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Groups of finite quasi-projective dimension

JAMES HOWIE and HANS RUDOLF SCHNEEBELI

1. Introduction

1.1. Lyndon's Identity Theorem [11] may be interpreted as a description of the structure of the relation module arising from a one-relator presentation – or, more generally, from a staggered presentation – of a group G . For such presentations, the relation module is the direct sum of the cyclic submodules generated by the images of the defining relators. Furthermore, each such submodule has the form $\mathbb{Z}G/C$, where C is the finite cyclic subgroup of G generated by the image of the root of the corresponding defining relator.

We say that a presentation has the Identity Property if its relation module has the above form. This is equivalent to the condition (I.1) of Lyndon and Schupp ([13], p. 158). We say that a group G has the Identity Property if some presentation of G has the Identity Property. In this paper, we consider a property which is weaker than the Identity Property in the following respect. Instead of considering a relation module, we look at the kernel of the n th boundary map in an RG -projective resolution of R , where R is a commutative ring with 1, and we allow this kernel to be a direct sum of cyclic modules of the form RG/S , where S is an arbitrary subgroup of G .

1.2 Let R be a commutative ring with unit and let G be a group. An exact sequence of left RG -modules

$$\mathcal{Q}: 0 \rightarrow Q \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

of finite length $n > 0$ is called an *RG -quasi-projective resolution* of A if the modules P, P_{n-1}, \dots, P_0 are RG -projective and there exists an indexed set $\{G_\alpha\}_I$ of subgroups of G such that

$$Q \cong \bigoplus_I RG/G_\alpha.$$

The set $\{G_\alpha\}_I$ can be chosen such that no RG/G_α is RG -projective. We then say that the set $\{G_\alpha\}_I$ is *associated to the resolution* \mathcal{Q} . We make the convention that

the sequence $0 \rightarrow A \rightarrow A \rightarrow 0$ is an RG -quasi-projective resolution of length 0 if and only if A is RG -projective.

We define the RG -quasi-projective dimension of A to be the shortest possible length of an RG -quasi-projective resolution $\mathfrak{Q} \rightarrow A$. If no such resolution exists, the quasi-projective dimension is said to be infinite. In particular $\text{qpd}_R G$ denotes the RG -quasi-projective dimension of the trivial module R . We write $\text{qpd}_{\mathbb{Z}} G$ as $\text{qpd } G$. Our convention for length 0 is necessary to exclude the sequence $0 \rightarrow R \rightarrow R \rightarrow 0$, if R is not RG -projective.

Our notation, using RG -, might suggest that the above definitions depend only on the ring RG . It is, however important to recognize the explicit group-ring structure of RG . For example, even if $RG \cong SH$ as rings, an RG -quasi-projective resolution need not be an SH -quasi-projective resolution.

EXAMPLES

1. For all groups G , the inequality $\text{qpd}_R G \leq \text{cd}_R G$ holds.
2. Suppose G is a group with the Identity Property, then $\text{qpd}_R G \leq 2$.

Particular instances of such groups are:

- one relator groups,
- groups with staggered presentations,
- certain small cancellation groups [12],
- groups with an “aspherical” presentation in the sense of Lyndon and Schupp [13].

3. If G is a finite group, and there exists a periodic RG -projective resolution of finite period k over R , then $\text{qpd}_R G$ is at most k .

1.3 We now describe the structure of the article and discuss our main results.

In Section 2 we deal with some general consequences of our definition for $\text{qpd}_R G$. The similarity between cd_R and qpd_R is a basic theme.

Theorem 1 states a fundamental subgroup property for groups of finite qpd over R . Suppose S is a subgroup of G and $\text{qpd}_R G < \infty$. Then any RG -quasi-projective resolution may be interpreted as an RS -quasi-projective resolution and an associated set of subgroups of S may be defined in terms of a given associated set of subgroups of G . In particular, we have $\text{qpd}_R S \leq \text{qpd}_R G$ whenever $S \subset G$.

