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On Tsuji's and Pommerenke's examples for fuchsian groups of convergence type

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Introduction

Let Γ be an infinitely generated fuchsian group acting on the unit disc Δ and Ω the set of ordinary points of Γ in the complex plane $\mathbb{C}$. In this paper we will consider only those groups Γ for which $\Omega \cap \partial\Delta = \emptyset$, where $\partial\Delta$ is the boundary of Δ and $\emptyset$ is the empty set. The classical problem to which this paper has relevance is the problem of determining the divergence or convergence of the series

$$\sum_{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \Gamma,\ c \neq 0} \frac{1}{|c|^2}.$$

This problem appears in [1, p. 176 and 6, p. 515]. This series is known to converge [6, p. 514] if the Lebesgue linear measure of the some fundamental domain D of Γ intersect $\partial\Delta$ is positive. Is the converse true?

No, in [6, p. 515] counterexamples for which $m(\bar{D}_0 \cap \partial\Delta) = 0$ are given, where $m(S)$ is the Lebesgue linear measure in $\partial\Delta$, D_0 is the Ford fundamental domain, and $\bar{D}_0$ is the closure of D_0. Although these examples are probably correct, there was a significant gap in Tsuji's claim that his constructions provide examples to anything at all. In particular, it is not obvious that if $m(\bar{D}_0 \cap \partial\Delta) = 0$, then $m(\bar{D} \cap \partial\Delta) = 0$ for all other fundamental polygons of Γ. This gap was filled by Pommerenke in [3], where he gives examples of his own. Actually Pommerenke proves that in fact if Γ is of convergence type, i.e. $\sum 1/|c|^2 < \infty$, and $m(\bar{D}_0 \cap \partial\Delta) = 0$, then for any Borel subset B of $\partial\Delta$ for which $V(B) \cap B = \emptyset$, if V is not the identity transformation, we have $m(B) = 0$.

The confusion to the validity or completeness of the Tsuji examples apparently stems from the notion that if $m(\bar{D} \cap \partial\Delta) = 0$ for one fundamental polygon of Γ then $m(\bar{D}' \cap \partial\Delta) = 0$ for all other fundamental polygons D' of Γ. In [4] examples are given to show that this notion is false. The theorem proved in this paper is an extension of these examples to all groups Γ for which $\Omega \cap \partial\Delta = \emptyset$.

THE MAIN THEOREM 1. *Let Γ be an infinitely generated fuchsian group for which $\Omega \cap \partial\Delta = \varnothing$. Then Γ has a fundamental polygon P such that $m(\bar{P} \cap \partial\Delta) = 0$.*

Remarks. It will be evident that the above theorem is true for any function $m^*: \mathfrak{B} \to [0, \infty)$, where $\mathfrak{B}$ is the set of Borel subsets of $\partial\Delta$, such that:

(a) $m^*(A \cup B) \leq m^*(A) + m^*(B)$;
(b) if $A \subseteq B$, then $m^*(A) \leq m^*(B)$;
(c) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $m(\{e^{i\theta} : \theta_0 - \delta < \theta < \theta_0 + \delta\}) < \varepsilon$ for all θ_0.

Preliminaries. We use the following two theorems.

THEOREM 2. *Let $\{C_i, C_i'\}_i$ be a collection of circles perpendicular to $\partial\Delta$ which satisfy: $(*)$ all circles are exterior or externally tangent to each other except for the possibility that $C_i \cap C_i' \cap \Delta \neq \varnothing$ in the special case mentioned below. Let Γ be a fuchsian group and $A_i \in \Gamma$ be such that $A_i(C_i) = C_i'$ with the outside of C_i going onto the inside of C_i'. If A_i is elliptic, then we assume A_i is of minimal rotation, i.e. $|\mathrm{tr}\,(A_i)| = 2\cos(\pi/n)$ for some n, and that $C_i \cap C_i' \cap \Delta$ is the fixed point of A_i, if $n \neq 2$. If $n = 2$, then $C_i = C_i'$. Then the polygon P formed by the intersection of Δ with the region exterior to all the circles from $\{C_i, C_i'\}_i$ is a fundamental polygon of Γ if and only if*

