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Twisted Hopf Algebras

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Introduction

Let R be a commutative ring with unit and let A be a commutative R-algebra. Let L be a twisted Lie algebra over A (see 3.1). As R-Lie algebra, L has a universal enveloping algebra, $U_R L$, as twisted Lie algebra over A it has a twisted universal enveloping algebra, $U_{R-A}L$. It is well known that $U_R L$ is a Hopf algebra with antipode, [3], while one easily realises that this is, in general, not true for $U_{R-A}L$. However, there are certain parallels between $U_R L$ and $U_{R-A}L$ pointing to a Hopf algebra-like structure on $U_{R-A}L$. First of all, the diagonal $\Delta : U_R L \to U_R L \otimes_R U_R L$ induces a canonical L-module structure on the R-tensor product $M_0 \otimes_R N_0$ of two L-modules M_0 and N_0. If M and N are modules of L considered as twisted Lie algebra (see 3.10), then there is a similar canonical L-module structure on $M \otimes_A N$, [4], so there should be some diagonal-like map for $U_{R-A}L$. If α is the antipode of $U_R L$, then the map $\nabla = (1 \otimes \alpha) \circ \Delta$ induces a canonical L-module structure on $Hom_R(M_0, N_0)$. There is a similar canonical L-module structure on $Hom_A(M, N)$ for the twisted Lie algebra L, which points to the existence of some sort of ∇-map for $U_{R-A}L$. As a main result of this paper we show that mappings Δ and ∇ exist for $U_{R-A}L$ with the appropriate properties, see 3.7. We call the resulting structure on $U_{R-A}L$ twisted Hopf algebra.

In [7], Sweedler uses twisted Hopf algebras, or $\times_A$-bialgebras, as he named them, to classify algebras over A for commutative A. Takeuchi obtained Sweedler's theory for non-commutative A, [8]. In our paper, we don't deal with classifications of algebras. Our approach to twisted Hopf algebras is via twisted Lie algebras. In particular we are interested in the question of when such a twisted Hopf algebra is isomorphic to $U_{R-A}L$ for some L, see 4.20.

In the paper we assume the reader to be familiar with ordinary Lie algebra and Hopf algebra theory. Basic material for developing the theory of twisted Hopf algebras is presented in chapter 1. It can be found in greater detail and with proofs in [7] and [8]. In chapter 2 we define twisted Hopf algebras and prove that the category of modules of a twisted Hopf algebra (co-commutative and admitting a ∇-map) is a closed abelian category, see 2.11. In chapter 3 we show that the twisted universal enveloping algebra of a twisted Lie algebra is a twisted Hopf algebra admitting a ∇-map. In chapter 4 we introduce some homological machinery to obtain a Poincaré-Birkhoff-Witt theorem for twisted Lie algebras and, above all, to investigate the relationships of twisted Hopf algebras with twisted Lie algebras using primitive elements.

One word about the notations. All our rings and algebras have units and all modules are unitary. Injective and surjective morphisms are denoted by the arrows $\rightarrowtail$ and $\twoheadrightarrow$, respectively. We also use the notations $\xrightarrow{\text{incl}}$ and $\xrightarrow{\text{proj}}$ to denote canonical inclusions and canonical projections.

1. Preliminaries

We list here some basic material needed in this paper. More details, in particular proofs, can be found in [7] and [8]. We adopt the notations of [7] and [8].

Let R be a fixed commutative ring with unit. We denote the category of R-modules by R-mod. Unadorned Hom, End and $\otimes$ mean R-morphisms, R-endomorphisms and tensor product over R, respectively.

Let A be a fixed commutative algebra over R. We denote the category of left A-modules by A-mod. An A-bimodule is an R-module M carrying a left and right A-module structure which commute with each other, i.e. for $a, a' \in A$, $m \in M$, $(a \cdot m) \cdot a' = a \cdot (m \cdot a')$. The category of A-bimodules is denoted by A-bimod. Via the ring map $R \to A$ defining the R-algebra structure on A we consider A-mod and A-bimod as subcategories of R-mod. M^0 denotes the opposite A-bimodule of M. M^0 is R-isomorphic with M via $m \in M \mapsto m^0 \in M^0$ with A-bimodule structure defined by $a \cdot m^0 \cdot a' = (a' \cdot m \cdot a)^0$, $a,\ a' \in A$, $m \in M$. For $M \in A$-bimod, the left and right A-module structure is indicated by ${}_xM$ and M_y, respectively, [7].

Let $M, N \in A$-bimod. Let

$$M_A \otimes N = \int_x {}_xM \otimes {}_xN \quad \left(M \otimes_A N = \int_x M_x \otimes {}_xN \right)$$

be the quotient R-module of $M \otimes N$ by the R-submodule generated by the elements $am \otimes n - m \otimes an$ $(ma \otimes n - m \otimes an)$, $m \in M$, $n \in N$, $a \in A$. As R-module, $M \otimes_A N \cong M^0 {}_A\otimes N$. Define

$$M \times_A N = \int^u \int_x {}_xM_u \otimes {}_xN_u = \int^u M_u {}_A\otimes N_u = \Big\{ \sum_i m_i {}_A\otimes n_i \in M_A \otimes N \,\Big|\, \sum_i m_i a {}_A\otimes n_i = \sum_i m_i {}_A\otimes n_i a, \forall a \in A \Big\},$$

the left (right) A-module structure being the xA-module structure (uA-module structure). $\times_A$ defines a bifunctor $\times_A : A\text{-bimod} \times A\text{-bimod} \to A\text{-bimod}$. For $f : M \to M'$, $g : N \to N'$ in A-bimod, $f \otimes g$ induces a morphism $f \times g : M \times_A N \to M \times_A N'$ in A-bimod, [7, (2.3)]. The twist map $M_A \otimes N \to N_A \otimes M$ induces an isomorphism "twist": $M \times_A N \to N \times_A M$ in A-bimod. The product $\times_A$ is not necessarily associative. However, for $M, N, P \in A$-bimod, define

$$M \times_A N \times_A P = \int^u M_{u\,A} \otimes N_{u\,A} \otimes P_u$$

with A-bimodule structure being given by ${}_x(M \times_A N \times_A P)_y = {}_x M_y \times_A N \times_A P$. Then there are canonical maps in A-bimod, in general not injective,

$$\alpha : (M \times_A N) \times_A P \to M \times_A N \times_A P$$

$$\alpha' : M \times_A (N \times_A P) \to M \times_A N \times_A P.$$

Later on, we shall have to consider

$$(M^0 \times_A N)^0 = \int^u \int_x {}_u M_x \otimes_x N_u = \int^u {}_u M \otimes_A N_u,$$

the left (right) A-module structure being the uA-module structure (xA-module structure). Let

$$Z(M, N, P) = \int^{u,v} {}_u M_v \otimes_A N \otimes_{A\,u} P_v$$

with left (right) A-module structure being given by ${}_u M$ (M_u). There is a canonical map in A-bimod (compare [8, §6]).

$$\beta : (M^0 \times_A N)^0 \times_A P \to Z(M, N, P).$$

Let $M, N, P, Q \in A$-bimod. The twist map $N \otimes P \to P \otimes N$ gives rise to $\varphi : \int_x (M \times_A N)_x \otimes_x Q \to \int_{xx} M \otimes_A P \otimes_x N \otimes_A Q$ which induces
$\xi : (M \times_A N) \otimes_A (P \times_A Q) \to (M \otimes_A P) \times_A (N \otimes_A Q)$, [7, (2.10)].

Let $M, N \in A$-bimod. $Hom_A(M, N)$ denotes A-linear morphisms from M to N, where both M and N are considered as left A-modules. For $M \in A$-mod, $\operatorname{End} M$ has A-bimodule structure defined by $a \cdot \varphi : m \mapsto a \cdot \varphi(m)$, $\varphi a : m \mapsto \varphi(a \cdot m)$, $a \in A$, $\varphi \in \operatorname{End} M$, $m \in M$. In particular, $\operatorname{End} A$ is in A-bimod. For

$M \in A$-bimod, there are maps in A-bimod

$$\theta : M \times_A \text{End } A \to M$$

$$\sum_i m_{i\,A} \otimes \varphi_i \mapsto \sum_i \varphi_i(1) \cdot m_i,$$

$$\theta' : \text{End } A \times_A M \to M$$

$$\sum_j \varphi_{j\,A} \otimes m_j \mapsto \sum_j \varphi_j(1) \cdot m_j,$$

[8, 2.2].

2. Algebras, coalgebras and twisted Hopf algebras

2.1 DEFINITION. An *algebra over* A is a pair $\mu : A \to B$ in the category R-alg of R-algebras. Algebras over A are denoted by (B, m, μ) or simply B, where $m : B \otimes_A B \to B$ denotes multiplication. There is a natural A-bimodule structure on B defined by left and right multiplication. Algebras over A form a category which will be denoted by $(R-A)$-alg. For (B, m, μ) in $(R-A)$-alg, (B^0, m^0, μ^0) denotes the opposite algebra, where $\mu^0(a) = (\mu(a))^0$, $m^0(b^0 \otimes_A b'^0) = (m(b' \otimes_A b))^0$. The underlying A-bimodule of B^0 is the opposite of the underlying A-bimodule of B.

