Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 55 (1980)

Artikel: On cohomological periodicity for infinite groups.

Autor: Talelli, Olympia

DOI: https://doi.org/10.5169/seals-42369

Comment. Math. Helvetici 55 (1980) 178–192 Birkhäuser Verlag, Basel

On cohomological periodicity for infinite groups

Olympia Talelli

Introduction

The phenomenon of cohomological periodicity for finite groups has long been understood. Here we introduce the notion of "a group G having period q after k-steps" so as to allow infinite groups to have periodic cohomology. The definition is given in terms of a projective resolution of G and, as expected, it is equivalent to having the functors $H^n(G,-)$ and $H^{n+q}(G,-)$ naturally isomorphic for all $n \geq k+1$. We then show that this definition coincides with the classical one for finite groups and moreover, we obtain that if an infinite group G has period q after k-steps then $k \geq 1$.

In §2 we investigate what it means for a countable locally finite group to have period q after k-steps. We obtain what one would expect, i.e. that a countable locally finite group G has period q after k-steps iff every finite subgroup of G has period q. Moreover, we have here that $k=1$.

Then we show that there is an element $g \in H^q(G, Z)$ such that cup product with g

$$\bigcup g : H^i(G,-) \to H^{i+q}(G,-)$$

induces the natural isomorphism for all $i \geq 2$.

Finally, in §3 we characterize the infinite locally finite p-groups which have period q after k-steps. First, we point out two obvious candidates, i.e. the infinite locally cyclic p-group and the infinite locally quaternion group, and then we show that these are the only ones. This result depends heavily on the well known similar statement for periodic finite p-groups [Cartan+Eilenberg].

I wish to thank K. W. Gruenberg for his help during the preparation of this paper.

§1. Periodicity after some "steps"

Let G be a group and ZG its integral group ring. We work in the category of left ZG-modules. If A is a ZG-module, by a resolution of A we shall always mean a projective resolution of A.

DEFINITION. *A group G is said to have period q after k-steps* if there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R_{k+q} \arrow[r, "\beta"'] & P_{k+q-1} \arrow[r] & \cdots \arrow[r] & P_k \arrow[rr, "\partial_k"] \arrow[dr, "\alpha"'] & & P_{k-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\
& & & & & R_k \arrow[ur] & & & & &
\end{tikzcd} \tag{1}$$

where Z is regarded as a trivial ZG-module, $R_{k+q} = R_k$ and P_i $0 \leq i \leq k+q-1$ are projective ZG-modules. We take $R_0 = Z$. If $k = 0$ then G is said to have period q. Having (1) we can form a resolution of G

$$\begin{tikzcd}
P'_* : \cdots \arrow[r] & P'_i \arrow[rr, "\partial_i"] \arrow[dr] & & P'_{i-1} \arrow[r] & \cdots \arrow[r] & P'_0 \arrow[r] & Z \arrow[r] & 0 \\
& & R'_i \arrow[ur] & & & & &
\end{tikzcd}$$

by defining

$$P'_i = P_i \qquad 0 \leq i \leq k-1 \qquad\qquad \partial'_i = \partial_i \qquad 1 \leq i \leq k$$

and

$$\begin{array}{llll}
P'_i = P_{k+\lambda_i} & i = k + nq + \lambda_i & \partial'_i = \partial_{k+\lambda_i} & i = k + nq + \lambda_i \\
& n \geq 0,\ 0 \leq \lambda_i < q & & n \geq 0,\ 0 < \lambda_i < q \\
& & \partial'_{k+\mu q} = \beta\alpha & \mu \geq 1
\end{array}$$

Such a resolution is naturally considered periodic after k-steps; we may refer to (1) itself as a resolution of G which is periodic after k-steps. If $k = 0$ we call such a resolution a periodic resolution of G.

Evidently the functors $H^n(G, -)$ and $H^{n+q}(G, -)$ are naturally isomorphic for all $n \geq k+1$.

Now let M, N be ZG-modules, $g \in \mathrm{Ext}^q_{ZG}(M, N)$ and consider the following diagram

$$\begin{tikzcd}
P_* : \cdots \arrow[r] & P_{i+q} \arrow[r] \arrow[dd] & P_{i+q-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & P_{q+1} \arrow[r] \arrow[dd] & P_q \arrow[rr] \arrow[dd] \arrow[dr] & & P_{q-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & & & & & R_q \arrow[ur] \arrow[d, "\theta"] & & & & & \\
E_* : \cdots \arrow[r] & E_i \arrow[r] & E_{i-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & N \arrow[r] & 0 & & & &
\end{tikzcd}$$

where $P_* \to M$, $E_* \to N$ are resolutions of M, N respectively and $\theta : R_q \to M$ a

q-cocycle representing g. Then θ lifts to a chain map $P_* \to E_*$ of degree $-q$ and this chain map induces cup product with g

$$\bigcup g : \operatorname{Ext}^i_{ZG}(N, -) \to \operatorname{Ext}^{i+q}_{ZG}(M, -) i \geq 0.$$

THEOREM 1.1. *Let G be a group. Then the following statements are equivalent:*

(i) *G has period q after k-steps*
(ii) *There is a resolution of G*

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[dr] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\
 & & R_i \arrow[u] & & & &
\end{tikzcd}$$

and an element $h \in \operatorname{Ext}^q_{ZG}(R_k, R_k)$ such that for every ZG-module A cup product with h

$$\bigcup h : \operatorname{Ext}^i_{ZG}(R_k, A) \longrightarrow \operatorname{Ext}^{i+q}_{ZG}(R_k, A)$$

is an isomorphism for $i \geq 1$ and an epimorphism for $i = 0$. Note that $\operatorname{Ext}^i_{ZG}(R_k, A) = H^{i+k}(G, A)$ for $i \geq 1$, $k \geq 0$.

