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Autor:	Purzitsky, N.
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On Tsuji's and Pommerenke's examples for fuchsian groups of convergence type

N. PURZITSKY

Introduction

Let Γ be an infinitely generated fuchsian group acting on the unit disc Δ and Ω the set of ordinary points of Γ in the complex plane \mathbb{C} . In this paper we will consider only those groups Γ for which $\Omega \cap \partial\Delta = \emptyset$, where $\partial\Delta$ is the boundary of Δ and \emptyset is the empty set. The classical problem to which this paper has relevance is the problem of determining the divergence or convergence of the series

$$\sum_{\substack{c \\ c \neq 0}} \frac{1}{|c|^2}.$$

This problem appears in [1, p. 176 and 6, p. 515]. This series is known to converge [6, p. 514] if the Lebesgue linear measure of the some fundamental domain D of Γ intersect $\partial\Delta$ is positive. Is the converse true?

No, in [6, p. 515] counterexamples for which $m(\bar{D}_0 \cap \partial\Delta) = 0$ are given, where $m(S)$ is the Lebesgue linear measure in $\partial\Delta$, D_0 is the Ford fundamental domain, and \bar{D}_0 is the closure of D_0 . Although these examples are probably correct, there was a significant gap in Tsuji's claim that his constructions provide examples to anything at all. In particular, it is not obvious that if $m(\bar{D}_0 \cap \partial\Delta) = 0$, then $m(\bar{D} \cap \partial\Delta) = 0$ for all other fundamental polygons of Γ . This gap was filled by Pommerenke in [3], where he gives examples of his own. Actually Pommerenke proves that in fact if Γ is of convergence type, i.e. $\sum 1/|c|^2 < \infty$, and $m(\bar{D}_0 \cap \partial\Delta) = 0$, then for any Borel subset B of $\partial\Delta$ for which $V(B) \cap B = \emptyset$, if V is not the identity transformation, we have $m(B) = 0$.

The confusion to the validity or completeness of the Tsuji examples apparently stems from the notion that if $m(\bar{D} \cap \partial\Delta) = 0$ for one fundamental polygon of Γ then $m(\bar{D}' \cap \partial\Delta) = 0$ for all other fundamental polygons D' of Γ . In [4] examples are given to show that this notion is false. The theorem proved in this paper is an extension of these examples to all groups Γ for which $\Omega \cap \partial\Delta = \emptyset$.

THE MAIN THEOREM 1. *Let Γ be an infinitely generated fuchsian group for which $\Omega \cap \partial\Delta = \emptyset$. Then Γ has a fundamental polygon P such that $m(\bar{P} \cap \partial\Delta) = 0$.*

Remarks. It will be evident that the above theorem is true for any function $m^* : \mathcal{B} \rightarrow [0, \infty)$, where \mathcal{B} is the set of Borel subsets of $\partial\Delta$, such that:

- (a) $m^*(A \cup B) \leq m^*(A) + m^*(B)$;
- (b) if $A \subseteq B$, then $m^*(A) \leq m^*(B)$;
- (c) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $m(\{e^{i\theta} : \theta_0 - \delta < \theta < \theta_0 + \delta\}) < \varepsilon$ for all θ_0 .

Preliminaries. We use the following two theorems.

THEOREM 2. *Let $\{C_i, C'_i\}_i$ be a collection of circles perpendicular to $\partial\Delta$ which satisfy: (*) all circles are exterior or externally tangent to each other except for the possibility that $C_i \cap C'_i \cap \Delta \neq \emptyset$ in the special case mentioned below. Let Γ be a fuchsian group and $A_i \in \Gamma$ be such that $A_i(C_i) = C'_i$ with the outside of C_i going onto the inside of C'_i . If A_i is elliptic, then we assume A_i is of minimal rotation, i.e. $|\text{tr}(A_i)| = 2 \cos(\pi/n)$ for some n , and that $C_i \cap C'_i \cap \Delta$ is the fixed point of A_i , if $n \neq 2$. If $n = 2$, then $C_i = C'_i$. Then the polygon P formed by the intersection of Δ with the region exterior to all the circles from $\{C_i, C'_i\}_i$ is a fundamental polygon of Γ if and only if*

- (1) *the group generated by the $\{A_i\}_{i=1}^\infty$, denoted by $\langle A_1, A_2, \dots \rangle$ is Γ and*
- (2) *$\Omega \cap \partial\Delta = \emptyset$ or \bar{P} contains a fundamental set of $\Omega \cap \partial\Delta$.*

Theorem 2 is proven in [5].

The next theorem is a special case of Theorem 6 in [4].