Let $M \in A$-mod. The A-module structure on M is given by an R-algebra map $\mu : A \to \text{End } M$. Hence $\text{End } M$ is in $(R-A)$-alg.

Let B' and B'' be in $(R-A)$-alg. By [7, (4.1)], $B' \times_A B''$ is in $(R-A)$-alg in a canonical way. Siminarily, $(B'^0 \times_A B'')^0 \in (R-A)$-alg. The $\times_A$-product preserves morphisms in $(R-A)$-alg, [7, (4.2)].

For $M, N \in A$-mod, there is a canonical morphism in $(R-A)$-alg

$$\text{End } M \times_A \text{End } N \to \text{End } (M_A \otimes N) \tag{2.1}$$

which is induced by $\text{End } M \otimes \text{End } N \to \text{End } (M \otimes N) \to \text{Hom } (M \otimes N, M_A \otimes N)$. The R-algebra map $((\text{End } N)^0 \otimes \text{End } M)^0 \to \text{End}(\text{Hom}(M, N))$, $(\psi^0 \otimes \varphi)^0 \mapsto (f \mapsto \psi \circ f \circ \varphi)$, $\varphi \in \text{End } M$, $\psi \in \text{End } N$, $f \in \text{Hom } (M, N)$, followed by $\text{End } (\text{Hom}(M, N)) \to \text{Hom } (\text{Hom}_A (M, N), \text{Hom } (M, N))$ induces an $(R-A)$-algebra morphism

$$((\text{End } N)^0 \times_A \text{End } M)^0 \to \text{End}(\text{Hom}_A (M, N)). \tag{2.2}$$

The proofs of (2.1) and (2.2) are straightforward.

2.2 DEFINITION. An *augmentation* of an algebra B over A is a pair $\varepsilon : B \to \operatorname{End} A$ in $(R-A)$-alg. This is the same as giving a left A-morphism $\varepsilon_1 : B \to A$ such that $\varepsilon_1 \circ \mu = id_A$, $\varepsilon_1(b \cdot b') = \varepsilon_1(b \cdot \varepsilon_1(b'))$. Then $\varepsilon_1(b) = \varepsilon(b)(1)$, [7, p. 108]. The *augmentation ideal* $I(B) = \ker \varepsilon_1$ is only a left ideal in B. Augmented algebras form a category, denoted by $(R-A)$-aalg. Clearly, $\operatorname{End} A \in (R-A)$-aalg.

2.3 DEFINITION. For $B \in (R-A)$-alg, a *left B-module* is a pair $\varphi : B \to \operatorname{End} M$ in $(R-A)$-alg, where $M \in A$-mod. We write (M, φ), or simply M, for this pair. This definition is equivalent to giving a map $B \otimes_A M \to M$ in A-mod with the usual associativity properties. A *morphism* of B-modules is a map $\phi : M \to N$ in A-mod such that

$$\begin{array}{ccccc} & & \operatorname{End} M & & \\ & \nearrow & & \searrow^{\phi_*} & \\ B & & & & \operatorname{Hom}(M, N) \\ & \searrow & & \nearrow_{\phi^*} & \\ & & \operatorname{End} N & & \end{array}$$

commutes. The category of B-modules is denoted by B-mod. This is an abelian category. If B is augmented, $\operatorname{End} A \in B$-mod.

2.4 DEFINITION [8, 4.1]. A *coalgebra* is a triple (C, Δ, ε), where $C \in A$-bimod, the diagonal $\Delta : C \to C \times_A C$ and the counit $\varepsilon : C \to \operatorname{End} A$ are in A-bimod satisfying

$$\begin{array}{ccccccc} & & C \times_A C & \xrightarrow{\Delta \times 1} & (C \times_A C) \times_A C & & \\ & {}^{\Delta}\nearrow & & & & \searrow^{\alpha} & \\ C & & & & & & C \times_A C \times_A C \\ & {}_{\Delta}\searrow & & & & \nearrow_{\alpha'} & \\ & & C \times_A C & \xrightarrow{1 \times \Delta} & C \times_A (C \times_A C) & & \end{array}$$

$$\begin{array}{ccccc} & & C \times_A C & & \\ & {}^{\varepsilon \times 1}\swarrow & \uparrow{\scriptstyle \Delta} & \searrow^{1 \times \varepsilon} & \\ \operatorname{End} A \times_A C & & & & C \times_A \operatorname{End} A. \\ & {}_{\theta'}\searrow & & \swarrow_{\theta} & \\ & & C & & \end{array}$$

We write

$$\Delta(c) = \sum c_{(1)}\,{}_A\!\otimes c_{(2)}, \quad c \in C.$$

C is called *co-commutative*, if

$$\begin{array}{ccc} C \times_A C & \xrightarrow{\text{twist}} & C \times_A C \\ {}^{\Delta}\nwarrow & & \nearrow^{\Delta} \\ & C & \end{array}$$

commutes. A morphism of coalgebras from $(C', \Delta', \varepsilon')$ to $(C'', \Delta'', \varepsilon'')$ is a map $\varphi : C' \to C''$ in A-bimod such that $\Delta'' \circ \varphi = (\varphi \times \varphi) \circ \Delta'$, $\varepsilon'' \circ \varphi = \varepsilon'$. We denote the category of coalgebras by $(R-A)$-coalg. Clearly, $A \in (R-A)$-coalg.

Let (C', Δ', ξ'), $(C'', \Delta'', \varepsilon'') \in (R-A)$-coalg. On $C = C' \otimes_A C''$ we define the diagonal Δ as the composite of $C' \otimes_A C'' \xrightarrow{\Delta' \otimes_A \Delta''} (C' \times_A C') \otimes_A (C'' \times_A C'') \xrightarrow{\xi} (C' \otimes_A C'') \times_A (C' \otimes_A C'')$ and the counit as the composite of $C' \otimes_A C'' \xrightarrow{\varepsilon' \otimes_A \varepsilon''} \text{End}\, A \otimes_A \text{End}\, A \xrightarrow{m} \text{End}\, A$. (C, Δ, ε) is in $(R-A)$-coalg, [8, 4.7].

2.5 DEFINITION. A *co-augmentation* of (C, Δ, ε) is a pair $\mu : A \to C$ in $(R-A)$-coalg. The category of co-augmented coalgebras is denoted by $(R-A)$-ccoalg. It is closed under the tensor product defined above.

2.6 DEFINITION. A *twisted Hopf algebra* is a 5-tuple $(H, m, \mu, \Delta, \varepsilon)$, where
1) $(H, \Delta, \varepsilon, \mu) \in (R-A)$-ccoalg
2) $(H, m, \mu, \varepsilon) \in (R-A)$-aalg
3) The multiplication $m : H \times_A H \to H$ is in $(R-A)$-coalg.
Conditions 1), 2) and 3) are equivalent to 1), 2) and
3′) The diagonal $\Delta : H \to H \times_A H$ is in $(R-A)$-alg.
H is called *co-commutative*, if the underlying $(R-A)$-coalgebra is co-commutative. A morphism of twisted Hopf algebras is a map which is both in $(R-A)$-aalg and in $(R-A)$ ccoalg. We denote the category of twisted Hopf algebras by $(R-A)$-Hopf. A left *H-module* is a left module of the underlying $(R-A)$-algebra of H.

For $R = A$ this definition yields the familiar Hopf algebras, [3]. The reason for the notation "twisted Hopf algebra" instead of "$\times_A$ -bialgebra" as in [7] and [8] is that the twisted universal enveloping algebra of a twisted Lie algebra is a twisted Hopf algebra, see section 3 below, while the universal enveloping algebra of an ordinary Lie algebra is an ordinary Hopf algebra, [3].

2.7 Example. The smash product. Let K be a co-commutative Hopf algebra over R. Let A be a commutative K-module algebra, [6, p. 153]. As left A-module, the smash product $A\#K$ is $A \otimes K$. Define $\Delta : A\#K \to (A\#K) \times_A (A\#K)$ by $a\#k \mapsto \sum a\#k_{(1)}\, {}_A\!\otimes 1\#k_{(2)}$. Then $A\#K$ is in $(R-A)$-Hopf, [7, p. 117].

The Δ-map implies the construction of a tensor product in H-mod as follows: Let (M, φ), $(N, \psi) \in H$-mod. Define $\varphi \otimes \psi : H \to \text{End}(M_A \otimes N)$ as the composite of $H \xrightarrow{\Delta} H \times_A H \xrightarrow{\varphi \times \psi} \text{End}\, M \times_A \text{End}\, N \to \text{End}(M_A \otimes N)$, the last arrow being the map (2.1). That is, $h(m_A \otimes n) = \sum \varphi h_{(1)}(m)\, {}_A\!\otimes \psi h_{(2)}(n)$, $h \in H$, $m \in M$, $n \in N$.

$(M_A \otimes N, \varphi \otimes \psi)$ is called the *tensor product* of (M, φ) with (N, ψ). Note that this product is associative. Consequently, H-mod is a monoidal category, the left and right unit being (A, ε). If H is co-commutative, the twist map $M_A \otimes N \rightarrow N_A \otimes M$ induces an isomorphism in H-mod $(M_A \otimes N, \varphi \otimes \psi) \rightarrow (N_A \otimes M, \psi \otimes \varphi)$.

