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## Groups of finite quasi-projective dimension

JAMES HOWIE and HANS RUDOLF SCHNEEBELI

### 1. Introduction

1.1. Lyndon's Identity Theorem [11] may be interpreted as a description of the structure of the relation module arising from a one-relator presentation – or, more generally, from a staggered presentation – of a group  $G$ . For such presentations, the relation module is the direct sum of the cyclic submodules generated by the images of the defining relators. Furthermore, each such submodule has the form  $\mathbb{Z}G/C$ , where  $C$  is the finite cyclic subgroup of  $G$  generated by the image of the root of the corresponding defining relator.

We say that a presentation has the Identity Property if its relation module has the above form. This is equivalent to the condition (I.1) of Lyndon and Schupp ([13], p. 158). We say that a group  $G$  has the Identity Property if some presentation of  $G$  has the Identity Property. In this paper, we consider a property which is weaker than the Identity Property in the following respect. Instead of considering a relation module, we look at the kernel of the  $n$ th boundary map in an  $RG$ -projective resolution of  $R$ , where  $R$  is a commutative ring with 1, and we allow this kernel to be a direct sum of cyclic modules of the form  $RG/S$ , where  $S$  is an arbitrary subgroup of  $G$ .

1.2 Let  $R$  be a commutative ring with unit and let  $G$  be a group. An exact sequence of left  $RG$ -modules

$$\mathcal{Q}: 0 \rightarrow Q \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

of finite length  $n > 0$  is called an  *$RG$ -quasi-projective resolution* of  $A$  if the modules  $P, P_{n-1}, \dots, P_0$  are  $RG$ -projective and there exists an indexed set  $\{G_\alpha\}_I$  of subgroups of  $G$  such that

$$Q \cong \bigoplus_I RG/G_\alpha.$$

The set  $\{G_\alpha\}_I$  can be chosen such that no  $RG/G_\alpha$  is  $RG$ -projective. We then say that the set  $\{G_\alpha\}_I$  is *associated to the resolution*  $\mathcal{Q}$ . We make the convention that

the sequence  $0 \rightarrow A \rightarrow A \rightarrow 0$  is an  $RG$ -quasi-projective resolution of length 0 if and only if  $A$  is  $RG$ -projective.

We define the  $RG$ -quasi-projective dimension of  $A$  to be the shortest possible length of an  $RG$ -quasi-projective resolution  $\mathfrak{Q} \rightarrow A$ . If no such resolution exists, the quasi-projective dimension is said to be infinite. In particular  $\text{qpd}_R G$  denotes the  $RG$ -quasi-projective dimension of the trivial module  $R$ . We write  $\text{qpd}_{\mathbb{Z}} G$  as  $\text{qpd } G$ . Our convention for length 0 is necessary to exclude the sequence  $0 \rightarrow R \rightarrow R \rightarrow 0$ , if  $R$  is not  $RG$ -projective.

Our notation, using  $RG$ -, might suggest that the above definitions depend only on the ring  $RG$ . It is, however important to recognize the explicit group-ring structure of  $RG$ . For example, even if  $RG \cong SH$  as rings, an  $RG$ -quasi-projective resolution need not be an  $SH$ -quasi-projective resolution.

### EXAMPLES

1. For all groups  $G$ , the inequality  $\text{qpd}_R G \leq \text{cd}_R G$  holds.
2. Suppose  $G$  is a group with the Identity Property, then  $\text{qpd}_R G \leq 2$ .

Particular instances of such groups are:

- one relator groups,
- groups with staggered presentations,
- certain small cancellation groups [12],
- groups with an “aspherical” presentation in the sense of Lyndon and Schupp [13].

3. If  $G$  is a finite group, and there exists a periodic  $RG$ -projective resolution of finite period  $k$  over  $R$ , then  $\text{qpd}_R G$  is at most  $k$ .

1.3 We now describe the structure of the article and discuss our main results.

In Section 2 we deal with some general consequences of our definition for  $\text{qpd}_R G$ . The similarity between  $\text{cd}_R$  and  $\text{qpd}_R$  is a basic theme.

Theorem 1 states a fundamental subgroup property for groups of finite  $\text{qpd}$  over  $R$ . Suppose  $S$  is a subgroup of  $G$  and  $\text{qpd}_R G < \infty$ . Then any  $RG$ -quasi-projective resolution may be interpreted as an  $RS$ -quasi-projective resolution and an associated set of subgroups of  $S$  may be defined in terms of a given associated set of subgroups of  $G$ . In particular, we have  $\text{qpd}_R S \leq \text{qpd}_R G$  whenever  $S \subset G$ .

It follows from Theorem 1 that any set of subgroups of  $G$  associated to some quasi-projective resolution consists of finite groups. Hence, for all  $R$ -torsion-free groups  $G$ ,  $\text{qpd}_R G = \text{cd}_R G$ . In particular  $\text{qpd}_{\mathbb{Q}} G = \text{cd}_{\mathbb{Q}} G$  for all groups  $G$ . Another consequence of Theorem 1 is the following: Suppose  $R$  is torsion-free as a  $\mathbb{Z}$ -module and  $\text{qpd}_R G$  is odd, then  $G$  is  $R$ -torsion-free.

Suppose  $G$  is a fundamental group of a graph of groups whose vertex groups have bounded  $\text{qpd}$  over  $R$  and whose edge groups are  $R$ -torsion-free. Theorem 5

says that  $\text{qpd}_R G$  is finite. This is an analogue for  $\text{qpd}_R$  of a result of Chiswell for  $\text{cd}_R$  (cf. [5], p. 70).