Proof. (i)$\Rightarrow$(ii). Consider a resolution P'_* of G which is periodic, or period q, after k-steps

$$\begin{tikzcd}
P'_* : \cdots \arrow[r] & P'_i \arrow[r] \arrow[dr] & P'_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_0 \arrow[r] & Z \arrow[r] & 0 \\
 & & R'_i & & & &
\end{tikzcd}$$

and the resolution $P'_* \mid R'_k$ of R'_k

$$P'_* \mid_{R_k} : \cdots \longrightarrow P'_i \longrightarrow P'_{i-1} \longrightarrow \cdots \longrightarrow P'_k \xrightarrow{\alpha} R'_k \longrightarrow 0.$$

Now there is an element $h \in \operatorname{Ext}^q_{ZG}(R'_k, R'_k)$ defined by $P'_{k+q} \xrightarrow{\alpha} R'_k$ or in Yoneda's interpretation of $\operatorname{Ext}^i_{ZG}(-, -)$ by the multiple extension

$$0 \longrightarrow R'_k \longrightarrow P'_{k+q-1} \longrightarrow \cdots \longrightarrow P'_k \xrightarrow{\alpha} R'_k \longrightarrow 0.$$

Clearly, for any ZG-module A cup product with h

$$\bigcup h : \operatorname{Ext}^i_{ZG}(R_k, A) \longrightarrow \operatorname{Ext}^{i+q}_{ZG}(R_k, A)$$

is the identity for $i \geq 1$ and an epimorphism for $i = 0$. (ii) $\Rightarrow$ (i) can be easily deduced from the following Theorem. It is due to C. T. C. Wall [[7], Theorem. 1.2]; however, we give here a different proof.

THEOREM 1.2. *Let A, B be ZG-modules. If $g \in \mathrm{Ext}^n_{ZG}(A, B)$ is such that for any ZG-module E cup product with g*

$$\cup g : \mathrm{Ext}^i_{ZG}(B, E) \longrightarrow \mathrm{Ext}^{i+n}_{ZG}(A, E)$$

is injective for $i = 1$ and surjective for $i = 0$, then g is represented by a multiple extension

$$0 \longrightarrow B \longrightarrow K \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with K, P_i $0 \leq i \leq n-2$ projective ZG-modules.

Proof. Let $P_* \to A$, $E_* \to B$ be resolutions of A and B respectively, and let $\theta : R_n \to B$ be a cocycle representing g

$$\begin{tikzcd}
P_* : \cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[dd] & P_n \arrow[rr] \arrow[dd] \arrow[dr] & & P_{n-1} \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0 \\
& & & R_n \arrow[ur] \arrow[d, "\theta"'] & & & & & & \\
E_* : \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & B \arrow[r] & 0. & & & & &
\end{tikzcd}$$

Now θ lifts to a chain map $P_* \to E_*$ of degree $-n$ which induces cup product with g.

Let

$$\begin{tikzcd}
R_n \arrow[r, "\gamma"] \arrow[d, "\theta"'] & P_{n-1} \arrow[d, "\alpha"] \\
B \arrow[r, "\beta"'] & K
\end{tikzcd}$$

be a push-out diagram. Then β is injective and coker $\beta \simeq$ coker $\gamma \simeq R_{n-1}$.

Now g is represented by $0 \to B \to K \to P_{n-2} \to \cdots \to P_0 \to A \to 0$. We shall show that K is a projective ZG-module.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r] \arrow[d, "\theta"'] & P_{n-1} \arrow[r] \arrow[d, "\alpha"] & R_{n-1} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & B \arrow[r] & K \arrow[r] & R_{n-1} \arrow[r] & 0.
\end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Hom}_{ZG}(B, E) \arrow[r] \arrow[d, "\theta^*"'] \arrow[dr] & \operatorname{Ext}_{ZG}(R_{n-1}, E) \arrow[r] \arrow[d, "id"'] & \operatorname{Ext}_{ZG}(K, E) \arrow[r] \arrow[d, "\alpha^*"'] & \operatorname{Ext}_{ZG}(B, E) \arrow[r] \arrow[d, "\theta^*"'] & \cdots \\
\cdots \arrow[r] & \operatorname{Hom}_{ZG}(R_n, E) \arrow[r, "\delta"'] & \operatorname{Ext}_{ZG}(R_{n-1}, E) \arrow[r] & \operatorname{Ext}_{ZG}(P_{n-1}, E) \arrow[r] & \operatorname{Ext}_{ZG}(R_n, E) \arrow[r] & \cdots
\end{tikzcd}$$