We thus obtained a functor ${}_A\otimes N$ from H-mod into itself. A suitable generalisation of "antipode" to the case of twisted Hopf algebras provides the right adjoint to this functor.

2.8 DEFINITION. Let $(H, m, \mu, \Delta, \varepsilon) \in (R-A)$-Hopf. A ∇-*map* is a morphism $\nabla : H \rightarrow (H^0 \times_A H)^0$ in $(R-A)$-alg such that the diagrams

$$\begin{array}{ccccc} H^0 \times_A H & \xrightarrow{\Delta^0 \times 1} & (H \times_A H)^0 \times_A H & \xrightarrow{\beta} & Z(H^0, H, H) \\ \uparrow{\scriptstyle \nabla^0} & & & & \downarrow{\scriptstyle 1^0 \otimes m} \\ H^0 & \xrightarrow{\simeq} & H^0 \times_A A & \xrightarrow{1^0 \otimes \mu} & H^0 \otimes_A H \end{array}$$

$$\begin{array}{ccccc} H \times_A H & \xrightarrow{\nabla \times 1} & (H^0 \times_A H)^0 \times_A H & \xrightarrow{\beta} & Z(H, H, H) \\ \uparrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle 1 \otimes m} \\ H & \xrightarrow{\simeq} & H \otimes_A A & \xrightarrow{1 \otimes \mu} & H \otimes_A H \end{array}$$

commute. Similar to 2.4 we write

$$\nabla(h) = \sum h^{(1)} \otimes_A h^{(2)}, \ h \in H.$$

Twisted Hopf algebras admitting a ∇-map form a category, denoted by $(R-A)$-aHopf, the morphisms being maps $\varphi : H' \rightarrow H''$ in $(R-A)$-Hopf such that $\nabla' \circ \varphi = (\varphi^0 \times \varphi)^0 \circ \nabla$.

The ∇-map generalises antipodes of ordinary Hopf algebras in the following sense:

2.9 Example. Let $R = A$ and let H be an ordinary Hopf algebra over R with antipode α. Thus $\alpha : H \rightarrow H$ is an algebra antimorphism, $\alpha(h \cdot h') = \alpha(h') \cdot \alpha(h)$, such that $m \circ (1 \otimes \alpha) \circ \Delta = \mu \circ \varepsilon$, $m \cdot (\alpha \otimes 1) \cdot \Delta = \mu \circ \varepsilon$. Define the ∇-map by

$$\nabla = (1 \otimes \alpha) \circ \Delta.$$

It is easy to see that this ∇ satisfies 2.8.

2.10 Example. Consider the smash product, 2.7. Denote by Δ_K the diagonal of K.

Suppose α is an antipode of K. Let $\nabla_K = (1 \otimes \alpha) \circ \Delta_K$. Define $\nabla : H \to H \otimes_A H$ as the composite of

$$A\#K \xrightarrow{1\#\nabla_K} A\#K \otimes K \xrightarrow{\text{incl}} A\#K \otimes A\#K \xrightarrow{\text{proj}} (A\#K) \otimes_A (A\#K).$$

Then $\nabla : H \to (H^0 \times_A H)^0$, [7, p. 142]. It is left to the reader to show that ∇ satisfies 2.8.

Sweedler, [7], and Takeuchi, [8], use a somewhat different ∇-map, called the Ess. Our ∇-map is, in some sense, adjoint to the diagonal Δ. To be more precise, we construct an H-module structure on $\text{Hom}_A(M, N)$, (M, φ), $(N, \psi) \in H$-mod, as the composite of

$$H \xrightarrow{\nabla} (H^0 \times_A H)^0 \xrightarrow{(\psi^0 \times \varphi)^0} ((\text{End } N)^0 \times_A \text{End } M)^0 \longrightarrow \text{End}(\text{Hom}_A(M, N),$$

the last arrow being the map (2.2). That is, $h(F) : m \mapsto \sum \psi h^{(1)}(F(\varphi h^{(2)}(m)))$, $h \in H$, $m \in M$, $F \in \text{Hom}_A(M, N)$. The adjointness of the Δ- and ∇-map is expressed in the following

2.11 THEOREM. *The functors* ${}_A\otimes N$ *and* $Hom_A(N, \ )$ *are adjoint, i.e. for* $M, N, P \in H$-mod, *there is a natural isomorphism in* A-mod $\text{Hom}_H(M_A \otimes N, P) \cong$ $Hom_H(M, Hom_A(N, P))$. *Therefore, if* H *is co-commutative,* H-mod *is a closed abelian category.*

Proof: Let $f \in Hom_H(M_A \otimes N, P)$, i.e. $h(f(m_A \otimes n)) = \sum f(h_{(1)} m_A \otimes h_{(2)} n)$. Define $m \in M \mapsto F_m \in Hom_A(N, P)$ by $F_m(n) = f(m_A \otimes n)$. By 2.8,

$$\begin{aligned}(hF_m)(n) &= \sum h^{(1)}(F_m(h^{(2)}n)) = \sum h^{(1)}(f(m_A \otimes h^{(2)}n)) \\ &= \sum\sum f(h^{(1)}{}_{(1)} m_A \otimes h^{(1)}{}_{(2)} h^{(2)} n) = f(hm_A \otimes n) = F_{hm}(n).\end{aligned}$$

Conversely, given $m \in M \mapsto F_m \in Hom_A(N, P)$ such that $F_{hm}(n) = (hF_m)(n) = \sum h^{(1)}(F_m(h^{(2)}n))$. Define $f : M_A \otimes N \to P$ by $f(m_A \otimes n) = F_m(n)$. Then, by 2.8,

$$\begin{aligned}\sum f(h_{(1)} m_A \otimes h_{(2)} n) &= \sum (F_{h_{(1)}m})(h_{(2)} n) = \sum\sum h_{(1)}{}^{(1)}(F_m(h_{(1)}{}^{(2)} h_{(2)} n)) \\ &= h(F_m(n)) = h(f(m_A \otimes n)). \quad \square\end{aligned}$$

3. Twisted Lie algebras

Twisted Lie algebras are studied in [2], [4]. We are concerned here with the twisted universal enveloping algebra of a twisted Lie algebra and show that this is a twisted Hopf algebra with ∇-map.

Let

$$\text{Der } A = \{\xi \in \text{End } A \mid \xi(ab) = \xi(a) \cdot b + a \cdot \xi(b)\}$$

be the A-module of R-derivations of A. This has an R-Lie algebra structure by defining the bracket as

$$[\xi, \eta](a) = \xi(\eta(a)) - \eta(\xi(a)).$$

3.1 DEFINITION. Let $L \in A$-mod. L is called *twisted Lie algebra over* A if the following holds:

1) L is an R-Lie algebra.

2) There exists a morphism of R-Lie algebras and left A-modules $\varphi : L \to \mathrm{Der}\, A$ such that, for all $a \in A$, $x, y \in L$, $[x, ay] = a[x, y] + \varphi(x)(a)y$.

We simply write $x(a)$ for $\varphi(x)(a)$. A *morphism* of twisted Lie algebras is a morphism $L' \to L''$ of R-Lie algebras and A-modules such that

$$\begin{array}{ccc} L' & \longrightarrow & L'' \\ & \searrow \; \swarrow & \\ & \mathrm{Der}\, A & \end{array}$$

commutes. The category of twisted Lie algebras over A is denoted by $(R-A)$-Lie; clearly, $\mathrm{Der}\, A \in (R-A)$-Lie.

For $R = A$, 3.1 reduces to the usual definition of the category of R-Lie algebras. On the other hand, if $L \to \mathrm{Der}\, A$ is the zero map we obtain an A-Lie algebra, whence A-Lie $\subseteq (R-A)$-Lie.

Consider the category $(R-A)$-alg. We construct a functor Der : $(R-A)$-alg $\to (R-A)$-Lie as follows:

3.2 DEFINITION. Let $A \to B \in (R-A)$-alg. Let

$$\mathrm{Der}(A \to B) = \{(\xi, b) \in \mathrm{Der}\, A \oplus B \mid ba = ab + \xi(a),\ \forall a \in A\}.$$

This is an A-module by setting $a \cdot (x, b) = (ax, ab)$. The bracket is defined by $[(\xi, b), (\xi', b')] = ([\xi, \xi'], [b, b'])$. The morphism $\mathrm{Der}(A \to B) \to \mathrm{Der}\, A$ is the projection onto the first factor. Therefore, $\mathrm{Der}(A \to B) \in (R-A)$-Lie. A morphism $\varphi : B' \to B''$ in $(R-A)$-alg induces a morphism $\mathrm{Der}(A \to B') \to \mathrm{Der}(A \to B'')$ by sending (ξ, b) to $(\xi, \varphi(b))$. Consequently, Der is a functor.

The functor Der has a left adjoint which is given by the universal enveloping algebra construction.