The results of Section 3 give information about the finite subgroups in groups of finite  $\text{qpd}$ . Our central result is the following.

**THEOREM 6.** *Suppose  $S \neq 1$  is a finite  $R$ -torsion subgroup of  $G$ , and  $\{G_\alpha\}_I$  is a set of subgroups associated to some  $RG$ -quasi-projective resolution of  $R$  of finite length. Then there exist a unique  $\alpha \in I$  and a unique left coset  $gG_\alpha$  of  $G_\alpha$  in  $G$ , such that  $S$  is contained in  $gG_\alpha g^{-1}$ .*

In the case  $R = \mathbf{Z}$ , it follows from Theorem 6 that the associated set of subgroups  $\{G_\alpha\}_I$  is a full representative set of conjugacy classes of maximal finite subgroups. Hence, up to conjugacy, the groups  $G_\alpha$  are determined by  $G$  independently of the particular choice of a  $\mathbf{Z}G$ -quasi-projective resolution.

Theorem 6 is reminiscent of a theorem of Serre [9] and of results of Wall ([16], Lemma 7, Proposition 8). In fact, if  $R = \mathbf{Z}$  in our situation, then the hypotheses of Serre's theorem hold, but we cannot prove this without the help of Theorems 6 and 7. Our proof of Theorem 6 is partly based on Wall's arguments.

In the special case where  $G$  is a one-relator group, Theorem 6 recovers a result of Karrass, Magnus and Solitar [10].

Further restrictions on the finite subgroups derive from Theorem 7. In the case  $R = \mathbf{Z}$ , it states that a finite group  $S$  satisfies  $\text{qpd } S = n$  if and only if its Tate cohomology has period  $n$ . In particular, if  $\text{qpd } G = 2$ , then any finite subgroup of  $G$  is cyclic. Hence  $G$  has a relation module which is a direct sum of cyclic modules of the form  $\mathbf{Z}G/C$ , where  $C$  is a finite cyclic subgroup of  $G$ . Note the formal resemblance with the module-theoretic interpretation of the Identity Property.

As stated in 1.2, if  $G$  has the Identity Property, then  $\text{qpd } G \leq 2$ . We make no attempt to answer the question of whether the converse also holds. For torsion-free groups, this reduces to the question of whether cohomological and geometric dimensions always coincide (Eilenberg–Ganea problem).

The topics of Section 4 may be motivated by general properties of one-relator groups.

Let  $G$  be a one-relator group with torsion. Then either  $G$  is finite cyclic or the centre of  $G$  is trivial. We show in Corollary 8.1 that any group  $G$  of finite  $\text{qpd}$  with torsion either is finite or has trivial centre. It is known that a one-relator group has only finitely many conjugacy classes of finite subgroups. Proposition 9 gives necessary and sufficient conditions for a group of finite  $\text{qpd}$  to have only finitely many conjugacy classes of finite subgroups. This is of interest in connection with Wall's question F7 in [17].



Recall [14] that a group  $G$  is virtually torsion-free if one of its subgroups of finite index is torsion-free. The virtual cohomological dimension  $\text{vcd } G$  of a virtually torsion-free group  $G$  is defined by  $\text{vcd } G = \text{cd } S$ , when  $S$  is a torsion-free subgroup of finite index. If  $G$  is not virtually torsion-free, then by definition  $\text{vcd } G = \infty$ .

Like  $\text{qpd}$ , the invariant  $\text{vcd}$  assumes finite values on certain classes of groups with torsion. If  $G$  is virtually torsion-free,  $\text{vcd } G \leq \text{qpd } G$  holds. Our examples show that this is the most one can say in general about the relationship between  $\text{vcd}$  and  $\text{qpd}$ .

Of particular interest are groups  $G$  with  $\text{vcd } G \leq \text{qpd } G < \infty$ . In this case, the Farrell–Tate cohomology of  $G$  is periodic and is completely determined by the cohomology of the maximal finite subgroups of  $G$ .

Two of our examples make it clear that certain general properties of one-relator groups do not follow from the Identity Property. These examples concern the inequality  $\text{vcd } G \leq \text{qpd } G$  and the structure of the subgroup generated by the torsion elements.

The methods of this paper are algebraic. We have left open the question of a suitable geometric interpretation of the property of having finite  $\text{qpd}$ . By analogy with cohomological dimension, one might expect to obtain sufficient criteria for finite  $\text{qpd}$  via such an interpretation.

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## 2. Groups of finite $\text{qpd}$ over a ring $R$

### 2.1 Restriction to subgroups

**THEOREM 1.** *Suppose  $\mathcal{Q} \rightarrow A$  is an  $RG$ -quasi-projective resolution of  $A$  of length  $n$ , and  $\{G_\alpha\}_I$  is an associated set of subgroups. Let  $S$  be a subgroup of  $G$ . For each  $\alpha \in I$ , choose a set  $\{t_\beta; \beta \in J_\alpha\}$  of representatives of the double cosets  $SgG_\alpha$  ( $g \in G$ ). For each  $\beta \in J_\alpha$ , define  $S_{\alpha\beta} = S \cap t_\beta G_\alpha t_\beta^{-1}$ . Then  $\mathcal{Q} \rightarrow A$  is an  $RS$ -quasi-projective resolution, and there is an associated set of subgroups consisting of those  $S_{\alpha\beta}$  for which  $RS/S_{\alpha\beta}$  is not  $RS$ -projective.*

**COROLLARY 1.1.** *If  $S$  is a subgroup of  $G$ , then  $\text{qpd}_R S \leq \text{qpd}_R G$ .*

*Proof of Theorem 1.* Suppose  $\mathcal{Q} \rightarrow A$  has the form  $0 \rightarrow Q \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$ , where  $P$  is an  $RG$ -projective, and  $Q \cong \bigoplus_I RG/G_\alpha$ . Now  $Q$  is the free

$R$ -module on a left  $G$ -set  $T$ , whose decomposition into  $G$ -orbits has the form  $T = \sqcup_I G/G_\alpha$ .