By hypotheses $\theta^*: \operatorname{Ext}_{ZG}(B, E) \to \operatorname{Ext}_{ZG}(R_n, E) \simeq \operatorname{Ext}_{ZG}^{1+n}(A, E)$ is injective and

$$\delta\theta^*: \operatorname{Hom}_{ZG}(B, E) \to \operatorname{Ext}_{ZG}(R_{n-1}, E) \simeq \operatorname{Ext}_{ZG}^{n}(A, E)$$

is surjective. Moreover, $\operatorname{Ext}_{ZG}(P_{n-1}, E) = 0$ since P_{n-1} is projective. Thus by the commutativity of the diagram we obtain that $\operatorname{Ext}_{ZG}(K, E) = 0$, and this holds for any ZG-module E. Hence K is a projective ZG-module.

Remark. If P_{n-1}, B are finitely generated ZG-modules then K is a finitely generated module since from the push-out diagram we have an epimorphism $P_{n-1} \oplus B \xrightarrow{\alpha+\beta} K \longrightarrow 0$.

PROPOSITION 1.3. *If G has period q after k-steps then so does every subgroup H of G.*

Proof. This follows from the fact that a projective ZG-module is a projective ZH-module.

PROPOSITION 1.4. *If G is an infinite group and has period q after k-steps then $k \geq 1$.*

Proof. Let H be a group which has period q and consider a periodic resolution of H

$$0 \longrightarrow Z \xrightarrow{\beta} P_{q-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow Z \longrightarrow 0.$$

Now im $\beta \rightarrowtail H^0(H, P_{q-1})$. But P_{q-1} is a direct summand of an induced ZH-module, and it is well known that if A is a non-trivial induced ZH-module then $H^0(H, A) \neq 0$ iff H is finite. The result follows.

LEMMA 1.5. *If G is a finite group and has period q after k-steps, then G has period q.*

Proof. By Theorem 1.1 (i)$\Rightarrow$(ii) there is a resolution of G

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & P_i \arrow[rr] \arrow[dr] & & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\
& & R_i \arrow[ur] & & & & &
\end{tikzcd} \tag{2}$$

and an element $h \in \operatorname{Ext}^q_{ZG}(R_k, R_k)$ such that for every ZG-module A cup product with h

$$\bigcup h : \operatorname{Ext}^i_{ZG}(R_k, A) \to \operatorname{Ext}^{i+q}_{ZG}(R_k, A)$$

is an isomorphism for $i \geq 1$.

Now since G is a finite group, a projective ZG-module P is a direct summand of a coinduced ZG-module; hence $H^i(G, P) = 0$ for all $i \geq 1$. Thus from (2) we obtain an isomorphism

$$\delta : \operatorname{Ext}^q_{ZG}(R_k, R_k) \xrightarrow{\simeq} H^q(G, Z).$$

Let $h' \in H^q(G, Z)$ be the image of $h \in \operatorname{Ext}^q_{ZG}(R_k, R_k)$ under δ. Then clearly for any ZG-module A cup product with h'

$$\bigcup h' : H^i(G, A) \to H^{i+q}(G, A)$$

is an isomorphism for $i \geq k+1$.

Let $0 \to C \to P \to A \to 0$ be a short exact sequence with P a projective ZG-module. Then we obtain the following commutative diagram

$$\begin{array}{ccc} H^{n-1}(G, A) & \xrightarrow{\cup h'} & H^{n+q-1}(G, A) \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_{n+q}} \\ H^n(G, C) & \xrightarrow[\cup h']{} & H^{n+q}(G, C) \end{array}$$

where δ_n, δ_{n+q} are connecting homomorphisms. But since $H^i(G, P) = 0$ for all $i \geq 1$, δ_i is an isomorphism for $i > 1$ and an epimorphism for $i = 1$. Thus it follows that cup product with h'

$$\bigcup h' : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for $i \geq 1$ and an epimorphism for $i = 0$. The result now follows from Theorem 1.1 (ii) $\Rightarrow$ (i).

PROPOSITION 1.6. *If G has period q after k-steps then every finite subgroup H of G has period q.*

Proof. This follows from Proposition 1.3 and Lemma 1.5.

COROLLARY 1.7. *If G has period 2 after k-steps then every finite subgroup H of G is cyclic.*

Proof. It is known [[6], Lemma 5.2] that a finite group H has period 2 iff it is cyclic; hence the result follows from Proposition 1.6.