3.3 DEFINITION. Let $L \in (R-A)$-Lie. Let TL be the tensor algebra of L over R. An element $x_1 \otimes \cdots \otimes x_r \in TL$ of degree r is simply denoted by $x_{1 \ldots r}$. Let $J_L \subseteq TL$ be the twosided ideal $J_L = (x \otimes y - y \otimes x - [x, y])$, $x, y \in L$. $U_R L = TL/J_L$ is the ordinary enveloping algebra of the R-Lie algebra L. On $TL \otimes A$

define an $(R-A)$-algebra structure by $(x_{1\ldots r}\otimes a)(y_{1\ldots s}\otimes a') = x_{1\ldots r}\otimes a y_1 \otimes y_{2\ldots s}\otimes a'$, $x_{1\ldots r}$, $y_{1\ldots s}\in TL$, a, $a'\in A$. Let $J_0 \subseteq TL\otimes A$ be the twosided ideal defined by $J_0 = (x\otimes a - ax - x(a))$, $a\in A$, $x\in L$, and put $J = J_0 + J_L\otimes A$. This is an ideal in $TL\otimes A$. The *twisted universal enveloping algebra* $U_{R-A}L$ *of* L is defined to be

$$U_{R-A}L = TL\otimes A/J.$$

U_{R-A} is a functor from $(R-A)$-Lie into $(R-A)$-alg.

3.4 *Remark.* $U_{R-A}L$ can also be obtained by $T^+(A\oplus L)/I$, where $T^+(A\oplus L)$ denotes the elements of positive degree in the R-tensor algebra $T(A\oplus L)$ and I is the ideal $I = (a\otimes a' - aa', a\otimes x - ax, x\otimes a - a\otimes x - x(a), x\otimes y - y\otimes x - [x, y])$, a, $a'\in A$, $x, y\in L$, [4].

3.5 PROPOSITION. U_{R-A} *is left adjoint to* Der.

Proof: Similar to the untwisted case. □

3.6 *Remark.* We can restate 3.5 in terms of the universal property of $U_{R-A}L$. Let $\phi : L \to B$ be a morphism of left A-modules and R-Lie algebras such that $(\varphi(x), \phi(x)) \in \mathrm{Der}(A\to B)$, φ being the map in 2), 3.1. Then there exists a unique morphism $U_{R-A}L \to B$ in $(R-A)$-alg such that

$$\begin{array}{ccc} L & \longrightarrow & B \\ \downarrow & & \uparrow \\ U_{R-A}L & & \end{array}$$

commutes.

In case $R = A$, $U_{R-A}L$ is just the ordinary universal enveloping algebra of L. $\mathrm{Der}(R\to B)$ reduces to the Lie algebra structure on B coming from the associative structure. 3.6 is just the familiar universal property of $U_{R-A}L$.

It is well known that the ordinary universal enveloping algebra U_RL of an R-Lie algebra is a co-commutative Hopf algebra with antipode, [3]. Our main result of this section is that the twisted universal enveloping algebra $U_{R-A}L$ of a twisted Lie algebra is a twisted Hopf algebra admitting a ∇-map. In the special case $L = \mathrm{Der}\, A$, where L satisfies an additional condition (e.g. if L is projective as A-module) Sweedler showed that $U_{R-A}(\mathrm{Der}\, A)$ is a twisted Hopf algebra with Ess, [7, 18.5].

3.7 THEOREM. *Let $L \in (R-A)$-Lie be arbitrary. Then $U_{R-A}L$ is a co-commutative Hopf algebra with ∇-map.*

Proof: On $TL\otimes A$ we define an augmentation $\varepsilon : TL\otimes A \to \operatorname{End} A$ by $\varepsilon(x_{1\ldots r}\otimes a)(a') = x_1(\cdots(x_r(aa'))$. We simply write $x_{1\ldots r}(a)$ for $\varepsilon(x_{1\ldots r}\otimes 1)(a)$. Since $x \in L$ acts as derivation, $\varepsilon(x\otimes a - ax - x(a))(a') = x(aa') - ax(a') - x(a)a' = 0$. Furthermore, $\varepsilon((x\otimes y - y\otimes x - [x,y])\otimes a)(a') = x(y(aa')) - y(x(aa')) - [x,y](aa') = 0$. By definition, ε is an algebra map. Hence, ε vanishes on J. Therefore, we obtain an algebra map $\varepsilon : U_{R-A}L \to \operatorname{End} A$.

Let $x_{1\ldots r}a$ denote the image of $x_{1\ldots r}\otimes a$ under the projection $TL\otimes A \to U_{R-A}L$. An easy induction argument shows that

$$x_{1\ldots r}a = \sum_{\substack{i_1<\cdots<i_p \\ i_{p+1}<\cdots<i_r}} x_{i_{p+1}\cdots i_r}(a)\cdot x_{i_1\cdots i_p}. \tag{3.1}$$

Frequently, we omit the indices under the summation symbol.

The tensor algebra TL has a diagonal $\Delta_0 : \mathrm{TL} \to \mathrm{TL}\otimes\mathrm{TL}$ defined by

$$\Delta_0(x_{1\ldots r}) = \Pi_{i=1}^r (x_i\otimes 1 + 1\otimes x_i) = \sum_{\substack{i_1<\cdots<i_p \\ i_{p+1}<\cdots<i_r}} x_{i_{p+1}\cdots i_r}\otimes x_{i_1\cdots i_p}.$$

Δ_0 is co-associative and co-commutative, thus defining an R-Hopf algebra structure on TL. Δ_0 induces the diagonal, Δ_L, of the R-Hopf algebra $U_R L = TL/J_L$. Consider the composite, Δ_1, of the maps $\mathrm{TL}\otimes A \xrightarrow{\Delta_0\otimes 1} \mathrm{TL}\otimes\mathrm{TL}\otimes A \xrightarrow{\text{incl}} \mathrm{TL}\otimes A\otimes \mathrm{TL}\otimes \mathrm{A} \xrightarrow{\text{proj}} U_{R-A}L\,{}_A\!\otimes U_{R-A}L$. Δ_1 is left and right A-linear. By (3.1), $\Delta_1(x_{1\ldots r}\otimes a)\cdot a' =$

$$\Delta_1(x_{1\ldots r}\otimes aa') = \sum x_{i_{p+1}\cdots i_r}\,{}_A\!\otimes x_{i_1\ldots i_p}aa' =$$

$$\sum\sum x_{i_{p+1}\cdots i_r}\,{}_A\!\otimes x_{i_{q+1}\cdots i_p}(a')\cdot x_{i_1\cdots i_q}a =$$

$$\sum\sum x_{i_{q+1}\cdots i_p}(a')\cdot x_{i_{p+1}\cdots i_r}\,{}_A\!\otimes x_{i_1\cdots i_q}a =$$

$$\sum x_{i_{q+1}\cdots i_r}a'\,{}_A\!\otimes x_{i_1\cdots i_q}a, \text{ whence } \Delta_1 : TL\otimes A \to U_{R-A}L \times_A U_{R-A}L.$$

Δ_1 is an R-algebra map. This is clear for the restriction of Δ_1 onto $\mathrm{TL} \subseteq \mathrm{TL}\otimes A$. Since Δ_1 is left and right A-linear, $\Delta_1((x_{1\ldots r}\otimes a)(y_{1\ldots s}\otimes a')) = \Delta_1((x_{1\ldots r}\otimes 1)(ay_1\otimes y_{2\ldots s}\otimes 1))\cdot a' = \Delta_1(x_{1\ldots r}\otimes 1)\cdot\Delta_1(ay_1\otimes y_{2\ldots s}\otimes 1)\cdot a' = \Delta_1(x_{1\ldots r}\otimes a)\cdot\Delta_1(y_{1\ldots s}\otimes a')$. Therefore, Δ_1 is an R-algebra map. Since Δ_0 induces the diagonal on TL/J_L it follows that $J_L\otimes A \subseteq \ker\Delta_1$. By definition,

$\Delta_1(x\otimes a-ax\otimes 1-x(a))=x_A\otimes a+1_A\otimes xa-ax_A\otimes 1-1_A\otimes ax-x(a)=0$. Therefore, $J_0\subseteq \ker \Delta_1$ and Δ_1 induces

$$\Delta : U_{R-A}L\to U_{R-A}L\times_A U_{R-A}L.$$

The remaining conditions for $U_{R-A}L$ to be in $(R-A)$-Hopf are now easily verified and will be left to the reader.