It follows from Theorem 1 that any set of subgroups of G associated to some quasi-projective resolution consists of finite groups. Hence, for all R -torsion-free groups G , $\text{qpd}_R G = \text{cd}_R G$. In particular $\text{qpd}_{\mathbb{Q}} G = \text{cd}_{\mathbb{Q}} G$ for all groups G . Another consequence of Theorem 1 is the following: Suppose R is torsion-free as a \mathbb{Z} -module and $\text{qpd}_R G$ is odd, then G is R -torsion-free.

Suppose G is a fundamental group of a graph of groups whose vertex groups have bounded qpd over R and whose edge groups are R -torsion-free. Theorem 5

says that $\text{qpd}_R G$ is finite. This is an analogue for qpd_R of a result of Chiswell for cd_R (cf. [5], p. 70).

The results of Section 3 give information about the finite subgroups in groups of finite qpd . Our central result is the following.

THEOREM 6. *Suppose $S \neq 1$ is a finite R -torsion subgroup of G , and $\{G_\alpha\}_I$ is a set of subgroups associated to some RG -quasi-projective resolution of R of finite length. Then there exist a unique $\alpha \in I$ and a unique left coset gG_α of G_α in G , such that S is contained in $gG_\alpha g^{-1}$.*

In the case $R = \mathbf{Z}$, it follows from Theorem 6 that the associated set of subgroups $\{G_\alpha\}_I$ is a full representative set of conjugacy classes of maximal finite subgroups. Hence, up to conjugacy, the groups G_α are determined by G independently of the particular choice of a $\mathbf{Z}G$ -quasi-projective resolution.

Theorem 6 is reminiscent of a theorem of Serre [9] and of results of Wall ([16], Lemma 7, Proposition 8). In fact, if $R = \mathbf{Z}$ in our situation, then the hypotheses of Serre's theorem hold, but we cannot prove this without the help of Theorems 6 and 7. Our proof of Theorem 6 is partly based on Wall's arguments.

In the special case where G is a one-relator group, Theorem 6 recovers a result of Karrass, Magnus and Solitar [10].

Further restrictions on the finite subgroups derive from Theorem 7. In the case $R = \mathbf{Z}$, it states that a finite group S satisfies $\text{qpd } S = n$ if and only if its Tate cohomology has period n . In particular, if $\text{qpd } G = 2$, then any finite subgroup of G is cyclic. Hence G has a relation module which is a direct sum of cyclic modules of the form $\mathbf{Z}G/C$, where C is a finite cyclic subgroup of G . Note the formal resemblance with the module-theoretic interpretation of the Identity Property.

As stated in 1.2, if G has the Identity Property, then $\text{qpd } G \leq 2$. We make no attempt to answer the question of whether the converse also holds. For torsion-free groups, this reduces to the question of whether cohomological and geometric dimensions always coincide (Eilenberg–Ganea problem).

The topics of Section 4 may be motivated by general properties of one-relator groups.

Let G be a one-relator group with torsion. Then either G is finite cyclic or the centre of G is trivial. We show in Corollary 8.1 that any group G of finite qpd with torsion either is finite or has trivial centre. It is known that a one-relator group has only finitely many conjugacy classes of finite subgroups. Proposition 9 gives necessary and sufficient conditions for a group of finite qpd to have only finitely many conjugacy classes of finite subgroups. This is of interest in connection with Wall's question F7 in [17].

Recall [14] that a group G is virtually torsion-free if one of its subgroups of finite index is torsion-free. The virtual cohomological dimension $\text{vcd } G$ of a virtually torsion-free group G is defined by $\text{vcd } G = \text{cd } S$, when S is a torsion-free subgroup of finite index. If G is not virtually torsion-free, then by definition $\text{vcd } G = \infty$.

Like qpd , the invariant vcd assumes finite values on certain classes of groups with torsion. If G is virtually torsion-free, $\text{vcd } G \leq \text{qpd } G$ holds. Our examples show that this is the most one can say in general about the relationship between vcd and qpd .

Of particular interest are groups G with $\text{vcd } G \leq \text{qpd } G < \infty$. In this case, the Farrell–Tate cohomology of G is periodic and is completely determined by the cohomology of the maximal finite subgroups of G .