Let G be a group such that the functors $H^j(G,-)$ and $H^{j+q}(G,-)$ are naturally isomorphic for some $j \geq 1$. Consider a resolution of G

$$\begin{tikzcd} \cdots \arrow[r] & P_i \arrow[r] \arrow[dr] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0. \\ & & R_i \arrow[u] & & & & \end{tikzcd} \tag{3}$$

Then clearly $\mathrm{Ext}_{ZG}(R_{j-1},-) \overset{\text{nat.}}{\simeq} \mathrm{Ext}_{ZG}(R_{j+q-1},-)$. Thus by [[4], Thm. 2.6] there exist projective ZG-modules Q_1, Q_2 such that

$$R_{j-1} \oplus Q_1 \xrightarrow[\alpha]{\cong} R_{j+q-1} \oplus Q_2. \tag{$*$}$$

Now (3) gives rise to the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & R_{j+q-1} \oplus Q_2 \arrow[r, "\varrho"] & P_{j+q-2} \oplus Q_2 \arrow[r] & P_{j+q-3} \arrow[r] & \cdots \arrow[r] & P_{j-2} \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & P_{j-1} \oplus Q_1 \arrow[r] \arrow[dr] & P_{j-2} \oplus Q_1 \arrow[r] \arrow[dr] & P_{j-3} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\ & & R_{j-1} \oplus Q_1 \arrow[u] & R_{j-2} \arrow[u] & & & & \end{tikzcd} \tag{4}$$

Hence it follows from (4) and ($*$) that:

PROPOSITION 1.8. *If the functors $H^\lambda(G,-)$ and $H^{\lambda+q}(G,-)$ are naturally isomorphic for some $\lambda \geq 2$, then G has period q after λ-1-steps.*

Now let $j=1$ and $q>0$. Clearly it follows that $H^n(G,-) \overset{\text{nat}}{\simeq} H^{n+q}(G,-)$ for all $n \geq 1$. Thus by Proposition 1.8 G has period q after 1-step. But G is finite since there is a monomorphism $\varrho\alpha : Z \oplus Q_1 \to P_{q-1} \oplus Q_2$ (same argument as for the proof of Proposition 1.4). Thus by Lemma 1.5 G has period q. Hence we have proved:

PROPOSITION 1.9. *If the functors $H^1(G,-)$ and $H^{1+q}(G,-)$ are naturally isomorphic for some $q>0$, then G is finite and has period q.*

2. Periodic countable locally finite groups

In this section we state and prove our main Theorem. The proof is based on direct limit arguments.

PROPOSITION 2.1 [[1]]. *Let $(P_i, \lambda_{ij})_I$ be a countable direct system of projective ZG-modules. Then there exists an exact sequence $0 \to Q \to Q \to P \to 0$ where $P = \varinjlim (P_i, \lambda_{ij})_I$ and Q is a projective ZG-module.*

LEMMA 2.2. *Let $\cdots \to A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\pi} \operatorname{coker} \alpha_1 \to 0$ be an exact sequence of ZG-modules such that for every $n \geq 0$ there is a resolution $P_{n*} \to A_n$ of the form $0 \to P_{n,1} \xrightarrow{\partial_1{}^n} P_{n,0} \xrightarrow{\partial_0{}^n} A_n \to 0$. If $\{\alpha_*^j\}: P_{j*} \to P_{j-1*}$ are chain maps lifting $\alpha_j : A_j \to A_{j-1}$ $j \geq 1$ then there is a resolution $P_* \to \operatorname{coker} \alpha_1$*

$$P_*: \cdots \longrightarrow P_k \xrightarrow{\partial_k} P_{k-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \operatorname{coker} \alpha_1 \longrightarrow 0$$

where

$$P_k = P_{k-1,1} \oplus P_{k,0} \quad k \geq 0$$

$$\partial_k = \begin{cases} P_{k,0} \xrightarrow[\alpha_0^k - (\partial_1^{k-2})^{-1} \alpha_0^{k-1} \alpha_0^k]{} P_{k-1,0} \oplus P_{k-2,1} \\ P_{k-1,1} \xrightarrow[\partial_1^{k-1} - \alpha_1^{k-1}]{} P_{k-1,0} \oplus P_{k-2,1} \end{cases} \qquad k \geq 1$$

and $\varepsilon = \pi\, \partial_0^0$.

Note that P_{ij} is understood to be zero if $i < 0$ or $j < 0$.

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "\alpha_n"] & A_{n-1} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r, "\alpha_1"] & A_0 \arrow[r, "\pi"] & \operatorname{coker} \alpha_1 \arrow[r] & 0 \\
\cdots \arrow[r] & P_{n,0} \arrow[r, "\alpha_0^n"] \arrow[u, "\partial_0^n"] & P_{n-1,0} \arrow[r] \arrow[u, "\partial_0^{n-1}"] & \cdots \arrow[r] & P_{1,0} \arrow[r, "\alpha_0^1"] \arrow[u, "\partial_0^1"] & P_{0,0} \arrow[u, "\partial_0^0"'] & & \\
\cdots \arrow[r] & P_{n,1} \arrow[r, "\alpha_1^n"] \arrow[u, "\partial_1^n"] & P_{n-1,1} \arrow[r] \arrow[u, "\partial_1^{n-1}"] & \cdots \arrow[r] & P_{1,1} \arrow[r, "\alpha_1^1"] \arrow[u, "\partial_1^1"] & P_{0,1} \arrow[u, "\partial_1^0"'] & &
\end{tikzcd}$$

Now it is easily seen that this gives rise to the resolution $P_* \to \operatorname{coker} \alpha_1$.

We shall need the following notion.