In order to define the ∇-map, let $\alpha : TL\to TL$ be the anti-algebra morphism generated by $\alpha(x)=-x$, $x\in L$. Let $\nabla_0=(1\otimes\alpha)\cdot\Delta_0$. We have $\nabla_0(x)=x\otimes 1-1\otimes x$, $x\in L$, $\nabla_0(x_1\ldots_r)=x_1\cdot\nabla_0(x_2\ldots_r)-\nabla_0(x_2\ldots_r)\cdot x_1$. Let ∇_1 be the composite

$$TL\otimes A\xrightarrow{\nabla_0\otimes 1} TL\otimes TL\otimes A\xrightarrow{1\otimes\text{twist}} TL\otimes A\otimes TL\xrightarrow{\text{incl}}$$

$$TL\otimes A\otimes TL\otimes A\xrightarrow{\text{proj}} U_{R-A}L\otimes_A U_{R-A}L.$$

It follows that

$$\nabla_1(x_1\ldots_r\otimes a)=x_1\cdot\nabla_1(x_2\ldots_r\otimes a)-\nabla_1(x_2\ldots_r\otimes a)\cdot x_1. \tag{3.2}$$

Let $u=x_1\ldots_r\otimes a$ and $x\in L$. By induction on r,

$$(\nabla_1(xu))a'=x\nabla_1(u)a'-\nabla_1(u)xa'=xa'\nabla_1(u)-a'\nabla_1(u)x-\nabla_1(u)x(a')$$

$$=a'(x\nabla_1(u)-\nabla_1(u)x)+x(a')\nabla_1(u)-\nabla_1(u)x(a')=a'\nabla_1(xu),$$

so that $\nabla_1 : TL\otimes A\to((U_{R-A}L)^0\times_A U_{R-A}L)^0$. From this and (3.2) it follows that ∇_1 is left and right A-linear and that it vanishes on J_0. Since ∇_0 vanishes on J_L it follows that ∇_1 vanishes on $J_L\otimes A$ hence inducing

$$\nabla : U_{R-A}L\to((U_{R-A}L)^0\times_A U_{R-A}L)^0.$$

The conditions in 2.8 hold sincs, essentially, they hold for Δ_0 and ∇_0. □

3.8 *Remark.* We point out that, in general, ∇ is not induced by a map $U_{R-A}L\to U_{R-A}L$ in the sense of 2.9.

3.9 Example Let $L_0\to\text{Der } A$ be an R-Lie algebra morphism. On $L=A\otimes L_0$

define a bracket by

$$[a\otimes x, a'\otimes y]=aa'\otimes[x,y]+ax(a')\otimes y-a'y(a)\otimes x.$$

L is a twisted Lie algebra called the *induced twisted Lie algebra*, [2, §3]. Let $K=U_RL_0$ be the universal enveloping algebra of L_0. It is easily seen that A is a K-module algebra and that $U_{R-A}L\cong A\#K$ in $(R-A)$-aHopf. Conversely, let $A\#K$ be a smash product and suppose $K=U_RL_0$ for a certain R-Lie algebra L_0. Then, with the above notations, $A\#K\cong U_{R-A}L$ in $(R-A)$-Hopf.

In particular, let R be a field of characteristic 0 and let $A=R[[x_1,\ldots,x_n]]$ (or $A=R[x_1,\ldots,x_n]$) be the ring of formal power series (or the ring of polynomials) in n indeterminants. Let $L_0\subseteq \operatorname{Der} A$ be the abelian subalgebra generated by the partial derivatives $\partial_i=\dfrac{\partial}{\partial x_i}$, $i=1,\ldots,n$. If $\xi\in\operatorname{Der} A$, let $\xi_i=\xi(x_i)$, $i=1,\ldots,n$. Then $\xi=\sum\xi_i\partial_i$. This induces an isomorphism $L=A\otimes L_0\cong\operatorname{Der} A$. Hence, $U_{R-A}L\cong A\#U_RL_0\cong A\#S_RL_0$, S_RL_0 being the symmetric algebra of L_0. For $R=\mathbb{R}$ we thus obtain that the universal enveloping algebra of the Lie algebra of formal vector fields is a smash product.

We turn now to the notion of left module of a twisted Lie algebra. Let $M\in A$-mod and let $A\to\operatorname{End} M$ be the map defining the A-module structure on M.

3.10 DEFINITION. M is called *left L-module*, if there is a morphism $L\to \operatorname{Der}(A\to\operatorname{End} M)$ in $(R-A)$-Lie. A morphism of L-modules is a map $M'\to M''$ in A-mod such that

$$\begin{array}{ccc} L & \longrightarrow & \operatorname{End} M' \\ \downarrow & & \downarrow \\ \operatorname{End} M'' & \longrightarrow & \operatorname{Hom}(M',M'') \end{array}$$

commutes.

3.11 THEOREM. *There is a natural equivalence of categories*

$$L\text{-mod}\simeq U_{R-A}L\text{-mod}.$$

Proof: Similar to the untwisted case. □

3.12 COROLLARY. L-mod *is a closed abelian category.* □

3.13 *Remark.* A *right* L-module structure on an A-module M is a morphism

$L \to \mathrm{Der}(A \to (\mathrm{End}\, M)^0)$ in $(R-A)$-Lie. The category of right L-modules is equivalent to the category of right $U_{R-A}L$-modules. In contrast to the case of ordinary R-Lie algebras, the categories of left and right L-modules are not equivalent. This is due to the fact that the ∇-map is not induced by a map $U_{R-A}L \to U_{R-A}L$.

4. PBW-theorem and primitively generated twisted Hopf algebras

For each A-module L we construct a certain homological invariant ("Baer invariant") $B_{R-A}(L)$ in A-mod. If $B_{R-A}(L)=0$ this implies that the Poincaré-Birkhoff-Witt theorem (PBW-theorem) holds for every twisted Lie algebra structure on L. Our methods are similar to the ones used in [1] for ordinary Lie algebras. Proofs can therefore often be omitted.

Let $L \in (R-A)$-Lie. The algebra $TL \otimes A$, as defined in 3.3, is a graded algebra, the elements of degree r being $T^rL \otimes A$, $T^rL = \otimes^r L$. The corresponding filtered algebra has filtration $F_r(TL \otimes A) = \sum_{k=0}^{r} T^kL \otimes A$. Via the projection $TL \otimes A \to U_{R-A}L$, this filtration induces a filtration $F_rU_{R-A}L$ such that $U_{R-A}L$ becomes a filtered algebra. Let Gr $U_{R-A}L$ denote the associated graded algebra. Now let S_AL be the symmetric algebra of the A-module L. It can be considered as the universal enveloping algebra of L with trivial bracket. One then has a natural and surjective morphism in $(R-A)$-aHopf

$$S_AL \to \mathrm{Gr}\, U_{R-A}L. \tag{4.1}$$

If this map is an isomorphism we say that L has the *PBW-property*.

Let $T = TL \otimes A$, $S = S_AL$ and let $K_{R-A}(L) = K(L) = K$ be the kernel

$$K \rightarrowtail T \twoheadrightarrow S.$$

Clearly, $K = (x \otimes a - ax,\ x \otimes y - y \otimes x)$ and K is a graded ideal. We will see that if K has certain universal properties, L has the PBW-property for every twisted Lie algebra structure on L.

4.1 DEFINITION. Let $T^+ \subseteq T$ be the ideal of positive degree elements. An *associative structure over* L is a morphism $T^+ \to M$ in T-bimod. An associative structure is therefore given by a morphism $\varphi: L \to M$ in A-mod such that $\varphi(x) * y = x * \varphi(y)$, where $*$ denotes the (left or right) T-module operation on M.

4.2 DEFINITION. Let $M \in T$-bimod. A *twisted Lie structure over* L is an R-module morphism $u \otimes v \in (A \oplus L) \otimes (A \oplus L) \mapsto \langle u, v\rangle \in M$ with the following properties:

1) $\langle u, v\rangle = -\langle v, u\rangle$.
2) $\langle a, b\rangle = 0$, $\forall a, b \in A$; $\langle u, av\rangle = \langle u, a\rangle * v + a * \langle u, v\rangle$.
3) $\langle u, v\rangle * t * (u'v' - v'u') = (uv - vu) * t * \langle u', v'\rangle$, $u, v, u', v' \in A \oplus L$, $t \in T$.
4) $\langle u, v\rangle * w - w * \langle u, v\rangle + \langle v, w\rangle * u - u * \langle v, w\rangle + \langle w, u\rangle * v - v * \langle w, u\rangle = 0$, u, v, $w \in A \oplus L$.

A twisted Lie structure is therefore an ordinary Lie structure in the sense of [1] satisfying the additional property 2) above.
A *morphism* of Lie structures is a map $M' \to M''$ in T-bimod such that

$$\begin{array}{ccc} M' & \longrightarrow & M'' \\ \nwarrow & & \nearrow \\ & (A \oplus L) \otimes (A \oplus L) & \end{array}$$

commutes. A twisted Lie structure, say $C(L)$, is called *universal* if for any Lie structure M there exists a unique morphism of twisted Lie structures $C(L) \to M$. $C(L)$ is clearly unique up to isomorphism, if it exists. Later we will construct it using some homological machinery.

4.3 Example. K is a twisted Lie structure by defining $\langle x, a\rangle = x \otimes a - ax$, $\langle x, y\rangle = x \otimes y - y \otimes x$.

The following proposition is easy.

4.4 PROPOSITION. *Let* $\varphi : T^+ \to M$ *be an associative structure. Then*

$$\langle x, a\rangle = \varphi(x) * a - a * \varphi(x), \langle x, y\rangle = \varphi(x) * y - y * \varphi(x)$$

defines a twisted Lie structure over L. □

4.5 DEFINITION. Let $(A \oplus L) \otimes (A \oplus L) \to M'$ be a twisted Lie structure. Suppose there is an associative structure $\varphi : T^+ \to M''$ and an injection $M' \rightarrowtail M''$ in T-bimod such that $\langle u, v\rangle = \varphi(u) * v - u * \varphi(v)$. Then we call M'' an *enveloping associative structure* of M'.

4.6 THEOREM. *If K is the universal Lie structure* (*cf.* 4.3) *then* (4.1) *is an isomorphism for all twisted Lie algebra structures on L.*

Proof. Although the proof of this theorem is essentially the same as the proof of the corresponding statement for ordinary Lie algebras (theorem 2 in [1]), we give an abstract version of it, the reason being that some of the arguments are used later.