The  $S$ -orbit decomposition of  $T$  has the form  $T = \sqcup_I (\sqcup_{J_\alpha} S/S_{\alpha\beta})$ , so as left  $RS$ -module

$$Q \cong \oplus_I (\oplus_{J_\alpha} RS/S_{\alpha\beta}).$$

Define  $J = \{(\alpha, \beta); \alpha \in I, \beta \in J_\alpha, RS/S_{\alpha\beta} \text{ is not } RS\text{-projective}\}$ . Then  $Q$  has an  $RS$ -direct sum decomposition  $Q \cong P' \oplus Q'$ , where  $P'$  is  $RS$ -projective and  $Q' = \oplus_J RS/S_{\alpha\beta}$ .

**COROLLARY 1.2.** *In the situation of Theorem 1, the subgroups  $G_\alpha$  are all finite.*

*Proof.* Fix  $\alpha \in I$  and let  $S = G_\alpha$  in the proof of Theorem 1. Then some  $S_{\alpha\beta}$  coincides with  $S$ , so  $Q$  contains the trivial module  $R$  as an  $RS$ -direct summand. Hence the  $RS$ -projective  $P_{n-1}$  contains  $R$  as a trivial  $RS$ -submodule. This is possible only if  $S$  is finite.

**COROLLARY 1.3.** *If  $G$  is  $R$ -torsion-free, then  $\text{qpd}_R G = \text{cd}_R G$ .*

*Proof.* It is sufficient to show that  $\text{cd}_R G \leq \text{qpd}_R G$  and we may assume that  $\text{qpd}_R G = n < \infty$ . For every finite subgroup  $S$  of  $G$ , the order  $|S| = |S| \cdot 1 \in R$  is a unit of  $R$ . Thus the canonical epimorphism  $RG \rightarrow RG/S$  splits via an  $RG$ -homomorphism  $\sigma: RG/S \rightarrow RG$ , where

$$\sigma(gS) = \frac{1}{|S|} \sum_{h \in S} gh,$$

and so  $RG/S$  is  $RG$ -projective.

It follows that any  $RG$ -quasi-projective resolution is an  $RG$ -projective resolution.

**COROLLARY 1.4.** *If  $S$  is an  $R$ -torsion-free subgroup of  $G$ , then  $\text{cd}_R S \leq \text{qpd}_R G$ .*

**COROLLARY 1.5.** *Suppose  $G$  is virtually  $R$ -torsion-free, then  $\text{vcd}_R G \leq \text{qpd}_R G$ .*

## 2.2 Quasi-projective and projective resolutions

Let  $0 \rightarrow Q \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$  be an  $RG$ -quasi-projective resolution of  $A$  and let  $0 \rightarrow K \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow A \rightarrow 0$  be an exact sequence of  $RG$ -modules with all the  $M_i$   $RG$ -projective. By Schanuel's lemma ([1], [15]) there

exist  $RG$ -projectives  $M'$  and  $P'$  such that  $(Q \oplus P) \oplus P' \cong K \oplus M'$ . Hence the exact sequence

$$\begin{array}{c} 0 \rightarrow K \rightarrow M_{n-1} \rightarrow M_{n-2} \rightarrow \cdots \rightarrow M_0 \rightarrow A \rightarrow 0 \\ \oplus \quad \cdot \quad \oplus \\ M \xrightarrow{1} M' \end{array}$$

is an  $RG$ -quasi-projective resolution of  $A$ .

If  $0 \rightarrow (\oplus_I RG/G_\alpha) \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$  is a quasi-projective resolution of  $A$  and  $\{G_\alpha\}_I$  is an associated set of subgroups, then an  $RG$ -projective resolution can be obtained by the following construction. For each  $\alpha \in I$ , choose an  $RG_\alpha$ -projective resolution of  $R$ ,  $M^\alpha \rightarrow RG_\alpha \rightarrow R \rightarrow 0$ . The functor  $RG \otimes G_\alpha -$  applied to this resolution gives an  $RG$ -projective resolution of  $RG/G_\alpha$ . We thus obtain an  $RG$ -projective resolution of  $A$  of the form

$$\cdots \rightarrow \oplus_I (RG \otimes_{RG_\alpha} M_1^\alpha) \xrightarrow{\partial_{n+1}} (\oplus_I RG) \oplus P \xrightarrow{\partial_n} P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0. \quad (*)$$

We use the resolution (\*) for  $R$  to calculate the homology and cohomology of  $G$  in high dimensions.