Two of our examples make it clear that certain general properties of one-relator groups do not follow from the Identity Property. These examples concern the inequality $\text{vcd } G \leq \text{qpd } G$ and the structure of the subgroup generated by the torsion elements.

The methods of this paper are algebraic. We have left open the question of a suitable geometric interpretation of the property of having finite qpd . By analogy with cohomological dimension, one might expect to obtain sufficient criteria for finite qpd via such an interpretation.

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2. Groups of finite qpd over a ring R

2.1 Restriction to subgroups

THEOREM 1. *Suppose $\mathcal{Q} \rightarrow A$ is an RG -quasi-projective resolution of A of length n , and $\{G_\alpha\}_I$ is an associated set of subgroups. Let S be a subgroup of G . For each $\alpha \in I$, choose a set $\{t_\beta; \beta \in J_\alpha\}$ of representatives of the double cosets SgG_α ($g \in G$). For each $\beta \in J_\alpha$, define $S_{\alpha\beta} = S \cap t_\beta G_\alpha t_\beta^{-1}$. Then $\mathcal{Q} \rightarrow A$ is an RS -quasi-projective resolution, and there is an associated set of subgroups consisting of those $S_{\alpha\beta}$ for which $RS/S_{\alpha\beta}$ is not RS -projective.*

COROLLARY 1.1. *If S is a subgroup of G , then $\text{qpd}_R S \leq \text{qpd}_R G$.*

Proof of Theorem 1. Suppose $\mathcal{Q} \rightarrow A$ has the form $0 \rightarrow Q \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$, where P is an RG -projective, and $Q \cong \bigoplus_I RG/G_\alpha$. Now Q is the free

R -module on a left G -set T , whose decomposition into G -orbits has the form $T = \sqcup_I G/G_\alpha$.

The S -orbit decomposition of T has the form $T = \sqcup_I (\sqcup_{J_\alpha} S/S_{\alpha\beta})$, so as left RS -module

$$Q \cong \oplus_I (\oplus_{J_\alpha} RS/S_{\alpha\beta}).$$

Define $J = \{(\alpha, \beta); \alpha \in I, \beta \in J_\alpha, RS/S_{\alpha\beta} \text{ is not } RS\text{-projective}\}$. Then Q has an RS -direct sum decomposition $Q \cong P' \oplus Q'$, where P' is RS -projective and $Q' = \oplus_J RS/S_{\alpha\beta}$.

COROLLARY 1.2. *In the situation of Theorem 1, the subgroups G_α are all finite.*

Proof. Fix $\alpha \in I$ and let $S = G_\alpha$ in the proof of Theorem 1. Then some $S_{\alpha\beta}$ coincides with S , so Q contains the trivial module R as an RS -direct summand. Hence the RS -projective P_{n-1} contains R as a trivial RS -submodule. This is possible only if S is finite.

COROLLARY 1.3. *If G is R -torsion-free, then $\text{qpd}_R G = \text{cd}_R G$.*

Proof. It is sufficient to show that $\text{cd}_R G \leq \text{qpd}_R G$ and we may assume that $\text{qpd}_R G = n < \infty$. For every finite subgroup S of G , the order $|S| = |S| \cdot 1 \in R$ is a unit of R . Thus the canonical epimorphism $RG \rightarrow RG/S$ splits via an RG -homomorphism $\sigma: RG/S \rightarrow RG$, where

$$\sigma(gS) = \frac{1}{|S|} \sum_{h \in S} gh,$$

and so RG/S is RG -projective.

It follows that any RG -quasi-projective resolution is an RG -projective resolution.

COROLLARY 1.4. *If S is an R -torsion-free subgroup of G , then $\text{cd}_R S \leq \text{qpd}_R G$.*

COROLLARY 1.5. *Suppose G is virtually R -torsion-free, then $\text{vcd}_R G \leq \text{qpd}_R G$.*

2.2 Quasi-projective and projective resolutions

Let $0 \rightarrow Q \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$ be an RG -quasi-projective resolution of A and let $0 \rightarrow K \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow A \rightarrow 0$ be an exact sequence of RG -modules with all the M_i RG -projective. By Schanuel's lemma ([1], [15]) there

exist RG -projectives M' and P' such that $(Q \oplus P) \oplus P' \cong K \oplus M'$. Hence the exact sequence

$$\begin{array}{c} 0 \rightarrow K \rightarrow M_{n-1} \rightarrow M_{n-2} \rightarrow \cdots \rightarrow M_0 \rightarrow A \rightarrow 0 \\ \oplus \quad \cdot \quad \oplus \\ M \xrightarrow{1} M' \end{array}$$

is an RG -quasi-projective resolution of A .