THEOREM 3. *Every infinitely generated group for which $\Omega \cap \partial\Delta = \emptyset$ has a Schottky fundamental domain P such that for each $A \in S(P)$ we have that A is either a handle generator, an ideal boundary generator, a parabolic transformation, an elliptic transformation, or a free hyperbolic element.*

We need to define some of the terms of Theorem 3. A *handle generator* $X \in \Gamma$ is a hyperbolic transformation which identifies a pair of sides of P for which there is another hyperbolic generator Y , also identifying a pair of sides of P , such that $C_X \cup C_Y \cup C'_X \cup C'_Y$ is expressable as a Jordan arc, where C_T and $C'_T = T(C_T)$ are sides of P identified by T . Note here that the axes of X and Y intersect.

An *ideal boundary generator* is a hyperbolic transformation which identifies a

pair of sides whose axis does not intersect the axis of any other hyperbolic transformation identifying a pair of sides of P .

A *Schottky fundamental polygon* of Γ is a hyperbolic convex region D in Δ which is the exterior of circles which satisfy $(*)$ such that $\Delta = \bigcup_{V \in \Gamma} V(\bar{D})$ and $V(D) \cap D = \emptyset$ for all $V \in \Gamma$, $V \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Cutting and Pasting. For the precise definition of cutting and pasting we refer to [1, p. 242]. We should say, however, that in this paper we will cut and paste only Schottky fundamental polygons and in such a way as to obtain only other Schottky fundamental polygons. With this restriction a rough definition of a cut and paste is the following. We draw a geodesic in P , say g , in such a way that $P \setminus g$ has two connected components, say R_1 and R_2 . We let $A \in \Gamma$ pair the sides C_A and $C'_A = A(C_A)$ of P . We assume g is drawn so that $C_A \subseteq \bar{R}_1$ and $C'_A \subseteq \bar{R}_2$. Then a *cut and paste by A along g* is the new polygon $P_1 = [R_2 \cup A(\bar{R}_1)]^0$, where S^0 is the interior of S . To insure P_1 is a Schottky polygon we choose g so that either both endpoints of g are in $\partial\Delta$, if A is not elliptic, or one endpoint of g is the elliptic fixed point of A , in the case A is elliptic.

Remark. Let the sides of P be labelled and enumerated by the sequence $\{C_i, C'_i\}_{i=1}^\infty$, where $C'_i = A_i(C_i)$ for some $A_i \in \Gamma$. Let $S(P) = \{A_1, A_2, \dots\}$. We note $S(P_1) = \{A^{\epsilon_X} X A^{\delta_X} : \epsilon_X, \delta_X = 0, \pm 1, \epsilon_A = \delta_A = 0, \text{ and } X \in S(P)\}$. The change of $S(P)$ to $S(P_1)$ is called a *Nielsen transformation*. It is classical [2] that $\langle S(P) \rangle = \langle S(P_1) \rangle$.

We now describe three ways of cutting and pasting a Schottky polygon P to a new Schottky polygon P_1 . When we say below to cut and paste P to say P_1 , we shall mean that: starting with P perform one of the three operations described below depending on X to obtain P_1 . Note that in the type 3 cutting and pasting we also prescribe the number ϵ .

Type 1. Let A be either an elliptic or parabolic transformation (by assumption there are no free hyperbolic generators) and $x \in \partial\Delta \cap \bar{P}$. Let g be the geodesic joining x to the fixed point of A . The type 1 cut and paste is the cutting and pasting of P along g by A .

Type 2. If A is a handle generator and $x \in \partial\Delta \cap \bar{P}$, then we need four cuts and pastes. We assume that as one walks $\partial\Delta$ counterclockwise one encounters the sets $\{x\}$, $C_A \cap \partial\Delta$, $C_B \cap \partial\Delta$, $C'_A \cap \partial\Delta$, $C'_B \cap \partial\Delta$, respectively. Draw g_A from x to the endpoint of C_A for which $C_A \cap \Delta$ is in one component of $\Delta \setminus g_A = R_1 \cup R_2$, say R_1 , and $C_B \cap \Delta$, $C'_A \cap \Delta$, $C'_B \cap \Delta$ are all in R_2 . Cut along g_A and glue $A(R_1)$ to R_2 along C'_A . Repeat this procedure in succession for C'_B and $A(x)$, $g'_A = A(g_A)$ and $B^{-1}A(x)$, $g_B = B^{-1}(g_{B^{-1}})$ and $A^{-1}B^{-1}A(x)$. In the resulting polygon the sides which are paired by A and B all have both endpoints equivalent to x .