**PROPOSITION 2.** *Suppose  $\{G_\alpha\}_I$  is a set of subgroups associated to some  $RG$ -quasi-projective resolution of  $R$  of Length  $n$ . Then for each  $q > n$ , there are natural isomorphisms*

$$H^q(G; -) \cong \prod_I H^{q-n}(G_\alpha; -); \quad H_q(G; -) \cong \oplus_I H_{q-n}(G_\alpha; -)$$

*of functors from  $RG$ -modules to  $R$ -modules.*

**PROPOSITION 3.** *If  $R$  admits an  $RG$ -quasi-projective resolution of length  $n$ , then  $H_n(G; R)$  embeds into a free  $R$ -module.*

**COROLLARY 3.1.** *If  $R$  is a PID and  $R$  admits an  $RG$ -quasi-projective resolution of length  $n$ , then  $H_n(G; R)$  is  $R$ -free.*

*Proof of Proposition 3.* We use the special resolution (\*) to calculate  $H_n(G; R)$ . For each  $\alpha \in I$ , the  $\alpha$ th direct summand  $RG \otimes_{RG_\alpha} M_1^\alpha$  is mapped under  $\partial_{n+1}$  into the augmentation ideal of the  $\alpha$ th direct summand  $RG \cong RG \otimes_{RG_\alpha} RG_\alpha$ . It follows that  $1_R \otimes_{RG} \partial_{n+1} = 0$ , and so  $H_n(G; R) \cong \ker(1_R \otimes_{RG} \partial_n)$  is isomorphic to a submodule of the  $R$ -projective  $(\oplus_I R) \oplus (R \otimes_{RG} P)$  and hence also of some free  $R$ -module.

**COROLLARY 3.2.** *Suppose  $R$  is torsion-free as an abelian group. If  $\text{qpd}_R G = 2k + 1$  is odd, then  $G$  is  $R$ -torsion-free, and so  $\text{cd}_R G = 2k + 1$ .*

*Proof.* Suppose  $C$  is a finite cyclic subgroup of  $G$  of order  $m > 1$ . By Theorem 1,  $R$  admits an  $RC$ -quasi-projective resolution of length  $2k + 1$ . By Proposition 3,  $H_{2k+1}(C, R)$  embeds into a free  $R$ -module and thus is  $\mathbf{Z}$ -torsion-free. On the other hand,  $R/mR = H_{2k+1}(C, R)$  has obvious  $\mathbf{Z}$ -torsion unless  $m \cdot 1$  is a unit in  $R$  and  $R/mR = 0$ .

If  $G$  is  $R$ -torsion-free, then Proposition 3 is relevant only for  $n = \text{cd}_R G$ . However if  $G$  has non-trivial information about  $H_i(G, R)$  for infinitely many  $i \geq \text{qpd}_R G$  as shown by the next result.

**PROPOSITION 4.** *Suppose there exists an  $RG$ -quasi-projective resolution of  $R$  of length  $i$ , let  $\{G_\alpha\}_I$  be its associated set of subgroups, and suppose, for each  $\alpha \in I$ , there exists an  $RG_\alpha$ -quasi-projective resolution of  $R$  of length  $k$ . Then there exists an  $RG$ -quasi-projective resolution of  $R$  of length  $k + i$ .*

*Proof.* In case  $G$  is  $R$ -torsion-free, there exist  $RG$ -projective resolutions of  $R$  of arbitrary length  $1 \geq \text{cd}_R G$ . Otherwise, the set  $\{G_\alpha\}_I$  is non-empty. Using the idea of the construction of the resolution (\*), but replacing each  $RG_\alpha$ -projective resolution  $\mathcal{M}^\alpha \rightarrow RG_\alpha \rightarrow R \rightarrow 0$  by an  $RG_\alpha$ -quasi-projective resolution of length  $k$ , we obtain an  $RG$ -quasi-projective resolution of length  $k + i$  by an analogous procedure. Here the following fact is used: Let  $U \subset V \subset G$  be subgroups, then  $RG \otimes_V RV/U \cong RG/U$ .

*Remark.* Inductive arguments based on Proposition 4 lead to the following:

- (i) In the circumstances of Proposition 4, there exist  $RG$ -quasi-projective resolutions of  $R$  of length  $m \cdot k + i$  for arbitrary integers  $m \geq 0$ .
- (ii) If  $\text{qpd}_R G = n$ , then there are  $RG$ -quasi-projective resolutions of  $R$  of length  $m \cdot n$  for all integers  $m > 0$ .

### 2.3. Graphs of groups of finite $\text{qpd}$

**THEOREM 5.** *Let  $\Gamma$  be a graph of groups,  $\{G_v\}_V$  its set of vertex groups,  $\{G_e\}_E$  its set of edge groups and  $G$  its fundamental group. Suppose there is an integer  $j$  such that  $\text{qpd}_R G_v < j$  for all  $v \in V$  and that all the groups  $G_e$  are  $R$ -torsion-free, then  $\text{qpd}_R G < \infty$ .*

*Proof.* Associated to the graph  $\Gamma$  there is an exact sequence of  $RG$ -modules  $A \twoheadrightarrow B \twoheadrightarrow R$  where  $A \cong \bigoplus_E RG/G_e$  and  $B \cong \bigoplus_V RG/G_v$  (cf. [5]). Let  $\mathcal{P}^e \twoheadrightarrow R$  be an  $RG_e$ -projective resolution, then  $RG \otimes_{RG_e} \mathcal{P}^e \rightarrow RG/G_e$  is an  $RG$ -projective resolution. Similarly  $RG \otimes_{RG_v} \mathcal{Q}^v \twoheadrightarrow RG/G_v$  is an  $RG$ -quasi-projective resolution, provided  $\mathcal{Q}^v \twoheadrightarrow R$  is  $RG_v$ -quasi-projective. We thus get an

$RG$ -projective resolution  $\mathcal{P} \twoheadrightarrow A$  of length  $p \leq j$ , and, using the remark after Proposition 4, an  $RG$ -quasi-projective resolution  $\mathcal{Q} \twoheadrightarrow B$  of length  $q > p$ . The monomorphism  $A \rightarrow B$  lifts to an  $RG$ -morphism of complexes  $F: \mathcal{P} \rightarrow \mathcal{Q}$ . The mapping cone construction described by Bass ([1] p. 30) yields an  $RG$ -quasi-projective resolution  $MC(F) \twoheadrightarrow R$  of length  $q$ .

