. $\partial c_{q-i} = 0$, $tc_{q-i} + \partial c_{q-i+1} = 0$, $Nc_{q-1-i+2} + \partial c_{q-i+2} = 0 \cdots Nc_{q-1-(i-r+2)} + \partial c_{q-(i-r+2)} = 0, tc_{q-1-(i-r+1)} + \partial c_{q-(i-r+1)} = 0$, where $t = e - g$. Setting $f_k(e) = c_k$ we define a chain map from

$$0 \longrightarrow Z[Z/p] \xrightarrow{e-g} \cdots \xrightarrow{e-g} Z[Z/p] \longrightarrow 0 \quad \text{into} \quad C_*. \tag{$*$}$$

On the first non-trivial homology group this map is an inclusion onto a direct summand and on the second one it is a multiplication by l relatively prime to p. Replacing the last differential $e-g$ by $e-g^k$ in $(*)$ we may assume that $l \equiv 1(p)$. We have that $f(e) = c_{q-1-(i-r)} = c$ and $Nc = 1 \cdot g_1 = (1 + p \cdot k) \cdot g_1$, where g_1 is a generator of a cyclic summand of an infinite order which is a boundary in $E_r^{*,*}$. If we set $f(e)=c-v\cdot g_1$ then the map induced on the homology will be an inclusion onto a direct summand. Performing the same construction for all generators of $H_*(X)$ which bound in $E_s^{i,q-i}$, we obtain a map from an elementary complex into C_*. This map will be an isomorphism onto torsion-free part since half of the generators of $H_*(C_*)$ are boundaries and for half of the generators we have $d_r x\neq 0$. If $Z/k\subset H_*(C_*)$ then there exists a chain map from

$$0\longrightarrow Z[Z/p]\xrightarrow{e+g+\cdots+g^k} Z[Z/p]\longrightarrow 0$$

into C_* which is an inclusion onto Z/k. This finishes the proof.

The next lemma is an easy exercise.

LEMMA 8. *Let A_* be an elementary complex, C_* be as in Lemma 7, $H_0(C_*)=Z$ and $H_1(C_*)=0$. Let $f: A_*\to C_*$ be a chain homotopy equivalence such that $H_0(f)[e]=[p]$, where $p\in C_0$ is an element of a base. Then f is chain homotopic to a O-admissible map.*

COROLLARY 9. *Let Y be a CW complex such that $\pi_1(Y)=Z/p$, $H_*(Y)$ is finitely generated and $H_*(\tilde{Y})$ is a finitely generated trivial $\pi_1(Y)$-module without p-torsion. Then Y is homotopy equivalent to a finite complex.*

It follows from [6] Theorem 2, Lemmas 7 and 8 and the fact that A_* is admissible.

The corollary and also a much stronger result follow immediately from Theorem A of the Mislin paper "Wall's obstruction for nilpotent spaces" Topology *14*, (1975) 311–317.

Now we give an example which illustrates Theorem 4 and 6. Let $\alpha\in\pi_{17}(S^{14})$ be an element of order 3. Set $X=S^{11}\vee S^{14}\cup_\alpha D^{18}$. Then we have $(p,X)=1$ for $p>5$. Aut gr $\tilde{H}_*(X)=\{\{\pm1\}\times\{\pm1\}\times\{\pm1\}\}$ and im $\alpha=\{\{(1,1),(-1,-1)\}\times\{\pm1\}\}$.

Let A_* be an elementary complex such that $A_i=0$ for $i=12$, 13. Then im $\beta=$ im $\beta'=\{\{(1,1),(-1,-1)\}\times\{1\}\}$. Therefore elements $(1,1,1)$ and $(1,-1,1)$ determine different homotopy types.

REFERENCES

[1] ECKMANN, B. and HILTON, P. J., *Décomposition homologique d'un polyèdre simplement connexe.* Comptes rendux Paris, *247* (1958) 2054–2056.
[2] MACLANE, SAUNDERS, *Homology*. Springer-Verlag 1963.
[3] QUINN, FRANK, *Finite nilpotent group actions on finite complexes.* Geometric Applications of Homotopy Theory I Lecture Notes in Math. 657 Springer Verlag.

[4] SMALLEN, D. L. and LIEBERMAN, G., *Localization and self-equivalence.* Duke Math. Journal *41* (1974) 183–186.
[5] THOMAS, C. B. and WALL, C. T. C., *The topological spherical space form problem I.* Compositio Math. *23* (1971), 101–114.
[6] WALL, C. T. C., *Finiteness conditions for* CW *complexes.* II Proc. Royal Society A *295* (1966) 129–139.
[7] WOJTKOWIAK, Z., *On Z/p free action on finite* CW *complexes.* Bull. L'acad. Pol. Scien. *25* (1977) 1175–1181.

Warsaw University, Institute of Math.
PKiN 9, 00-901 *Warsaw, Poland*

Received June 1, 1979