Let $(G_i, \gamma_{ij})_I$ be a direct system of groups. For $i \in I$ let (C_*^i, ∂_*^i) be a ZG_i-chain complex. If $i \leq j$ then let $c_{ij} : (C_*^i, \partial_*^i) \to (C_*^j, \partial_*^j)$ be a ZG_i-chain map, where (C_*^j, ∂_*^j) is considered as a ZG_i-chain complex via $\gamma_{ij} : G_i \to G_j$, such that

(1) c_{ii} is the identity chain map of (C_*^i, ∂_*^i), for all $i \in I$ and
(2) if $i \leq j \leq k$ then $c_{jk} c_{ij} = c_{ik}$ as ZG_i-chain maps.

We call $((C^i_*, \partial^i_*), c_{ij})_I$ a direct system of chain complexes over the direct system of groups $(G_i, \gamma_{ij})_I$. In particular let $(A_i, \alpha_{ij})_I$ be a direct system of modules over the direct system of groups $(G_i, \gamma_{ij})_I$. Clearly $(A_i, \alpha_{ij})_I$ is a direct system of abelian groups and if $A = \varinjlim (A_i, \alpha_{ij})_I$ then $G = \varinjlim (G_i, \gamma_{ij})_I$ acts on A in a natural way [[2], ch. 2, §6, nos 6, 7].

Moreover, if $B_i = ZG \underset{ZG_i}{\otimes} A_i$ and $\beta_{ij} : B_i \to B_j$

$$x \underset{ZG_i}{\otimes} a_i \to x \underset{ZG_j}{\otimes} \alpha_{ij} a_i$$

then it follows that $(B_i, \beta_{ij})_I$ is a direct system of ZG-modules and $\varinjlim (B_i, \beta_{ij})_I \simeq A$ as ZG-modules.

Recall that if X is a class of groups, then a group G is said to be locally an X-group if every finite set of elements of G is contained in some X-subgroup of G.

THEOREM 2.3. *Let G be a countable locally finite group all of whose finite subgroups have period q. Then G has period q after* 1-*step.*

Proof. It is easily seen that there is a direct system of finite subgroups of G over $I = \{1, 2, 3, \ldots\}$, $(G_i, e_{ij})_I$, with $e_{ij} : G_i \to G_j$ inclusions and $G = \varinjlim (G_i, e_{ij})_I$. By hypothesis for each $i \in I$ we have a periodic resolution $P^{i'}_*$ of G_i

$$P^{i'}_* : 0 \longrightarrow Z \longrightarrow P^{i'}_{q-1} \longrightarrow \cdots \longrightarrow P^{i'}_1 \longrightarrow ZG_i \xrightarrow{\varepsilon_i} Z \longrightarrow 0.$$

Now consider the following diagram

$$\begin{array}{cccccccccccccccc}
P^{i'}_* & 0 & \longrightarrow & Z & \longrightarrow & P^{i'}_{q-1} & \longrightarrow & \cdots & \longrightarrow & P^{i'}_1 & \longrightarrow & ZG_i & \xrightarrow{\varepsilon_i} & Z & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varrho'_{i\,i+1}} & & & \downarrow{\scriptstyle h} & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id_Z} & & \\
P^{i+1'}_* & 0 & \longrightarrow & Z & \longrightarrow & P^{i+1'}_{q-1} & \longrightarrow & \cdots & \longrightarrow & P^{i+1'}_1 & \longrightarrow & ZG_{i+1} & \xrightarrow{\varepsilon_{i+1}} & Z & \longrightarrow & 0.
\end{array}$$

Since $P^{i+1'}_*$ is evidently a resolution of G_i via $e_{i\,i+1} : G_i \to G_{i+1}$ we can lift id_Z to a ZG_i-chain map $\varrho'_{i\,i+1} : P^{i'}_* \to P^{i+1'}_*$.

Then $\varrho'_{i\,i+1}$ induces a map $h : Z \to Z$ (multiplication by h) which need not be the identity on Z.

Our aim is to find a periodic resolution P^{i+1}_* of G_{i+1} and a ZG_i-chain map $\varrho_{i\,i+1} : P^{i'}_* \to P^{i+1}_*$ which lifts id_Z on the right and induces id_Z on the left. Now the map $\varrho'_{i\,i+1}$ induces an isomorphism between the cohomology groups of G_i defined using $P^{i'}_*$ and those defined using $P^{i+1'}_*$. On $H^q(G_i, Z)$ this map is obviously multiplication by h. Since $H^q(G_i, Z) \simeq Z/|G_i|\, Z$ by the periodicity, it follows that $(h, |G_i|) = 1$; hence there are integers λ_1, λ_2 such that $1 = \lambda_1 h + \lambda_2 |G_i|$.