Let $U = U_{R-A}L$, so that $U \cong T/J$. Let $T_r = F_r(T)$ and define $K_r = K \cap T_r$, $S_r = (T_r + K)/K$, $J_r = J \cap T_r$, $U_r = (T_r + J)/J$. Then $U_r/U_{r-1} \cong T_r/(T_{r-1} + J_r)$, $S_r/S_{r-1} = T_r/(T_{r-1} + K_r)$. From the definition of J it follows that $T_{r-1} + K_r \subseteq T_{r-1} + J_r$. Hence we have a natural surjection

$$S_r/S_{r-1} \rightarrow U_r/U_{r-1}$$

with kernel $(T_{r-1} + J_r)/(T_{r-1} + K_r)$. This map is, of course, just (4.1). If we can show that $J_r \subseteq T_{r-1} + K_r$, we are done.

Let $\langle x, a\rangle = x \otimes a - ax - x(a)$, $\langle x, y\rangle = x \otimes y - y \otimes x - [x, y]$, $x, y \in L$, $a \in A$, so that $J = (\langle x, a\rangle, \langle x, y\rangle)$. Put $J_{(r)} = \sum_{p+q=r-1} T_p\langle x, a\rangle T_q + \sum_{p+q=r-2} T_p\langle x, y\rangle T_q \subseteq J$. In particular, $J_{(0)} = 0$, $J_{(1)} = A$-bimodule generated by all $\langle x, a\rangle$, $x \in L$, $a \in A$, and $J_{(2)} = A$-bimodule generated by all $x\langle y, a\rangle$, $\langle y, a\rangle x$, $\langle x, y\rangle$, $x, y \in L$, $a \in A$. Clearly, $J_{(r-1)} \subset J_{(r)}$ and $\bigcup J_{(r)} = J$. Put $M_r = J_{(r)}/J_{(r-1)}$. $M = \sum M_r$ is a T-bimodule and $\langle , \rangle$ induces a map $\langle , \rangle'$: $(A \oplus L) \otimes (A \oplus L) \rightarrow M$. Exactly as in [1, Lemma § 3] one proves that $(M, \langle , \rangle')$ is a twisted Lie structure over L. Since, by assumption, K is universal, it exists a morphism $K \rightarrow M$ of twisted Lie structures which, in fact, is an isomorphism (same argument as in [1]), i.e. $K_r/K_{r-1} \cong J_{(r)}/J_{(r-1)}$. Therefore every element of $J_{(r)}$ not in $J_{(r-1)}$ has leading term of degree exactly r. Consequently, since $\bigcup J_{(r)} = J$, $J_{(r)} = J \cap T_r = J_r$. But $J_{(r)} \subseteq T_{r-1} + K_r$. □

We give now a homological construction of the universal twisted Lie structure over L. Let

$$Q \rightarrowtail P \twoheadrightarrow L$$

be an exact sequence of A-modules with P projective. We have the following commutative diagram

$$\begin{array}{ccccc}
K_{R-A}(P) \cap (Q) & \rightarrowtail & (Q) & & \\
\downarrow & & \downarrow & & \\
K_{R-A}(P) & \rightarrowtail & T(P) \otimes A & \twoheadrightarrow & S_A(P) \\
\downarrow & & \downarrow & & \downarrow \\
K_{R-A}(L) & \rightarrowtail & T(L) \otimes A & \twoheadrightarrow & S_A(L)
\end{array}$$

where (Q) is the two sided ideal in $T(P)\otimes A$ generated by Q. Let $Z_{R-A} = ([T(P)\otimes A, (Q)])$ be the two sided ideal in $T(P)\otimes A$ generated by $[u, q]$, $u\in T(P)\otimes A$, $q\in(Q)$. Since $Z_{R-A}\subseteq K_{R-A}(P)$, we can define

$$B_{R-A}(L) = K_{R-A}(P)\cap(Q)/Z_{R-A},\quad C_{R-A}(L) = K_{R-A}(P)/Z_{R-A}.$$

A similar argument as in [1] shows that $B_{R-A}(L)$ and $C_{R-A}(L)$ are well defined functors from A-mod to A-mod. We have an exact sequence in A-mod

$$B_{R-A}(L) \rightarrowtail C_{R-A}(L) \twoheadrightarrow K_{R-A}(L).$$

Note that $B_{R-A}(L) = \sum B^n_{R-A}(L)$, $C_{R-A}(L) = \sum C^n_{R-A}(L)$ are graded A-modules. Frequently, we simply write $B(L)$ for $B_{R-A}(L)$ and $C(L)$ for $C_{R-A}(L)$.

By 4.3, $K_{R-A}(P) = K(P)$ is a twisted Lie structure on P. It induces a twisted Lie structure on $C(L) = K(P)/Z$ which will be denoted by $(C(L), [,])$.

4.7 THEOREM. *$(C(L), [,])$ is the universal twisted Lie structure over L.*

Proof. We first consider the case where L is a free A-module. Let $\{x_i\}_{i\in I}$ be a set of generators for L and let I be given an ordering such that ord $x_i > 0$, all $i\in I$. Let $\mathcal{H}$ be the free R-module on $\{x_i\}_{i\in I}$. Then $S_A L \cong A\otimes S_R\mathcal{H}$ as A-algebras. We assign the elements of A the ordering 0. An A-base for $S_A L$ thus consists of tuples $1\otimes x_{i_1\cdots i_r}$, $i_1\leqslant\cdots\leqslant i_r$.

4.8 LEMMA. 1) *Let $(M, \langle,\rangle)$ be a Lie structure over L. Then M has an enveloping associative structure.*
2) *K is the universal twisted Lie structure.*

Proof of Lemma: Let $N = M\oplus S_A L$. The T-bimodule structure on N is given as follows: On M it is just the given one. Let $u = a\otimes x_{1\cdots r}\in A\otimes S_R\mathcal{H}$. Write $x_0 = a$ and let $x, y\in\mathcal{H}$ or $\in A$. Define $x\circ u$, $u\circ y\in M$ by

$$x\circ u = \sum_{x>x_i} x_{o\cdots i-1}\langle x, x_i\rangle x_{i+1\cdots n}$$

$$u\circ y = \sum_{y<x_i} x_{0\cdots i-1}\langle x_i, y\rangle x_{i+1\cdots r}.$$

The T-bimodule action on N on elements $u \in S_A L$ is then defined by

$$x * u = x \circ u + xu$$
$$u * y = u \circ y + uy$$

where $xu = a \otimes xx_{1 \ldots r}$, $uy = a \otimes x_{1 \ldots r} y$, if $x, y \in \mathcal{H}$, or $xu = xa \otimes x_{1 \ldots r}$, $uy = ay \otimes x_{1 \ldots r} y$, if $x, y \in A$.

By definition bx and cy act as

$$(bx) * u = b * (x * u), \quad u * (cy) = (u * c) * y.$$

In order to show that this defines a left and right T-module structure on N we need to check that $c * (b * u) = (cb) * u$, $(u * b) * c = u * (bc)$. The first equality is obvious. Using 2) and 3), 4.2, and induction on r we obtain

$$\begin{aligned}(u * b) * c = &\sum_{i=1}^{r} ax_{1 \ldots i-1} \langle x_i, b \rangle x_{i+1 \ldots r}\, c + \sum_{i=1}^{r} abx_{1 \ldots i-1} \langle x_i, c \rangle x_{i+1 \ldots r} \\ &+ abc \otimes x_{1 \ldots r} = \sum_{i=1}^{r-1} ax_{1 \ldots i-1} \langle x_i, b \rangle x_{i+1 \ldots r-1}\, cx_r \\ &\qquad + \sum_{i=1}^{r-1} ax_{1 \ldots i-1} (x_i b - bx_i) x_{i+1 \ldots r-1} \langle x_r, c \rangle \\ &+ ax_{1 \ldots r-1} \langle x_r, b \rangle c + \sum_{i=1}^{r} abx_{1 \ldots i-1} \langle x_i, c \rangle x_{i+1 \ldots r} + abc \otimes x_{1 \ldots r} \\ = &\left(\sum_{i=1}^{r-1} ax_{1 \ldots i-1} \langle x_i, b \rangle x_{i+1 \ldots r-1} c + \sum_{i=1}^{r-1} abx_{1 \ldots i-1} \langle x_i, c \rangle x_{i+1 \ldots r-1} \right) x_r \\ &+ ax_{1 \ldots r-1} (\langle x_r, b \rangle c + b \langle x_r, c \rangle) + abc \otimes x_{1 \ldots r} \\ = &\sum_{i=1}^{r-1} ax_{1 \ldots i-1} \langle x_i, bc \rangle x_{i+1 \ldots r} + ax_{1 \ldots r-1} \langle x_r, bc \rangle + abc \otimes x_{1 \ldots r} \\ = &\; u * (bc).\end{aligned}$$

The proof of the corresponding statement in [1] now shows that $(x * u) * y = x * (u * y)$ and $(x * u) * b = x * (u * b)$. The equalities $(a * u) * y = a * (u * y)$ and $(a * u) * b = a * (u * b)$ are easily established. It follows from this that N is in T-bimod. The map $\varphi : x \in L \mapsto x \in S_A^1 L \subseteq N$ satisfies $\varphi(x) * y = x * \varphi(y)$, $x, y \in L$.