If $0 \rightarrow (\oplus_I RG/G_\alpha) \oplus P \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$ is a quasi-projective resolution of A and $\{G_\alpha\}_I$ is an associated set of subgroups, then an RG -projective resolution can be obtained by the following construction. For each $\alpha \in I$, choose an RG_α -projective resolution of R , $M^\alpha \rightarrow RG_\alpha \rightarrow R \rightarrow 0$. The functor $RG \otimes G_\alpha -$ applied to this resolution gives an RG -projective resolution of RG/G_α . We thus obtain an RG -projective resolution of A of the form

$$\cdots \rightarrow \oplus_I (RG \otimes_{RG_\alpha} M_1^\alpha) \xrightarrow{\partial_{n+1}} (\oplus_I RG) \oplus P \xrightarrow{\partial_n} P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0. \quad (*)$$

We use the resolution (*) for R to calculate the homology and cohomology of G in high dimensions.

PROPOSITION 2. *Suppose $\{G_\alpha\}_I$ is a set of subgroups associated to some RG -quasi-projective resolution of R of Length n . Then for each $q > n$, there are natural isomorphisms*

$$H^q(G; -) \cong \prod_I H^{q-n}(G_\alpha; -); \quad H_q(G; -) \cong \oplus_I H_{q-n}(G_\alpha; -)$$

of functors from RG -modules to R -modules.

PROPOSITION 3. *If R admits an RG -quasi-projective resolution of length n , then $H_n(G; R)$ embeds into a free R -module.*

COROLLARY 3.1. *If R is a PID and R admits an RG -quasi-projective resolution of length n , then $H_n(G; R)$ is R -free.*

Proof of Proposition 3. We use the special resolution (*) to calculate $H_n(G; R)$. For each $\alpha \in I$, the α th direct summand $RG \otimes_{RG_\alpha} M_1^\alpha$ is mapped under ∂_{n+1} into the augmentation ideal of the α th direct summand $RG \cong RG \otimes_{RG_\alpha} RG_\alpha$. It follows that $1_R \otimes_{RG} \partial_{n+1} = 0$, and so $H_n(G; R) \cong \ker(1_R \otimes_{RG} \partial_n)$ is isomorphic to a submodule of the R -projective $(\oplus_I R) \oplus (R \otimes_{RG} P)$ and hence also of some free R -module.

COROLLARY 3.2. *Suppose R is torsion-free as an abelian group. If $\text{qpd}_R G = 2k + 1$ is odd, then G is R -torsion-free, and so $\text{cd}_R G = 2k + 1$.*

Proof. Suppose C is a finite cyclic subgroup of G of order $m > 1$. By Theorem 1, R admits an RC -quasi-projective resolution of length $2k + 1$. By Proposition 3, $H_{2k+1}(C, R)$ embeds into a free R -module and thus is \mathbf{Z} -torsion-free. On the other hand, $R/mR = H_{2k+1}(C, R)$ has obvious \mathbf{Z} -torsion unless $m \cdot 1$ is a unit in R and $R/mR = 0$.

If G is R -torsion-free, then Proposition 3 is relevant only for $n = \text{cd}_R G$. However if G has non-trivial information about $H_i(G, R)$ for infinitely many $i \geq \text{qpd}_R G$ as shown by the next result.