(1) *the group generated by the $\{A_i\}_{i=1}^{\infty}$, denoted by $\langle A_1, A_2, \ldots\rangle$ is Γ and*
(2) *$\Omega \cap \partial\Delta = \varnothing$ or $\bar{P}$ contains a fundamental set of $\Omega \cap \partial\Delta$.*

Theorem 2 is proven in [5].

The next theorem is a special case of Theorem 6 in [4].

THEOREM 3. *Every infinitely generated group for which $\Omega \cap \partial\Delta = \varnothing$ has a Schottky fundamental domain P such that for each $A \in S(P)$ we have that A is either a handle generator, an ideal boundary generator, a parabolic transformation, an elliptic transformation, or a free hyperbolic element.*

We need to define some of the terms of Theorem 3. A *handle generator* $X \in \Gamma$ is a hyperbolic transformation which identifies a pair of sides of P for which there is another hyperbolic generator Y, also identifying a pair of sides of P, such that $C_X \cup C_Y \cup C_X' \cup C_Y'$ is expressable as a Jordan arc, where C_T and $C_T' = T(C_T)$ are sides of P identified by T. Note here that the axes of X and Y intersect.

An *ideal boundary* generator is a hyperbolic transformation which identifies a pair of sides whose axis does not intersect the axis of any other hyperbolic transformation identifying a pair of sides of P.

A *Schottky fundamental polygon* of Γ is a hyperbolic convex region D in Δ which is the exterior of circles which satisfy (*) such that $\Delta = \bigcup_{V\in\Gamma} V(\bar{D})$ and $V(D)\cap D=\varnothing$ for all $V\in\Gamma$, $V\neq \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$.

Cutting and Pasting. For the precise definition of cutting and pasting we refer to [1, p. 242]. We should say, however, that in this paper we will cut and paste only Schottky fundamental polygons and in such a way as to obtain only other Schottky fundamental polygons. With this restriction a rough definition of a cut and paste is the following. We draw a geodesic in P, say g, in such a way that $P\backslash g$ has two connected components, say R_1 and R_2. We let $A\in\Gamma$ pair the sides C_A and $C'_A = A(C_A)$ of P. We assume g is drawn so that $C_A\subseteq \bar{R}_1$ and $C'_A\subseteq \bar{R}_2$. Then a *cut and paste by A along* g is the new polygon $P_1=[R_2\cup A(\bar{R}_1)]^0$, where S^0 is the interior of S. To insure P_1 is a Schottky polygon we choose g so that either both endpoints of g are in $\partial\Delta$, if A is not elliptic, or one endpoint of g is the elliptic fixed point of A, in the case A is elliptic.

Remark. Let the sides of P be labelled and enumerated by the sequence $\{C_i, C'_i\}_{i=1}^{\infty}$, where $C'_i = A_i(C_i)$ for some $A_i\in\Gamma$. Let $S(P)=\{A_1, A_2, \ldots\}$. We note $S(P_1)=\{A^{\varepsilon_X}XA^{\delta_X} : \varepsilon_X, \delta_X = 0, \pm 1,\ \varepsilon_A=\delta_A=0, \text{ and } X\in S(P)\}$. The change of $S(P)$ to $S(P_1)$ is called a *Nielsen transformation.* It is classical [2] that $\langle S(P)\rangle = \langle S(P_1)\rangle$.

We now describe three ways of cutting and pasting a Schottky polygon P to a new Schottky polygon P_1. When we say below to cut and paste P to say P_1, we shall mean that: starting with P perform one of the three operations described below depending on X to obtain P_1. Note that in the type 3 cutting and pasting we also prescribe the number ε.