Type 3. Let A be an ideal boundary generator $x \in \partial P \cap \partial \Delta$, $\varepsilon > 0$ and x_A, y_A be the fixed points of A . We first observe that there exists a $\delta > 0$ such that $|A(z) - z| < \varepsilon$ if either $|z - y_A| < \delta$ or $|z - x_A| < \delta$. Let $C_A, C'_A = A(C_A)$ be the sides of P , a Schottky polygon, which are paired by A . Let x_A be inside the circle determined by C'_A and let w be the endpoint of C_A which lies in the connected component of $\partial \Delta \setminus \{x_A, y_A\}$ which does *not* contain x . Let g_1 be the geodesic joining x to w . Cut P along g_1 and paste along C'_A to obtain a new polygon P_1 . If $|A(w) - A^2(w)| < \varepsilon$, we stop. If not repeat this procedure. By the above remarks, since $\lim_{n \rightarrow \infty} A^n(w) = x_A$, we see that after finitely many applications we do indeed obtain a fundamental polygon for which a pair of the equivalent endpoints, $w_1, w_2 = A(w_1)$, of the sides identified by A are within ε of each other. Since the minor arc determined by w_1, w_2 has length $\theta = 2 \arcsin [|w_1 - w_2|/2]$ we see that θ can be made arbitrarily small by this cut and paste.

Proof of the Main Theorem. We start with the fundamental domain P of Theorem 3. We let $\{s_i, s'_i\}_{i=0}^\infty$ be the sides of P enumerated so that $s'_i = A_i(s_i)$ for some $A_i \in \Gamma$. We set $S(P) = \{A_i\}_{i=0}^\infty$. We assume that $m = m(\bar{P} \cap \partial \Delta) > 0$, otherwise we take P as our fundamental domain. We also assume that -1 is an endpoint of s_0 and $1 = A_0(-1)$ is the endpoint of s'_0 equivalent to -1 . If A_0 is a handle generator we let A_{j_0} be the member of $S(P)$ whose axis intersects the axis of A_0 . Set $S_0 = \{A_0\}$, if A_0 is not a handle generator, or $S_0 = \{A_0, A_{j_0}\}$, if A_0 is a handle generator.

We do the case that A_0 is not an ideal boundary generator. This means that the sides of P paired by the elements of $S(P) \setminus S_0$ lie in either $\bar{H}^+ = \{x + iy : y \geq 0\}$ or $\bar{H}^- = \{x + iy : y \leq 0\}$; say \bar{H}^+ . In the general case the proof given below would be repeated for the sides which lie in \bar{H}^- .

We let i_1 be the least positive integer among the indices of $S(P) \setminus S_0$. We choose $x = 1$ and cut and paste, as dictated by the type of generator A_{i_1} is, in such a way that 0 is in the resulting polygon P_1 . We set $S_1 = S_0 \cup \{A_{i_1}\}$, if A_{i_1} is not a handle generator. If A_{i_1} is a handle generator, then we set $S_1 = S_0 \cup \{A_{i_1}, A_{j_1}\}$, where A_{j_1} is the unique element of $S(P)$ whose axis intersects the axis of A_{i_1} . Let T_1 be the set of circles determined by the sides of P_1 which are paired by elements of S_1 . If A_{i_1} is an ideal boundary generator than we choose $\varepsilon_{i_1} = \frac{1}{2}$ in the type 3 cut and paste.

We next endow $S(P_1)$ with the *enumeration derived from $S(P)$* . This is best described by observing that each $Y \in S(P_1)$ is of the form $Y = WA_jV$, where $W, V \in \langle S_1 \rangle$. We index Y by j and write $S(P_1) = \{A_0 = Y_0, Y_1, \dots, A_{i_0} = Y_{i_0}, \dots\}$, where $Y_j = WA_jV$ for some $A_j \in S(P)$.

We continue inductively. Given $n \geq 1$, P_n , S_n , T_n we enumerate $S(P_n) = \{X_0 = A_0, X_1, \dots\}$ by the enumeration derived from $S(P_{n-1})$. We note that if S_n

contains precisely k_n ideal boundary generators then $J_n = \partial\Delta \cap \text{Ext}(T_n)$, where $\text{Ext}(T_n)$ is the closed region outside all the circles determined by the sides of P_n in T_n , has $k_n + 1$ connected components, $J_{1,n}, \dots, J_{k_n+1,n}$ all of which are in $\bar{H}^+ \cap \partial\Delta$.