### 3. Finite subgroups in groups of finite qpd

#### 3.1. Conjugacy classes of finite subgroups

**THEOREM 6.** *Suppose  $S \neq 1$  is a finite  $R$ -torsion subgroup of  $G$ , and  $\{G_\alpha\}_I$  is a set of subgroups associated to some  $RG$ -quasi-projective resolution of  $R$  of finite length. Then there exist a unique  $\alpha \in I$  and a unique left coset  $gG_\alpha$  of  $G_\alpha$  in  $G$ , such that  $S$  is contained in  $gG_\alpha g^{-1}$ .*

The proof is split into three parts.

(a) The conclusion of the theorem holds if  $p = |S|$  is prime.

Let  $J$  denote the set of all pairs  $(\alpha, \beta)$  with  $\alpha \in I$  and  $\beta = gG_\alpha \in G/G_\alpha$  such that  $S_{\alpha\beta} = S \cap gG_\alpha g^{-1} \neq 1$ . Since  $p$  is not invertible in  $R$ , it follows from Theorem 1 that  $\{S_{\alpha\beta}\}_J$  is a set of subgroups associated to some  $RS$ -quasi-projective resolution of finite length  $n$ , say. Now  $S_{\alpha\beta} = S$  for all  $(\alpha, \beta) \in J$  and so applying Proposition 2 twice, we get  $R$ -isomorphisms

$$R/pR \cong H_{2n+1}(S, R) \cong \bigoplus_J H_{n+1}(S, R) \cong \bigoplus_J \bigoplus_J H_1(S, R) \cong \bigoplus_J \bigoplus_J R/pR.$$

Comparing the ranks of the free  $R/pR$ -modules  $R/pR$  and  $\bigoplus_J \bigoplus_J R/pR$ , we find that  $J$  is a singleton, as required.

(b) If  $\alpha, \alpha' \in I$ ,  $g \in G$  are such that  $G_{\alpha'} \cap gG_\alpha g^{-1}$  is not  $R$ -torsion-free, then  $\alpha = \alpha'$  and  $g \in G_\alpha$ .

Otherwise, choose an  $R$ -torsion subgroup  $S$  of  $G_{\alpha'} \cap gG_\alpha g^{-1}$  of prime order, and apply (a).

(c) The general case follows by induction on the order of  $S$ . For the inductive step, we apply an argument of Wall ([16], Proposition 8). Here (a) is the initial case of the induction and (b) plays the rôle of Wall's Lemma 7. Note that if  $S$  is an  $R$ -torsion group, so is any subgroup of  $S$ .

**COROLLARY 6.1.** *Suppose  $R = \mathbb{Z}$  and  $G, \{G_\alpha\}_I$  are as in Theorem 6. Then*

(i) *The set  $\{G_\alpha\}_I$  is a complete set of representatives of conjugacy classes of maximal finite subgroups of  $G$ .*

(ii) *If  $G \neq 1$  is finite, then  $I$  is a singleton, say  $I = \{0\}$ , and  $G_0 = G$ .*

Thus, in the particular case  $R = \mathbb{Z}$ , the subgroups  $G_\alpha$  are determined up to conjugacy by  $G$  itself independently of the choice of a  $\mathbb{Z}G$ -quasi-projective

resolution. In this sense we may speak about “a set of subgroups  $\{G_\alpha\}_I$  associated to  $G$ .”

In the case  $R = \mathbf{Z}_{(p)}$ , the localisation of  $\mathbf{Z}$  at a prime  $p$ , a weaker form of Corollary 6.1 holds. We state it as a second corollary, since we refer to it in the proof of the next theorem.

**COROLLARY 6.2.** *Suppose  $R = \mathbf{Z}_{(p)}$  and  $G, \{G_\alpha\}_I$  are as in Theorem 6. For each  $\alpha \in I$ , choose a  $p$ -Sylow subgroup  $S_\alpha$  in  $G_\alpha$ . Then*

(i) *The set  $\{S_\alpha\}$  is a complete set of representatives of conjugacy classes of maximal finite  $p$ -subgroups of  $G$ .*

(ii) *If  $G$  is finite and not  $p$ -torsion-free, then  $I$  is a singleton, say  $I = \{0\}$ , and  $S_0$  is a  $p$ -Sylow subgroup of  $G$ .*

### 3.2. Periodicity

Recall ([4] Ch. XII) that a finite group  $G$  has  $p$ -period  $k > 0$  if the  $p$ -component of its Tate cohomology satisfies  $\hat{H}^i(G, -)_{(p)} \cong \hat{H}^{i+k}(G, -)_{(p)}$  and  $k$  is minimal with this property. This is equivalent to  $\hat{H}^*(G, -)$  having period  $k$  on the category of  $\mathbf{Z}_{(p)}G$ -modules.

If  $\pi$  is a non-empty set of primes, then  $G$  is  $\pi$ -periodic if and only if it is  $p$ -periodic for each  $p \in \pi$ . The  $\pi$ -period of  $G$  is the least common multiple of the  $p$ -periods for all  $p \in \pi$ .

**THEOREM 7.** *Suppose  $R = \mathbf{Z}_{(\pi)}$ , where  $\pi$  is a set of primes. Let  $G$  be a finite group which is not  $R$ -torsion-free. Then  $G$  has  $\pi$ -period  $k$  if and only if  $\text{qpd}_R G = k$ .*

*Proof.* Suppose that  $G$  has finite  $\pi$ -period  $k$ . By a theorem of Swan [15], there exists an  $RG$ -quasi-projective resolution of the form

$$0 \rightarrow R \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_0 \rightarrow R \rightarrow 0.$$

Therefore,  $\text{qpd}_R G \leq k$ .