Therefore it defines an associative structure $T^+ \to N$. Now, $x * a = \langle x, a\rangle + a \otimes x$, $a * x = a \otimes x$, hence $\langle x, a\rangle = x * a - a * x$. Let $x > y$. Then $ax * by = (ax * b) * y = \langle ax, b\rangle y + ab\langle x, y\rangle + ab \otimes xy$, $by * ax = \langle by, a\rangle x + ab \otimes xy$, whence $ax * by - by * ax = \langle ax, by\rangle$. This proves that N is an enveloping associative structure for M.

The map $T^+ \to N$ induces a map $K \to N$, also in T-bimod, sending $x \otimes a - ax$ into $\langle x, a\rangle$ and $x \otimes y - y \otimes x$ into $\langle x, y\rangle$. This map is clearly unique, hence K is the universal twisted Lie structure on L. □

We return now to the general case where L is an arbitrary A-module. Let $(M, \langle,\rangle)$ be a twisted Lie structure over L. Let P be a free A-module. By 4.8, $K(P)$ is the universal Lie structure over P. Therefore we have a unique map $K(P) \to M$ in $T(P) \otimes A$-bimod. The kernel of this map contains Z, so that we have an induced map $K(P)/Z = C(L) \to M$ which is unique and has the desired properties (cf. [1]). □□

4.9 COROLLARY. *The following statements are equivalent*:
1) $B_{R-A}(L) = 0$
2) $K_{R-A}(L)$ *is the universal twisted Lie structure over* L.
3) *Every twisted Lie structure over* L *has an enveloping associative structure.*

Therefore, a twisted Lie algebra L over A has the PBW-property, if, in particular, $B_{R-A}(L) = 0$.

Proof. 1)⇔2) is obvious. For 3)⇔2) use the proof of the corresponding statement for ordinary Lie algebras in [1]. □

From the definition of $B_{R-A}(L)$ it is clear that every twisted Lie algebra whose underlying A-module is projective has the PBW-property. (This result has been obtained by Rinehart, [4], using different methods.) A similar argument as in the proof of theorem 8 in [1] shows that B_{R-A} commutes with direct limits. Hence we have

4.10 THEOREM. $B_{R-A}(L) = 0$, *if* L *is a flat* A-module. □

4.11 *Remark.* There is a relation between the Baer-invariants $B_A(L)$ of [1] and $B_{R-A}(L)$. There is an exact sequence in A-mod

$$B^0_{R-A}(L) \rightarrowtail B_{R-A}(L) \twoheadrightarrow B_A(L).$$

$B^0_{R-A}(L)$ is the Baer-invariant of L obtained by dropping the bracket in all the definitiones and constructions for $B_{R-A}(L)$. That is, given an A-module X and a map $X \to \mathrm{Der}\, A$ in A-mod, the twisted Hopf algebra $U(X) = TX \otimes A/J_0$ is the twisted universal enveloping algebra of the "free" twisted Lie algebra $L(X)$, where $L(X) \subseteq U(X)$ is the Lie algebra generated by X as in the untwisted case, [3, 6.18]. If $B^0_{R-A}(X) = 0$ this implies that the canonical surjection $T_A X \twoheadrightarrow \mathrm{Gr}\, U(X)$ is an isomorphism for all maps $X \to \mathrm{Der}\, A$. The vanishing of $B^0_{R-A}(X)$ is harder to enforce than the vanishing of $B_A(X)$. For instance, if A contains a copy of the rationals $\mathbb{Q}$, then $B_A(X) = 0$ for every A-module X, [1], while, in general, $B^0_{R-A}(X) \neq 0$ for $\mathbb{Q} \subseteq A$.

In section 3 we constructed the functor $U_{R-A}: (R-A)$-Lie$\to(R-A)$-Hopf. We now construct a functor P_{R-A}: $(R-A)$-Hopf$\to(R-A)$-Lie in the reversed direction and study the composites $P_{R-A} \circ U_{R-A}$ and $U_{R-A} \circ P_{R-A}$.

4.12 DEFINITION. Let $H \in (R-A)$-Hopf. Let $J(H)$ be the co-kernel of $\mu: A \to H$. The diagonal Δ then induces a map $J(H) \to J(H)_A \otimes J(H)$ in A-mod.

$$P_{R-A}(H) = \ker(J(H) \to J(H)_A \otimes J(H))$$

is called the A-module of *primitive elements* of H. As left A-module, $H = A \oplus I(H)$, where $I(H) = \ker \varepsilon_1$ is the augmentation ideal. The primitive elements can be described by

$$P_{R-A}(H) = \{x \in I(H) \mid \Delta(x) = x_A \otimes 1 + 1_A \otimes x\}.$$

Our first result is

4.13 PROPOSITION. *P_{R-A} is a functor from $(R-A)$-Hopf into $(R-A)$-Lie.*

Proof. Let $x, y \in P_{R-A}(H)$. By 3), 2.6, $\Delta(xy) = xy_A \otimes 1 + x_A \otimes y + y_A \otimes x + 1_A \otimes xy$. Consequently, $\Delta[x, y] = [x, y]_A \otimes 1 + 1_A \otimes [x, y]$. Hence, $P_{R-A}(H)$ is an R-Lie algebra. Let $x \in P_{R-A}(H)$ and $a \in A$. By 2.4, $xa = (\varepsilon_{1A} \otimes 1) \circ \Delta(xa) = \varepsilon_{1A} \otimes 1(xa_A \otimes 1 + 1_A \otimes ax) = \varepsilon_1(xa) + ax$, where $\varepsilon_1(x) = \varepsilon(x)(1)$ as in 2.2. Then,

$$\begin{aligned}\varepsilon(x)(ab) &= \varepsilon_1(xab) = \varepsilon_1(axb + \varepsilon_1(xa) \cdot b) = a \cdot \varepsilon_1(xb) + \varepsilon_1(xa) \cdot b \\ &= a\varepsilon_1(x)(b) + b\varepsilon_1(x)(a)\end{aligned}$$

Consequently, ε restricts to an admissible map $P_{R-A}(H) \to \mathrm{Der}\, A$. Now, $[x, ay] = xay - ayx = \varepsilon_1(xa)y + axy - ayx = a[x, y] + x(a)y$ whence $P_{R-A}(H) \in (R-A)$-Lie. Let $\varphi: H_1 \to H_2$ be a morphism in $(R-A)$-Hopf. Since φ commutes with the diagonals, it takes primitive elements into primitive elements. Since φ commutes with augmentations, the induced map $P_{R-A}(H_1) \to P_{R-A}(H_2)$ is compatible with $P_{R-A}(H_i) \to \mathrm{Der}\, A$. $\square$

4.14 COROLLARY. *There is a natural inclusion* $P_{R-A}(H) \to \mathrm{Der}(A \to H)$. *Hence we obtain a map in* $(R-A)$-Hopf

$$UP(H) = U_{R-A}(P_{R-A}(H)) \to H. \tag{4.2}$$

Proof. The first map is given by sending $x \in P_{R-A}(H)$ into $(\varphi(x), x) \in \mathrm{Der}(A \to H)$, where $\varphi : P_{R-A}(H) \to \mathrm{Der}\, A$. By 3.6, (4.2) is an algebra morphism. It follows from 3), 2.6, that it commutes with diagonals. By construction of $P_{R-A}(H)$, it commutes also with augmentations. □

In general, (4.2) is neither surjective nor injective. If (4.2) is surjective, H is called *primitively generated.* A necessary condition for this is that H is co-commutative. Later we will give sufficient conditions for (4.2) to be injective.

Let $L \in (R-A)$-Lie. There is a canonical map $L \to PU(L) = P_{R-A}(U_{R-A}L)$, induced by $L \to U_{R-A}L$, which, in general, is not surjective. If $B_{R-A}(L) = 0$, it is injective, since, in this case, $L \to U_{R-A}L$ is injective.

In order to obtain more informations about the primitive elements we need the following.

4.15 LEMMA. *Assume* $B_{R-A}L = 0$. *Then* $\mathrm{Gr}\,(U_{R-A}L\,{}_A\!\otimes U_{R-A}L) \cong S_AL\,{}_A\!\otimes S_AL$.