Now for each component $J_{r,n}$ let $S(P_n)_r = \{X \in S(P_n) \setminus S_n : C_X \cap J_{r,n} \neq \emptyset\}$, where $C_X, C'_X = X(C_X)$ are the sides of P_n paired by X . We see that from the way we cut and the definition of boundary generator, that $C'_X \cap J_{r,n} \neq \emptyset$ if and only if $C_X \cap J_{r,n} \neq \emptyset$. We let $i_{r,n}$ be the least positive index among the indices of $S(P_n)_r$ and x_r the endpoint of $J_{r,n}$ closest to 1. We now cut and paste P_n by $X_{i_{1,n}}$ to $P_{1,n}$ and then for $j = 1, 2, \dots, k_n$ in succession we cut and paste $P_{j,n}$ by $X_{i_{j,n}}$ to $P_{j+1,n}$ and finally $P_{k_n,n}$ is cut and pasted by $X_{i_{k_n+1,n}}$ to P_{n+1} . By our assumption on ± 1 and s_0, s'_0 all geodesics drawn in the cutting and pasting have 0 outside the circle determined by them. Therefore we can and do choose all cuts and pastes so that $0 \in \bigcap_{n=1}^{\infty} P_n$. Naturally the type of cut and paste used in each step will be determined by the type of generator $X_{i_{r,n}}$ is. In the case $X_{i_{r,n}}$ is an ideal boundary generator we choose $\varepsilon_{i_{r,n}} = 1/(k_n + 1)2^n$.

To complete the induction we set $S_{n+1} = S_n \cup \{X_{i_{1,n}}, \dots, X_{i_{r,n}}, X_{j_{r,n}}, \dots, X_{i_{k_n+1,n}}\}$ and T_{n+1} to be the circles determined by the sides paired by elements of S_{n+1} . Now set $S = \bigcup_{n=1}^{\infty} S_n$, $T = \bigcup_{n=1}^{\infty} T_n$. We note that if the index j has an element $X_j \in S_n$, then $A_j = WX_jV$ for some $W, V \in \langle S_n \rangle$ is in $\langle S_n \rangle$ and hence $A_j \in \langle S \rangle$. Since each P_n and $P_{r,n}$ is a Schottky polygon each j has such an X_j . For the only way we could miss setting $i_{r,n} = j$ for some r, n is that the index j vanished from $S(P_n) = \{X_0, \dots\}$. This would mean that if $S(P_{n-1}) = \{Y_0, Y_1, \dots\}$, then $WY_jV = I$ for some $W, V \in \langle S_n \rangle$ and $Y_j \notin S_n$, and this is not possible. Thus $\langle S \rangle = \Gamma$. It is also clear 0 is outside all the circles $C \in T$. So by Theorems 2, the region P' exterior to all the $C \in T$ is a fundamental region of Γ .

It remains to show $m(\bar{P}' \cap \partial\Delta) = 0$. We show that for any $\varepsilon > 0$, $m(\bar{P}' \cap \partial\Delta) < \varepsilon$. We let $\varepsilon > 0$ and choose N such that $\sum_{j=N}^{\infty} 1/2^j < \varepsilon/2$; then $\varepsilon/2 > 1/2^N$. We consider P_N . Let $J_{1,N}, \dots, J_{k_N+1,N}$ be, as before, the connected components of $\partial\Delta$ outside the circles from T_N . Since $\Omega \cap \partial\Delta = \emptyset$, for each $r = 1, 2, \dots, k_N + 1$ there exists $N_r > N$ such that $\max \{m(J_{t,N} \cap J_{r,N}) : t = 1, 2, \dots, k_{N_r} + 1\} < \varepsilon/2(k_N + 1)$. Let $N' = \max \{N_1, N_2, \dots, N_{k_N+1}\}$. We observe that, if $k_{N'} > k_N$, there are at most $k_j + 1$, $j = N, N+1, \dots, N'-1, J_{r,N'}$ for which

$$C_j = \frac{1}{(k_j + 1)2^j} \leq m(J_{r,N'}) > C_{j+1} = \frac{1}{(k_{j+1} + 1)2^{j+1}}$$

and there are at most $(k_N + 1) J_{r,N'}$ such that

$$C_N = \frac{1}{(k_N + 1)} \frac{1}{2^N} \leq m(J_{r,N'}) \leq \frac{\varepsilon}{2(k_N + 1)} = d, \quad \text{since} \quad \varepsilon \geq \frac{1}{2^{N-1}}.$$

This follows from our choice of the “ ε ” in the type 3 cutting and pasting. Hence,

$$\begin{aligned}
 m(\partial\Delta \cap P') &\leq \sum_{r=1}^{k_{N'}+1} m(J_{r,N'}) = \sum_{j=N}^{N'-1} \sum_{C_{j+1} < m(J_{r,N'}) \leq C_j} m(J_{r,N'}) \\
 &\quad + \sum_{m(J_{r,N'}) \leq C_{N'}} m(J_{r,N'}) + \sum_{C_N \leq m(J_{r,N'}) \leq d} m(J_{r,N'}) \\
 &\leq \sum_{j=N}^{N'} \frac{1}{2^j} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
 \end{aligned}$$

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*Dept. of Mathematics
York University
Downsview, Ont. M3J 1P3, Canada*

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