Conversely, suppose  $\text{qpd}_R G = k < \infty$  and let  $p \in \pi$ . If  $G$  is  $p$ -torsion-free, then  $G$  has  $p$ -period  $1 < k$ , so we may assume that  $G$  has  $p$ -torsion.

Choose an  $RG$ -quasi-projective resolution  $\mathfrak{Q} \rightarrow R$  of length  $k$  and apply the exact functor  $\mathbf{Z}_{(p)} \otimes_R -$  to  $\mathfrak{Q}$  to obtain a  $\mathbf{Z}_{(p)}G$ -quasi-projective resolution  $\mathfrak{Q}_{(p)} \rightarrow \mathbf{Z}_{(p)}$  of length  $k$ .

Since  $G$  has  $p$ -torsion, it follows from Corollary 6.2. that any set of subgroups associated to  $\mathfrak{Q}_{(p)}$  consists of a single subgroup  $G_0$  whose index in  $G$  is prime to  $p$ . By Proposition 2, we have natural isomorphisms

$$(1) \ H^{2k+1}(G; -) \cong H^{k+1}(G_0; -) \quad (2) \ H^{k+1}(G; -) \cong H^1(G_0; -).$$

By Theorem 1 and Corollary 6.2, the set  $\{G_0\}$  is also associated to  $\mathfrak{Q}_{(p)}$  regarded as a  $\mathbf{Z}_{(p)}G_0$ -quasi-projective resolution. By another application of Proposition 2, we have a natural isomorphism

$$(3) \quad H^{k+1}(G_0; -) \cong H^1(G_0; -)$$

Combining (1), (2) and (3) and interpreting the result as Tate cohomology, we have a natural isomorphism

$$\hat{H}^{2k+1}(G, -) \cong \hat{H}^{k+1}(G, -).$$

Now dimension shifts may be used to establish, for all  $i \in \mathbf{Z}$ , the natural isomorphisms

$$\hat{H}^{i+k}(G, -) \cong \hat{H}^i(G, -)$$

of functors on  $\mathbf{Z}_{(p)}G$ -modules. Hence the  $p$ -period of  $G$  divides  $k$  for all  $p \in \pi$ .

**COROLLARY 7.1.** *If  $G \leq 2$ , then every finite subgroup of  $G$  is cyclic.*

*Proof.* Let  $F$  be a finite subgroup of  $G$ . Then  $F$  has period 1 or 2 and hence  $F_{ab} \cong \hat{H}^{-2}(F, \mathbf{Z}) \cong \hat{H}^0(F, \mathbf{Z}) \cong \mathbf{Z}/|F|\mathbf{Z}$

**COROLLARY 7.2.** *Suppose  $\text{qpd } G < \infty$  and  $\{G_\alpha\}_I$  is a full set of representatives of conjugacy classes of maximal finite subgroups of  $G$ . Then for all  $q > \text{qpd } G$ , there are natural isomorphisms*

$$H_q(G, -) \cong \bigoplus_I H_q(G_\alpha, -) \quad H^q(G, -) \cong \prod_I H^q(G_\alpha, -)$$

*of functors from  $\mathbf{Z}G$ -modules to  $\mathbf{Z}$ -modules.*

**COROLLARY 7.3.** *Suppose  $G, \{G_\alpha\}$  are as in Corollary 7.2. Then for each  $\alpha \in I$ ,  $\text{qpd } G_\alpha$  divides  $\text{qpd } G$ .*

## 4. Applications, Examples and Comments

### 4.1. Groups of finite qpd with torsion

Our theory provides some insight into the structure of groups with non-trivial torsion and finite qpd. The following results include some known facts about one-relator groups with torsion. No new results about torsion-free groups are to be expected, since in this case  $\text{qpd } G = \text{cd } G$ .

**PROPOSITION 8.** *Suppose  $\text{qpd } G < \infty$  and  $N$  is the normaliser in  $G$  of a non-trivial finite subgroup. Then  $N$  is finite.*

*Proof.* By hypothesis,  $N$  contains a non-trivial finite normal subgroup  $S$ . By Corollary 1.1,  $\text{qpd } N < \infty$ . Let  $\{N_\alpha\}_I$  be set of subgroups of  $N$  associated to some  $\mathbf{Z}N$ -quasi-projective resolution of  $\mathbf{Z}$ . By Theorem 6, there exist a unique  $\alpha \in I$  and a unique left coset  $nN_\alpha$  of  $N_\alpha$  in  $N$  such that  $S \subset nN_\alpha n^{-1}$ . Since  $S$  is normal in  $N$ , we have  $S \subset xN_\alpha x^{-1}$  for all  $x$  in  $N$ . It follows that  $N = N_\alpha$ , so by Corollary 1.2,  $N$  is finite.

**COROLLARY 8.1.** *Suppose  $\text{qpd } G < \infty$  and  $G$  has non-trivial torsion. Then either  $G$  is finite or  $G$  has trivial centre.*

**COROLLARY 8.2.** *Let  $G$  be an abelian group satisfying the hypotheses of Corollary 8.1. Then  $G$  is finite cyclic.*

*Proof.* Since  $H_{2n}(\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}; \mathbf{Z})$  has non-trivial torsion, it follows from Corollary 3.1 that  $\text{qpd } (\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}) = \infty$ . Hence any finite abelian group of finite qpd is cyclic. (cf. [4], p. 262).