Type 1. Let A be either an elliptic or parabolic transformation (by assumption there are no free hyperbolic generators) and $x\in\partial\Delta\cap\bar{P}$. Let g be the geodesic joining x to the fixed point of A. The type 1 cut and paste is the cutting and pasting of P along g by A.

Type 2. If A is a handle generator and $x\in\partial\Delta\cap\bar{P}$, then we need four cuts and pastes. We assume that as one walks $\partial\Delta$ counterclockwise one encounters the sets $\{x\}$, $C_A\cap\partial\Delta$, $C_B\cap\partial\Delta$, $C'_A\cap\partial\Delta$, $C'_B\cap\partial\Delta$, respectively. Draw g_A from x to the endpoint of C_A for which $C_A\cap\Delta$ is in one component of $\Delta\backslash g_A = R_1\cup R_2$, say R_1, and $C_B\cap\Delta$, $C'_A\cap\Delta$, $C'_B\cap\Delta$ are all in R_2. Cut along g_A and glue $A(R_1)$ to R_2 along C'_A. Repeat this procedure in succession for C'_B and $A(x)$, $g'_A = A(g_A)$ and $B^{-1}A(x)$, $g_B = B^{-1}(g_{B^{-1}})$ and $A^{-1}B^{-1}A(x)$. In the resulting polygon the sides which are paired by A and B all have both endpoints equivalent to x.

Type 3. Let A be an ideal boundary generator $x \in \partial P \cap \partial\Delta$, $\varepsilon > 0$ and x_A, y_A be the fixed points of A. We first observe that there exists a $\delta > 0$ such that $|A(z) - z| < \varepsilon$ if either $|z - y_A| < \delta$ or $|z - x_A| < \delta$. Let C_A, $C'_A = A(C_A)$ be the sides of P, a Schottky polygon, which are paired by A. Let x_A be inside the circle determined by C'_A and let w be the endpoint of C_A which lies in the connected component of $\partial\Delta \backslash \{x_A, y_A\}$ which does *not* contain x. Let g_1 be the geodesic joining x to w. Cut P along g_1 and paste along C'_A to obtain a new polygon P_1. If $|A(w) - A^2(w)| < \varepsilon$, we stop. If not repeat this procedure. By the above remarks, since $\lim_{n\to\infty} A^n(w) = x_A$, we see that after finitely many applications we do indeed obtain a fundamental polygon for which a pair of the equivalent endpoints, w_1, $w_2 = A(w_1)$, of the sides identified by A are within ε of each other. Since the minor arc determined by w_1, w_2 has length $\theta = 2 \arcsin[|w_1 - w_2|/2]$ we see that θ can be made arbitrarily small by this cut and paste.

Proof of the Main Theorem. We start with the fundamental domain P of Theorem 3. We let $\{s_i, s'_i\}_{i=0}^{\infty}$ be the sides of P enumerated so that $s'_i = A_i(s_i)$ for some $A_i \in \Gamma$. We set $S(P) = \{A_i\}_{i=0}^{\infty}$. We assume that $m = m(\bar{P} \cap \partial\Delta) > 0$, otherwise we take P as our fundamental domain. We also assume that -1 is an endpoint of s_0 and $1 = A_0(-1)$ is the endpoint of s'_0 equivalent to -1. If A_0 is a handle generator we let A_{j_0} be the member of $S(P)$ whose axis intersects the axis of A_0. Set $S_0 = \{A_0\}$, if A_0 is not a handle generator, or $S_0 = \{A_0, A_{j_0}\}$, if A_0 is a handle generator.

We do the case that A_0 is not an ideal boundary generator. This means that the sides of P paired by the elements of $S(P)\backslash S_0$ lie in either $\bar{H}^+ = \{x + iy : y \geqq 0\}$ or $\bar{H}^- = \{x + iy : y \leqq 0\}$; say $\bar{H}^+$. In the general case the proof given below would be repeated for the sides which lie in $\bar{H}^-$.