Let $\lambda = \lambda_1 + x\,|G_i|$ where $x = p_1 \cdots p_m$ with p_j $1 \leq j \leq m$ primes such that $p_j \mid |G_{i+1}|$ and $p_j \nmid |G_i|$, $p_j \nmid \lambda_1$ for all $1 \leq j \leq m$. If no such primes exist, take $x = 1$. Then

$$(\lambda, |G_{i+1}|) = 1 \quad \text{and} \quad 1 = \lambda h + (\lambda_2 - xh)\,|G_i|\ (*).$$

Consider $p_*^{i+1'}$ as a projective resolution of G_{i+1}. Then id_Z is a q-cocycle which defines an element $g_{i+1} \in H^q(G_{i+1}, Z)$. Clearly $\lambda g_{i+1} \in H^q(G_{i+1}, Z)$ is represented by the q-cocycle $\lambda : Z \to Z$ (multiplication by λ). We shall show that $\lambda g_{i+1} \in H^q(G_{i+1}, Z) = \mathrm{Ext}^q_{ZG_{i+1}}(Z, Z)$ is represented by a multiple extension

$$P_*^{i+1} : 0 \longrightarrow Z \longrightarrow P_{q-1}^{i+1} \longrightarrow P_{q-2}^{i+1} \longrightarrow \cdots \longrightarrow P_1^{i+1} \longrightarrow ZG_{i+1} \longrightarrow Z \longrightarrow 0$$

with P_k^{i+1} $1 \leq k \leq q-1$ projective ZG_{i+1}-modules. By Theorem 1.2 it is enough to show that for any ZG_{i+1}-module A cup product with λg_{i+1}

$$\cup \lambda g_{i+1} : H^k(G_{i+1}, A) \longrightarrow H^{k+q}(G_{i+1}, A)$$

is injective for $k = 1$ and surjective for $k = 0$. This follows easily since cup product with λg_{i+1} is multiplication by λ for $k = 0, 1$ and we have that $(\lambda, |G_{i+1}|) = 1$. Note that $|G_{i+1}|\, H^k(G_{i+1}, A) = 0$ for all $k \geq 1$ and by the periodicity $H^q(G_{i+1}, A) \simeq A^{G_{i+1}}/(\sum_{g \in G_{i+1}} g)A$ where $A^{G_{i+1}} \simeq \mathrm{Hom}_{ZG_{i+1}}(Z, A)$.

Moreover, from the proof of Theorem 1.2, we obtain the following commutative diagram

$$\begin{tikzcd}
P_*^{i'} \arrow[d, "\varrho'_{i\,i+1}"'] & 0 \arrow[r] & Z \arrow[r, "\mu_i"] \arrow[d, "h"'] & P_{q-1}^{i'} \arrow[rr] \arrow[d, "\alpha"] \arrow[ddl, "\gamma"'] & & \cdots \arrow[r] & P_1^{i'} \arrow[r] \arrow[d] & ZG_i \arrow[r] \arrow[d, tail] & Z \arrow[r] \arrow[d, "id_Z"] & 0 \\
P_*^{i+1'} \arrow[d, "L'"'] & 0 \arrow[r] & Z \arrow[r] \arrow[d, "\lambda"'] & P_{q-1}^{i+1'} \arrow[r] \arrow[d, "\beta"] & P_{q-2}^{i+1'} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_1^{i+1'} \arrow[r] \arrow[d, equal] & ZG_{i+1} \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "id_Z"] & 0 \\
P_*^{i+1} & 0 \arrow[r] & Z \arrow[r, "\mu_{i+1}"'] & P_{q-1}^{i+1} \arrow[r] & P_{q-2}^{i+1} \arrow[r] & \cdots \arrow[r] & P_1^{i+1} \arrow[r] & ZG_{i+1} \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

Clearly $L'\varrho'_{i\,i+1} : P_*^{i'} \to P_*^{i+1}$ is a ZG_i-chain map.

Now consider $P_*^{i'}$ as a resolution of G_i. Then id_Z is a q-cocycle which defines an element $g_i \in H^q(G_1, Z)$.

Clearly the q-cocycle $\lambda h : Z \to Z$ represents $\lambda h g_i$. But from $(*)$ $\lambda h g_i = g_i$ since $|G_i|\, g_i = 0$. Hence the cocycles h and id_Z represent the same element. So there exists $\gamma : P'_{q-1} \to Z$ such that $id_Z - \lambda h = \gamma\mu_i$. If we now take $\beta\alpha + \mu_{i+1}\gamma$ instead of $\beta\alpha$ we still have a chain map $P^{i'}_* \to P^{i+1}_*$ which now induces id_Z on the left. We call this chain map $\varrho_{i\,i+1}$, i.e. we have the following commutative diagram

$$\begin{array}{cccccccccccccccc}
P^{i'}_* & & 0 & \longrightarrow & Z & \longrightarrow & P^{i'}_{q-1} & \longrightarrow \cdots \longrightarrow & P^{i'}_1 & \longrightarrow & ZG_i & \xrightarrow{\varepsilon_i} & Z & \longrightarrow & 0 \\
{\scriptstyle \varrho_{i\,i+1}}\big\downarrow & & & & \big\downarrow{\scriptstyle id_Z} & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle id_Z} & & \\
P^{i+1}_* & & 0 & \longrightarrow & Z & \longrightarrow & P^{i+1}_{q-1} & \longrightarrow \cdots \longrightarrow & P^{i+1}_1 & \longrightarrow & ZG_{i+1} & \xrightarrow{\varepsilon_{i+1}} & Z & \longrightarrow & 0.
\end{array}$$

Take $P^{1'}_* = P^1_*$. Then given P^1_* and $P^{2'}_*$ we construct as above a periodic resolution P^2_* of G_2 and a ZG_1-chain map $\varrho_{1\,2} : P^1_* \to P^2_*$ which lifts id_Z on the right and induces id_Z on the left. In this way we construct inductively a direct system $(P^i_*, \varrho_{ij})_I$ of periodic resolutions over the direct system of groups $(G_i, e_{ij})_I$. Clearly $\varrho_{ij(i\leq j)} : P^i_* \to P^j_*$ are given as $\varrho_{ij} = \varrho_{j-1j} \cdots \varrho_{i\,i+1}$ and they lift id_Z on the right and induce id_Z on the left.