**COROLLARY 8.3.** *Suppose  $G \cong H \times K$  with  $H \neq 1 \neq K$  and  $G$  satisfies the hypotheses of Corollary 8.1. Then  $H$  and  $K$  are finite of coprime order.*

#### 4.2. Conjugacy classes of finite subgroups

**PROPOSITION 9.** *Suppose  $\text{qpd } G = n < \infty$ . Then the following are equivalent:*

- (i) *There are only finitely many conjugacy classes of finite subgroups in  $G$ .*
- (ii) *There exists an integer  $j$  such that for all  $i > j$  the groups  $H^i(G, \mathbf{Z})$  are finite.*
- (iii)  *$H^{2n}(G, \mathbf{Z})$  is finite.*

*Proof.* Use Theorems 6, 7 and Corollary 7.1 together with the isomorphisms

$$H^{2n}(G, \mathbf{Z}) \cong \prod_I H^{2n}(G_\alpha, \mathbf{Z}) \cong \prod_I \hat{H}^0(G_\alpha, \mathbf{Z}) \cong \prod_I \mathbf{Z}/|G_\alpha| \mathbf{Z}.$$

*Remark.* Proposition 9 is a partial answer to Wall's question F7 in [17]. It implies, for example, that a group of type  $FP_\infty$  and of finite qpd admits only finitely many conjugacy classes of finite subgroups.

#### 4.3. Virtually torsion-free groups of finite qpd

##### (a) Virtual cohomological dimension

**PROPOSITION 10.** *Let  $G$  be a one-relator group and  $R$  a ring. Then*

- (i)  $\text{vcd}_R G \leq 2$ .
- (ii) *For any  $R$ -torsion-free subgroup  $H$  of  $G$ ,  $\text{cd}_R H \leq 2$  holds.*



*Proof.* Corollary 1.5 applies since  $G$  is virtually torsion-free [7]. Hence  $\text{vcd}_R G \leq \text{qpd}_R G \leq 2$ . Statement (ii) follows from Corollary 1.4. A proof based on combinatorial arguments is given in ([2], Theorem 7.7).

The inequality  $\text{vcd}_R G \leq \text{qpd}_R G$  holds whenever  $\text{vcd}_R G$  is finite. In the case  $R = \mathbf{Z}$ , this is the most one can say about the relationship between  $\text{vcd } G$  and  $\text{qpd } G$ . We refer to the examples 1, 2 below and also Example 3 in 4.4.

**EXAMPLE 1.** Let  $G$  and  $H$  be finite groups. Then  $\text{vcd}(G * H) = 1$  and  $\text{qpd}(G * H)$  equals the least common multiple of  $\text{qpd } G$  and  $\text{qpd } H$ .

**EXAMPLE 2.** The matrix group  $SL(2, \mathbf{Z})$  is infinite, has torsion and non-trivial centre. Hence  $\text{qpd } SL(2, \mathbf{Z}) = \infty$  by Corollary 8.1. However  $\text{vcd}(SL(2, \mathbf{Z})) = 1$ .

There is a decomposition for  $SL(2, \mathbf{Z})$ :

$$SL(2, \mathbf{Z}) \cong \mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$$

This shows that some care is needed if one wishes to weaken the hypothesis of Theorem 5. However, the condition set there on the edge-groups is probably stronger than is necessary for the conclusions to hold.

### (b) *Farrell–Tate cohomology*

Recall that the Farrell–Tate cohomology  $\hat{H}^*$  is defined in [6] for any group of finite  $\text{vcd}$ . Now suppose that  $G$  is virtually torsion-free and  $\text{qpd } G$  is finite, then by Corollary 1.5 we have  $\text{vcd } G \leq \text{qpd } G$ . Since any finite subgroup of  $G$  has periodic Tate cohomology, the Farrell–Tate functors  $\hat{H}^*(G, -)$  are periodic ([3], § 14). For all  $q > \text{vcd } G$ , there are natural isomorphisms  $\hat{H}^q(G, -) = H^q(G, -)$ . Hence the next proposition is a consequence of our results in § 3.

**PROPOSITION 11.** Suppose  $G$  is virtually torsion-free,  $\text{qpd } G < \infty$  and  $\{G_\alpha\}_I$  is a set of associated subgroups of  $G$ . Let  $k$  be the least common multiple of  $\{\text{qpd } G_\alpha\}_I$  and  $m$  the least common multiple of  $\{|G_\alpha|\}_I$ . Then

- (i) The Farrell–Tate cohomology  $\hat{H}^*(G, -)$  has period  $k$  dividing  $\text{qpd } G$ .
- (ii) For any  $\mathbf{Z}G$ -module  $M$ , the groups  $\hat{H}^*(G, M)$  are annihilated by  $m$ .

*Remarks.* (1) Part (ii) of Proposition 11 answers a question of ([3], § 11) in the special case of groups of finite  $\text{qpd}$ .

(2) The group  $SL(2, \mathbf{Z})$  has periodic Farrell–Tate cohomology with period 2, but  $\text{qpd } SL(2, \mathbf{Z}) = \infty$ .

#### 4.4 Groups with the Identity Property

Subgroups of one-relator groups have certain properties not shared by groups of finite qpd in general. We give two examples of groups of finite qpd, each of which violates a general property of subgroups of one-relator groups. In each case, the group in question has the Identity Property.