We let i_1 be the least positive integer among the indices of $S(P)\backslash S_0$. We choose $x = 1$ and cut and paste, as dictated by the type of generator A_{i_1} is, in such a way that 0 is in the resulting polygon P_1. We set $S_1 = S_0 \cup \{A_{i_1}\}$, if A_{i_1} is not a handle generator. If A_{i_1} is a handle generator, then we set $S_1 = S_0 \cup \{A_{i_1}, A_{j_1}\}$, where A_{j_1} is the unique element of $S(P)$ whose axis intersects the axis of A_{i_1}. Let T_1 be the set of circles determined by the sides of P_1 which are paired by elements of S_1. If A_{i_1} is an ideal boundary generator than we choose $\varepsilon_{i_1} = \frac{1}{2}$ in the type 3 cut and paste.

We next endow $S(P_1)$ with the *enumeration derived from* $S(P)$. This is best described by observing that each $Y \in S(P_1)$ is of the form $Y = WA_jV$, where $W, V \in \langle S_1 \rangle$. We index Y by j and write $S(P_1) = \{A_0 = Y_0, Y_1, \ldots, A_{i_0} = Y_{i_0}, \ldots\}$, where $Y_j = WA_jV$ for some $A_j \in S(P)$.

We continue inductively. Given $n \geq 1$, P_n, S_n, T_n we enumerate $S(P_n) = \{X_0 = A_0, X_1, \ldots\}$ by the enumeration derived from $S(P_{n-1})$. We note that if S_n

contains precisely k_n ideal boundary generators then $J_n = \partial\Delta \cap \operatorname{Ext}(T_n)$, where $\operatorname{Ext}(T_n)$ is the closed region outside all the circles determined by the sides of P_n in T_n, has k_n+1 connected components, $J_{1,n}, \ldots J_{k_n+1,n}$ all of which are in $\bar{H}^+ \cap \partial\Delta$.

Now for each component $J_{r,n}$ let $S(P_n)_r = \{X \in S(P_n) \backslash S_n : C_X \cap J_{r,n} \neq \varnothing\}$, where C_X, $C'_X = X(C_X)$ are the sides of P_n paired by X. We see that from the way we cut and the definition of boundary generator, that $C'_X \cap J_{r,n} \neq \varnothing$ if and only if $C_X \cap J_{r,n} \neq \varnothing$. We let $i_{r,n}$ be the least positive index among the indices of $S(P_n)_r$ and x_r the endpoint of $J_{r,n}$ closest to 1. We now cut and paste P_n by $X_{i_{1,n}}$ to $P_{1,n}$ and then for $j = 1, 2, \ldots, k_n$ in succession we cut and paste $P_{j,n}$ by $X_{i_{j,n}}$ to $P_{j+1,n}$ and finally $P_{k_n,n}$ is cut and pasted by $X_{i_{k_n+1,n}}$ to P_{n+1}. By our assumption on ± 1 and s_0, s'_0 all geodesics drawn in the cutting and pasting have 0 outside the circle determined by them. Therefore we can and do choose all cuts and pastes so that $0 \in \bigcap_{n=1}^{\infty} P_n$. Naturally the type of cut and paste used in each step will be determined by the type of generator $X_{i_{r,n}}$ is. In the case $X_{i_{r,n}}$ is an ideal boundary generator we choose $\varepsilon_{i_{r,n}} = 1/(k_n+1)2^n$.