Proof. Let $T^2 = T_A\!\otimes T$, $S^2 = S_A\!\otimes S$, $U^2 = U_A\!\otimes U$, $J^2 = T_A\!\otimes J + J_A\!\otimes T$, $K^2 = T_A\!\otimes K + K_A\!\otimes T$ (actually, J^2 has to be thought of as the image of $T_A\!\otimes J + J_A\!\otimes T$ in T^2, similar for K^2 and others below). Then $U^2 = T^2/J^2$, $S^2 = T^2/K^2$. Using the notations of the proof of 4.6, let $T^2_r = \sum_{p+q=r} T_{pA}\!\otimes T_q$ be the filtration on T^2 and define $K^2_r = K^2 \cap T^2_r$, $S^2_r = (T^2_r + K^2)/K^2$, $J^2_r = J^2 \cap T^2_r$, $U^2_r = (T^2_r + J^2)/J^2$. Since K^2 is graded it follows that $K^2_r = \sum_{p+q=r} T_{qA}\!\otimes K_p + K_{pA}\!\otimes T_q$. As in 4.6, $U^2_r/U^2_{r-1} \cong T^2_r/(T^2_{r-1} + J^2_r)$, $S^2_r/S^2_{r-1} \cong T^2_r/(T^2_{r-1} + K^2_r)$ and there is a natural morphism

$$S^2_r/S^2_{r-1} \to U^2_r/U^2_{r-1} \tag{4.3}$$

with kernel $(T^2_{r-1} + J^2_r)/(T^2_{r-1} + K^2_r)$. (4.3) is just the natural map $S^2 \to \mathrm{Gr}\,(U^2)$. We must show that $J^2_r \subseteq T^2_{r-1} + K^2_r$. For this let $J^2_{(r)} = \sum_{p+q=r} T_{qA}\!\otimes J_{(p)} + J_{(p)A}\!\otimes T_q$. Now we repeat the argument used in the proof of 4.6. Since $B_{R-A}L = 0$, we have $K_r/K_{r-1} \cong J_{(r)}/J_{(r-1)}$ It follows that if $u \in J^2_{(r)}$, $u \notin J^2_{(r-1)}$, u has leading term of degree exactly r. Since $\cup J^2_{(r)} = J^2$ this implies that $J^2_{(r)} = T^2_r \cap J = J^2_r$. But $J^2_{(r)} \subseteq T^2_{r-1} + K^2_r$ so we are done. □

Note that in case $A = R$ this theorem is easy, since $U_RL \otimes U_RL \cong U_R(L \oplus L)$ and $B_R(L \oplus L) = 0$, if $B_R(L) = 0$.

Let $u \in F_r(U)$ be primitive, $u \notin F_{r-1}(U)$. Since Δ is a filtered morphism $\Delta(u) \in F_r(U_A \otimes U)$, $\Delta(u) \notin F_{r-1}(U_A \otimes U)$. Let $\bar{u}$ be the class in $F_r(U)/F_{r-1}(U)$ represented by u. It follows that $\bar{u}$ is primitive in $\mathrm{Gr}(U)$. Suppose $B_{R-A}L = 0$. Then $\mathrm{Gr}\,(U) \cong S$, $\mathrm{Gr}\,(U_A \otimes U) \cong S_A \otimes S$ and primitive elements in U give rise to primitives in S. In particular, we have

4.16 PROPOSITION. *Let* $B_{R-A}L = 0$. *Then* $PU(L) = L$ *if* $P_{R-A}(S_A L) = L$. □

In order to take advantage of this proposition we need some facts about the De Rham complex of the A-module L. Let $\wedge = \wedge_A L$ be the exterior algebra of the A-module L. In the graded A-module $S \otimes_A \wedge = \sum_{q \geq 0} S \otimes_A \wedge^q$ define a differential d of degree $+1$ by

$$d(x_{1 \cdots p} \otimes_A y) = \textstyle\sum_{i=1}^{p} x_{1 \cdots i-1} \hat{x}_i x_{i+1 \cdots p} \otimes_A x_i \wedge y,$$

$x_{1 \cdots p} \in S^p$, $y \in \wedge^q$. We thus obtain the *De Rham complex* of L. The cohomology modules are denoted by $H^q(S \otimes_A \wedge)$. Let $H^{p,q} \subseteq H^q$ be the submodule generated by cocycles in $S^p \otimes_A \wedge^q$. Thus $H^q = \sum_{p \geq 0} H^{p,q}$.

4.17 PROPOSITION. $H^{p,q}$ *is a torsion group bounded by* $p+q$.

Proof. Define a map $h : S \otimes_A \wedge \to S \otimes_A \wedge$ of degree -1 by

$$h(x_{1 \cdots p} \otimes y_1 \wedge \cdots \wedge y_q) = \sum_{j=1}^{q} (-1)^{j-1} x_{1 \cdots r} y_q \otimes_A y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_q.$$

Then $(hd + dh)(x_{1 \cdots p} \otimes_A y_1 \wedge \cdots \wedge y_q) = (p+q) x_{1 \cdots p} \otimes_A y_1 \wedge \cdots \wedge y_q$. Let $z = \sum x_{1 \cdots p} \otimes_A y_1 \wedge \cdots \wedge y_q$ represent a class $[z] \in H^{p,q}$. Then $(hd + dh)(z) = d(h(z)) = (p+q)z$. Therefore $(p+q)[z] = 0$ in $H^{p,q}$. □

We can now state our main results of this section.

4.18 THEOREM. If $H^0(S \otimes_A \wedge) = A$ *(i.e. if* L *is "connected"), then* $P_{R-A}(S_A L) = L$.

Proof. The diagonal $\Delta : S \to S_A \otimes S$ is a morphism of graded modules, i.e. $\Delta(S^r) \subseteq \sum_{p+q=r} S^p \otimes_A S^q$. Therefore, $P_{R-A}(S_A L) = \sum_{r \geq 1} P^r(S)$ is a graded A-module. Consider $\Delta | S^r$, $r \geq 2$. Let Δ_p^2 be the component of $\Delta \mid S^r$ in $S^{r-p}{}_A \otimes S^p$ so that $\Delta \mid S^r = \sum_{p=0}^{r} \Delta_p^r$. It follows that $x \in P^r(S)$ iff $\Delta_p^r(x) = 0$, $p = 1, \ldots, r-1$. In particular, $\Delta_1^r(x)=0$. Let $S^+=\sum_{r\geq 2} S^r$. We then have a commutative diagram

$$\begin{array}{ccc} S^+ & \xrightarrow{\Delta_1} & S\otimes_A S^1 \\ \downarrow & & \downarrow\cong \\ S\otimes_A \wedge^0 & \xrightarrow{d} & S\otimes_A \wedge^1 \end{array}$$

where $\wedge^0=A$, $S^1=\wedge^1=L$ and $\Delta_1=\sum_{r\geq 2}\Delta_1^r$. Suppose $x\in P^r(S)$, $r\geq 2$. Then $\Delta_1(x)=0$ or, equivalently, $d(x)=0$. By assumption, $H^0(S\otimes_A \wedge)=A$, so $x=dy$. But $y=0$, hence $x=0$. □

4.19 COROLLARY. *Let* $L\in(R-A)$-Lie *be such that* $B_{R-A}(L)=0$ *and* $H^0(S\otimes_A \wedge)=A$. *Then* $PU(L)=L$.

Proof. Combine 4.18 with 4.16. □

4.20. THEOREM. Let $H\in(R-A)$*-Hopf with primitive elements* $L=P_{R-A}(H)$. *Assume* $B_{R-A}(L)=0$ *and* $H^0(S\otimes_A \wedge)=A$. *Then* $UP(H)\to H$ *is injective. In particular if* H *is primitively generated,* $UP(H) \longrightarrow H$ *is an isomorphism in* $(R-A)$*-Hopf.*

Proof. Let $V=\ker UP(H)\to H$ and let $x\in V$, $x\in F_r(UP(H))$, $x\notin F_{r-1}(UP(H))$. Assume r is minimal with this property. $B_{R-A}(L)=0$ implies that $L\to U(L)$ is injective; consequently $r\geq 2$. Since $UP(H)\to H$ commutes with diagonals, $\Delta(x)=x_A\otimes 1+1_A\otimes x+\sum(u_A\otimes v+v'_A\otimes u)$ where $v, v'\in V$, having filtration degree smaller than r. The minimality condition on r thus implies $\sum(u_A\otimes v+v'_A\otimes u)=0$, hence $x\in P_{R-A}(UP(H))$. But $P_{R-A}(UP(H))=L$, by 4.19. Consequently $x=0$. □

The following propositions give conditions under which the assumptions in 4.19 and 4.20 are satisfied.

4.21 PROPOSITION. *If* A *is torsionfree as abelian group and* L *is flat as* A-module, *then* $B_{R-A}(L)=0$ *and* $H^0(S\otimes_A \wedge)=A$.

Proof. The statement about $B_{R-A}(L)$ is just 4.10. If A is torsion free as abelian group and L is a flat A-module, then $S_A L$ is torsion free as abelian group. Therefore, $H^0(S\otimes_A \wedge)=A$ by 4.17. □

In the particular case $R=A$ we are dealing with ordinary Lie and Hopf algebras. $U_{R-A}L=U_A L$ is the ordinary universal enveloping algebra of L,

$P_{R-A}(H) = P_A(H)$ is the Lie algebra of primitive elements in the usual sense [3] and $B_{R-A}(L) = B_A(L)$. 4.21, and therefore 4.19 and 4.20, can then be extended as follows:

4.22. THEOREM. *$B_A(L) = 0$ and $H^0(S \otimes_A \wedge) = A$ in the following cases:*
1) *A is torsion free as abelian group and L is a flat A*-module.
2) *A contains a copy of $\mathbb{Q}$, L arbitrary.*

Proof. 1) is 4.21 in case $R = A$. By [1], $B_A(L) = 0$, if A contains a copy of $\mathbb{Q}$, and $H^0(S \otimes_A \wedge) = A$ by 4.17. □

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