EXAMPLE 3. Let  $H$  denote Higman's group

$$\langle a, b, c, d \mid a^2 = a^b, b^2 = b^c, c^2 = c^d, d^2 = d^a \rangle,$$

and let  $S$  denote the one-relator group

$$\langle x, y \mid (x^{-1}y)^r = 1 \rangle, \quad (r > 1).$$

Choose non-trivial elements  $h, h' \in H$  and define

$$G = H \underset{h=x}{*} S \underset{y=h'}{*} H$$

The two canonical embeddings of  $H$  into  $G$  extend to an epimorphism of  $H * H$  onto  $G$ . Since  $H$  has no proper subgroups of finite index [8], neither has  $G$ . But the element  $x^{-1}y$  of  $G$  has finite order  $r > 1$ , so  $G$  is not virtually torsion-free and  $\text{vcd } G = \infty$ .

However  $\text{cd } H = 2$  ([2], p. 167) and  $\text{qpd } S = 2$  since  $S$  is a one-relator group. Now the proof of Theorem 5 gives a  $\mathbf{Z}G$ -quasi-projective resolution of  $\mathbf{Z}$  of length 2, so  $\text{qpd } G = 2$ .

EXAMPLE 4. Let  $D$  denote the infinite dihedral group  $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$  and let  $C$  denote the infinite cyclic subgroup of index 2 in  $D$ . Define  $G = D *_C D$ . Then  $\text{qpd } G$  is finite, by Theorem 5. In fact, the mapping cone construction in the proof of Theorem 5 yields a  $\mathbf{Z}G$ -quasi-projective resolution of  $\mathbf{Z}$  of length 2. It has four associated subgroups, each of order 2, and together they generate  $G$ . By Corollary 6.1, these four subgroups of order 2 form a complete set of representatives of conjugacy classes of finite subgroups.

Now suppose  $G$  is isomorphic to a free product  $*_I G_\alpha$  of finite groups  $G_\alpha \neq 1$  ( $\alpha \in I$ ). From the above discussion, we must have  $\text{card}(I) = 4$  and, for each  $\alpha \in I$ ,  $|G_\alpha| = 2$ . In other words,  $G \cong *_4 \mathbf{Z}/2\mathbf{Z}$ , and so  $G_{ab} \cong \bigoplus_4 \mathbf{Z}/2\mathbf{Z}$ . However, it follows from the decomposition  $G \cong D *_C D$  that  $G_{ab} \cong \bigoplus_3 \mathbf{Z}/2\mathbf{Z}$ . Thus we have obtained a contradiction, and so  $G$  cannot be expressed as a free product of finite groups.

This example contrasts with the following general fact about one-relator groups. Let  $H$  be a one-relator group and let  $N$  be the subgroup of  $H$  generated

by all the torsion elements of  $H$ . Then  $N$  is a free product of finite cyclic subgroups of  $H$ . ([7], Theorem 1). More generally, if  $S$  is any subgroup of  $H$ , generated by torsion elements, then  $S \subset N$ . Applying the Kurosh subgroup theorem, we deduce that  $S$  is a free product of finite cyclic groups.

## REFERENCES

- [1] BASS, H., *Algebraic K-theory*. Math. Lecture Note Series. Benjamin 1968.
- [2] BIERI, R., *Homological dimension of discrete groups*. Queen Mary College Mathematics Notes. 1976.
- [3] BROWN, K. S., *Groups of virtually finite dimension*. In Proceedings of the September 1977 Durham Conference on homological and combinatorial techniques in group theory. (C.T.C. Wall ed.)
- [4] CARTAN, H., EILENBERG, S., *Homological algebra*. Princeton University Press 1956.
- [5] CHISWELL, I. M., *Exact sequences associated with a graph of groups*. J. Pure Appl. Alg. **8** (1976) 63–74.
- [6] FARRELL, F. T. *An Extension of Tate cohomology to a class of Infinite groups*. J. Pure and Appl. Alg. **10** (1977) 153–166.
- [7] FISCHER, J., KARRASS, A. and SOLITAR, D., *On one-relator groups having elements of finite order*. Proc. Am. Math. Soc. **33** (1972) 297–301.
- [8] HIGMAN, G., *A finitely generated infinite simple group*. J. London Math. Soc. **26** (1951) 61–64.
- [9] HUEBSCHMANN, J., *Cohomology theory of aspherical groups and of small cancellation groups*. J. Pure and Appl. Algebra **14** (1979) 137–143.
- [10] KARRASS, A., MAGNUS, W. and SOLITAR, D., *Elements of finite order in groups with a single defining relation*. Comm. Pure Appl. Math. **13** (1960) 57–66.
- [11] LYNDON, R. C., *Cohomology theory of groups with a single defining relation*. Ann. of Math. **52** (1950) 650–665.
- [12] LYNDON, R. C., *On Dehn's algorithm*. Math. Ann. **166** (1966) 208–228.
- [13] LYNDON, R. C. and SCHUPP, P. E. *Combinatorial group theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete 89. Springer 1977.
- [14] SERRE, J. P., *Cohomologie des groupes discrets*. Annals of Math. Study **70** (1971) 77–169.
- [15] SWAN, R. G., *Periodic resolutions for finite groups*. Annals of Math. **72** (1960) 267–291.
- [16] WALL, C. T. C., *Pairs of relative cohomological dimension one*. J. Pure Appl. Alg. **1** (1971) 141–154.
- [17] WALL, C. T. C., *Homological and combinatorial techniques in group theory*. List of Problems. Preprint 1977.

*Note added in proof:* Recently, K. W. Gruenberg has informed us of a version of Serre's Theorem stronger than the form presented in [9]. For groups of finite qpd over  $\mathbb{Z}$  it contains our Theorem 6.

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