Now since direct limit preserves exactness, taking the direct limit of $(P^i_*, \varrho_{ij})_I$ we obtain the following exact sequence of ZG-modules

$$0 \longrightarrow Z \xrightarrow{\theta} \varinjlim P^i_{q-1} \xrightarrow{\alpha_{q-1}} \varinjlim P^i_{q-2} \longrightarrow \cdots \longrightarrow \varinjlim P^i_1 \xrightarrow{\alpha_1} ZG \xrightarrow{\varepsilon} Z \longrightarrow 0 \tag{$*$}'$$

By splicing together copies of $(*)'$ we obtain an exact sequence of ZG-modules

$$\cdots \xrightarrow{\alpha_1} ZG \xrightarrow{\alpha_q} \varinjlim P^i_{q-1} \xrightarrow{\alpha_{q-1}} \cdots \longrightarrow \varinjlim P^i_1 \xrightarrow{\alpha_1} ZG \longrightarrow$$
$$\xrightarrow{\alpha_q} \varinjlim P^i_{q-1} \xrightarrow{\alpha_{q-1}} \cdots \longrightarrow \varinjlim P^i_1 \xrightarrow{\alpha_1} ZG \xrightarrow{\varepsilon} Z \longrightarrow 0 \tag{1}$$

where $\alpha_q = \theta\varepsilon$.

By Proposition 2.1 we have resolutions $Q_{j*} \to \varinjlim P^i_j$ of the form

$$0 \longrightarrow Q_{j1} \longrightarrow Q_{j0} \longrightarrow \varinjlim P^i_j \longrightarrow 0 \quad \text{for all} \quad 1 \leq j \leq q-1.$$

Now the hypotheses of Lemma 2.2 hold for (1). Moreover, it is clear that we can choose here the chain maps $\{a^n_*\}$ of Lemma 2.2 so as

$$\{\alpha^{j+kq}_*\} = \{\alpha^j_*\}, \qquad \{\alpha^{\lambda q}_*\} = \{\alpha^q_*\} \qquad 1 \leq j \leq q-1, \qquad k \geq 0, \qquad \lambda \geq 1.$$

Thus by Lemma 2.2 we obtain a resolution of G

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & Q_{2,0}\oplus Q_{1,1} \arrow[r, "\partial_{q+2}"] & Q_{1,0} \arrow[r, "\partial_{q+1}"] \arrow[d] & ZG\oplus Q_{q-1,1} \arrow[r] & \cdots \\ & & R_{q+1} \arrow[ur] & & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \arrow[r] & Q_{2,0}\oplus Q_{1,1} \arrow[r, "\partial_{2}"] & Q_{1,0} \arrow[r, "\partial_{1}"] \arrow[d] & ZG \arrow[r, "\varepsilon"] & Z \arrow[r] & 0 \\ & & R_{1} \arrow[ur] & & & \end{tikzcd}$$

with $\partial_{q+2}=\partial_2$: $Q_{2,0}\oplus Q_{1,1}\to Q_{1,0}$. Hence $R_{q+1}\simeq R_1$; whence the result follows.

Remark. Note that in view of Proposition 1.4, Theorem 2.3 is the best possible we can obtain if G is an infinite countable locally finite group.

COROLLARY 2.4. *Let G be a countable locally finite group. Then the following statements are equivalent:*
(i) *G has period q after k-steps.*
(ii) *Every finite subgroup of G has period q.*
Moreover, $k=0$ if G is finite and $k=1$ if G is infinite.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 1.6, (ii) $\Rightarrow$ (i) is a consequence of Theorem 2.3.

Now if G is a finite group and P a projective ZG-module then $H^i(G, P)=0$ for all $i\geq 1$. We have an analogous result for a countable locally finite group:

PROPOSITION 2.5. *Let G be a countable locally finite group and P a projective ZG-module. Then $H^j(G, P)=0$ for all $j\geq 2$.*

Proof. It follows easily from the following result [[5], p. 158, Lemma 4.1].
If G is a countable locally finite group and A a ZG-module such that for some $n\geq 2$
$H^n(K, A)=0=H^{n-1}(K, A)$ for all finite subgroups K of G, then $H^n(G, A)=0$.

Now let G be a countable locally finite group which has period q after k-steps. Then by Corollary 2.4 G has period q after 1-step. Thus by Theorem 1.1 (i) $\Rightarrow$ (ii) there is a resolution of G

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & P_i \arrow[r] \arrow[d] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & Z \arrow[r] & 0 \\ & R_i \arrow[ur] & & & & & \end{tikzcd}$$

and an element $g' \in \operatorname{Ext}^q_{ZG}(R_1, R_1)$ such that for every ZG-module A cup product with g'

$$\cup g' : \operatorname{Ext}^i_{ZG}(R_1, A) \longrightarrow \operatorname{Ext}^{i+q}_{ZG}(R_1, A)$$

is an isomorphism for all $i \geq 1$.