To complete the induction we set $S_{n+1} = S_n \cup \{X_{i_{1,n}}, \ldots, X_{i_{r,n}}, X_{j_{r,n}}, \ldots, X_{i_{k_n+1,n}}\}$ and T_{n+1} to be the circles determined by the sides paired by elements of S_{n+1}. Now set $S = \bigcup_{n=1}^{\infty} S_n$, $T = \bigcup_{n=1}^{\infty} T_n$. We note that if the index j has an element $X_j \in S_n$, then $A_j = WX_jV$ for some $W, V \in \langle S_n \rangle$ is in $\langle S_n \rangle$ and hence $A_j \in \langle S \rangle$. Since each P_n and $P_{r,n}$ is a Schottky polygon each j has such an X_j. For the only way we could miss setting $i_{r,n} = j$ for some r, n is that the index j vanished from $S(P_n) = \{X_0, \ldots\}$. This would mean that if $S(P_{n-1}) = \{Y_0, Y_1, \ldots\}$, then $WY_jV = I$ for some $W, V \in \langle S_n \rangle$ and $Y_j \notin S_n$, and this is not possible. Thus $\langle S \rangle = \Gamma$. It is also clear 0 is outside all the circles $C \in T$. So by Theorems 2, the region P' exterior to all the $C \in T$ is a fundamental region of Γ.

It remains to show $m(\bar{P}' \cap \partial\Delta) = 0$. We show that for any $\varepsilon > 0$, $m(\bar{P}' \cap \partial\Delta) < \varepsilon$. We let $\varepsilon > 0$ and choose N such that $\sum_{j=N}^{\infty} 1/2^j < \varepsilon/2$; then $\varepsilon/2 > 1/2^N$. We consider P_N. Let $J_{1,N}, \ldots, J_{k_N+1,N}$ be, as before, the connected components of $\partial\Delta$ outside the circles from T_N. Since $\Omega \cap \partial\Delta = \varnothing$, for each $r = 1, 2, \ldots, k_N+1$ there exists $N_r > N$ such that $\max\{m(J_{t,N_r} \cap J_{r,N}) : t = 1, 2, \ldots, k_{N_r}+1\} < \varepsilon/2(k_N+1)$. Let $N' = \operatorname{Max}\{N_1, N_2, \ldots, N_{k_{N+1}}\}$. We observe that, if $k_{N'} > k_N$, there are at most k_j+1, $j = N, N+1, \ldots, N'-1$, $J_{r,N'}$ for which

$$C_j = \frac{1}{(k_j+1)2^j} \geqq m(J_{r,N'}) > C_{j+1} = \frac{1}{(k_{j+1}+1)2^{j+1}}$$

and there are at most (k_N+1) $J_{r,N'}$ such that

$$C_N = \frac{1}{(k_N+1)}\frac{1}{2^N} \leqq m(J_{r,N'}) \leqq \frac{\varepsilon}{2(k_N+1)} = d, \quad \text{since} \quad \varepsilon \geqq \frac{1}{2^{N-1}}.$$

This follows from our choice of the "ε" in the type 3 cutting and pasting. Hence,

$$m(\partial\Delta \cap P') \leqq \sum_{r=1}^{k_{N'}+1} m(J_{r,N'}) = \sum_{j=N}^{N'-1} \sum_{C_{j+1}<m(J_{r,N'})\leqq C_j} m(J_{r,N'})$$

$$+ \sum_{m(J_{r,N'})\leq C_{N'}} m(J_{r,N'}) + \sum_{C_N \leqq m(J_{r,N'}) \leqq d} m(J_{r,N'})$$

$$\leqq \sum_{j=N}^{N'} \frac{1}{2^j} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

REFERENCES

[1] LEHNER, J., *Discontinuous groups and automorphic functions*, AMS Surveys No. 8, Providence, R.I., 1964.

[2] MAGNUS, W., KARRASS, A., SOLITAR, D., *Combinatorial groups theory*, Wiley Interscience Publishers, New York, 1966.

[3] POMMERENKE, CH., *On the Green's function of fuchsian groups*, Ann. Acad. Sci. Fennicae, Ser. A I Math. 2, 409–427 (1976).

[4] PURZITSKY, N., *Fricke polygons for infinitely generated groups I.* (to appear).

[5] ——, *A cutting and pasting of infinite sided polygons with applications to fuchsian groups.* (to appear Acta Mathematica).

[6]. TSUJI, M., *Potential Theory*, Chelsea, New York, 1959.

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