By Proposition 2.5 if P is a projective ZG-module then $H^j(G, P) = 0$ for all $j \geq 2$. Having this result one obtains from (2) that $\operatorname{Ext}^q_{ZG}(R_1, R_1) \xrightarrow[\cong]{\delta} H^q(G, Z)$; note that $q \geq 2$ since by Proposition 1.6 q is a period of every finite subgroup of G, and that is known to be even [[3], ch. XII, p. 261].

Now if $g \in H^q(G, Z)$ corresponds to $g' \in \operatorname{Ext}^q_{ZG}(R_1, R_1)$ under δ then one clearly has that for any ZG-module A cup product with g

$$\cup g : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for all $i \geq 2$. Thus we have proved that:

PROPOSITION 2.6. *Let G be a countable locally finite group which has period q after k-steps. Then there is an element $g \in H^q(G, Z)$ such that for any ZG-module A cup product with g*

$$\cup g : H^i(G, A) \longrightarrow H^{i+q}(G, A)$$

is an isomorphism for all $i \geq 2$.

§3. Periodic locally finite p-groups

The following well known theorem characterizes the finite p-groups which have period $q > 0$.

THEOREM 3.1 [[3], ch. XII, Thm. 11.6]. *For a finite p-group G the following statements are equivalent:*
(i) *G has period $q > 0$*
(ii) G is either cyclic or is a (generalized) quaternion group.
Moreover, a cyclic group has period 2 and a (generalized) quaternion group has period 4.

We shall characterize the infinite locally finite p-groups which have period $q > 0$ after k-steps.

I. Consider an infinite locally cyclic p-group. It is easily seen that such a group is uniquely determined up to isomorphism by the prime p and is given by

$$C_p\infty = \langle c_1, c_2, \ldots, c_i, \ldots; c_1^p = 1, c_2^p = c_1, \ldots, c_{i+1}^p = c_i, \ldots \rangle.$$

II. Recall that a (generalized) quaternion group $Q_2 i$ is given by

$$Q_2 i = \langle x, y; x^{2^{i-2}} = y^2, xyx = y \rangle \quad i \geq 3.$$

One easily shows that a (generalized) quaternion group $Q_2 i$ contains a normal cyclic subgroup C of index 2, which is characteristic if $i > 3$, and every element of $Q_2 i \setminus C$ is of order four and inverts every element of C. Having this result and following the proof of Proposition 1.I.2 [8], one shows:

PROPOSITION 3.2. *Let G be a locally quaternion group. Then G has a locally cyclic normal subgroup C of index* 2, *and every element of $G\setminus C$ is of order four and inverts every element of C.*

COROLLARY 3.3. *Up to isomorphism there exists only one infinite locally quaternion group, namely*

$$Q_2\infty = \langle c_1, \ldots, c_n, \ldots, y; c_1^2 = 1, \ldots, c_{n+1}^2 = c_n, \ldots, y^2 = c_1, \\ c_i^y c_i = 1 \quad \text{all} \quad i \geq 1 \rangle.$$

Proof. This follows easily from Proposition 3.2 and the fact that up to isomorphism there exists only one infinite locally cyclic 2-group, namely, $C_2\infty$.

PROPOSITION 3.4. (i) *An infinite locally cyclic p-group has period* 2 *after* 1-*step.*
(ii) *An infinite locally quaternion group has period* 4 *after* 1-*step.*

Proof. (i) Clearly it is enough to consider $C_{p\infty}$. Now $C_{p\infty}$ is a countable group. Moreover, every finite subgroup of $C_{p\infty}$ is cyclic, hence by Theorem 3.1 it has period 2. The result now follows from Theorem 2.3.
(ii) By Corollary 3.3 it is enough to consider $Q_{2\infty}$. Clearly $Q_{2\infty}$ is countable. Moreover, if K is a finite subgroup of $Q_{2\infty}$ then K is contained in some (generalized) quaternion group; hence by Theorem 3.1 and Proposition 1.3 K has period 4. Now the result follows from Theorem 2.3.

THEOREM 3.5. *For an infinite locally finite p-group G the following statements are equivalent:*
(i) *G has period $q > 0$ after k-steps.*
(ii) *G is either C_{p^∞} or Q_{2^∞}.*
Moreover, $k = 1$. If $G = C_{p^\infty}$ then $q = 2$ and if $G = Q_{2^\infty}$ then $q = 4$.

Proof. (i) $\Rightarrow$ (ii). Let G have period q after k-steps. By Proposition 1.6 every finite subgroup of G has period q and therefore by Theorem 3.1 every finite subgroup of G is either cyclic or is a (generalized) quaternion group. Thus G is either an infinite locally cyclic p-group or an infinite locally quaternion group, i.e. G is either C_{p^∞} of by Corollary 3.3, G is Q_{2^∞}. (ii) $\Rightarrow$ (i) follows from Proposition 3.4.

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Forschungsinstitut für Mathematik
ETH Zentrum
CH-8092 Zürich

Received March 10, 1979.