PROPOSITION 4. *Suppose there exists an RG -quasi-projective resolution of R of length i , let $\{G_\alpha\}_I$ be its associated set of subgroups, and suppose, for each $\alpha \in I$, there exists an RG_α -quasi-projective resolution of R of length k . Then there exists an RG -quasi-projective resolution of R of length $k + i$.*

Proof. In case G is R -torsion-free, there exist RG -projective resolutions of R of arbitrary length $1 \geq \text{cd}_R G$. Otherwise, the set $\{G_\alpha\}_I$ is non-empty. Using the idea of the construction of the resolution (*), but replacing each RG_α -projective resolution $\mathcal{M}^\alpha \rightarrow RG_\alpha \rightarrow R \rightarrow 0$ by an RG_α -quasi-projective resolution of length k , we obtain an RG -quasi-projective resolution of length $k + i$ by an analogous procedure. Here the following fact is used: Let $U \subset V \subset G$ be subgroups, then $RG \otimes_V RV/U \cong RG/U$.

Remark. Inductive arguments based on Proposition 4 lead to the following:

- (i) In the circumstances of Proposition 4, there exist RG -quasi-projective resolutions of R of length $m \cdot k + i$ for arbitrary integers $m \geq 0$.
- (ii) If $\text{qpd}_R G = n$, then there are RG -quasi-projective resolutions of R of length $m \cdot n$ for all integers $m > 0$.

2.3. Graphs of groups of finite qpd

THEOREM 5. *Let Γ be a graph of groups, $\{G_v\}_V$ its set of vertex groups, $\{G_e\}_E$ its set of edge groups and G its fundamental group. Suppose there is an integer j such that $\text{qpd}_R G_v < j$ for all $v \in V$ and that all the groups G_e are R -torsion-free, then $\text{qpd}_R G < \infty$.*

Proof. Associated to the graph Γ there is an exact sequence of RG -modules $A \twoheadrightarrow B \twoheadrightarrow R$ where $A \cong \bigoplus_E RG/G_e$ and $B \cong \bigoplus_V RG/G_v$ (cf. [5]). Let $\mathcal{P}^e \twoheadrightarrow R$ be an RG_e -projective resolution, then $RG \otimes_{RG_e} \mathcal{P}^e \rightarrow RG/G_e$ is an RG -projective resolution. Similarly $RG \otimes_{RG_v} \mathcal{Q}^v \twoheadrightarrow RG/G_v$ is an RG -quasi-projective resolution, provided $\mathcal{Q}^v \twoheadrightarrow R$ is RG_v -quasi-projective. We thus get an

RG -projective resolution $\mathcal{P} \twoheadrightarrow A$ of length $p \leq j$, and, using the remark after Proposition 4, an RG -quasi-projective resolution $\mathcal{Q} \twoheadrightarrow B$ of length $q > p$. The monomorphism $A \rightarrow B$ lifts to an RG -morphism of complexes $F: \mathcal{P} \rightarrow \mathcal{Q}$. The mapping cone construction described by Bass ([1] p. 30) yields an RG -quasi-projective resolution $MC(F) \twoheadrightarrow R$ of length q .

3. Finite subgroups in groups of finite qpd

3.1. Conjugacy classes of finite subgroups

THEOREM 6. *Suppose $S \neq 1$ is a finite R -torsion subgroup of G , and $\{G_\alpha\}_I$ is a set of subgroups associated to some RG -quasi-projective resolution of R of finite length. Then there exist a unique $\alpha \in I$ and a unique left coset gG_α of G_α in G , such that S is contained in $gG_\alpha g^{-1}$.*

The proof is split into three parts.

(a) The conclusion of the theorem holds if $p = |S|$ is prime.

Let J denote the set of all pairs (α, β) with $\alpha \in I$ and $\beta = gG_\alpha \in G/G_\alpha$ such that $S_{\alpha\beta} = S \cap gG_\alpha g^{-1} \neq 1$. Since p is not invertible in R , it follows from Theorem 1 that $\{S_{\alpha\beta}\}_J$ is a set of subgroups associated to some RS -quasi-projective resolution of finite length n , say. Now $S_{\alpha\beta} = S$ for all $(\alpha, \beta) \in J$ and so applying Proposition 2 twice, we get R -isomorphisms

$$R/pR \cong H_{2n+1}(S, R) \cong \bigoplus_J H_{n+1}(S, R) \cong \bigoplus_J \bigoplus_J H_1(S, R) \cong \bigoplus_J \bigoplus_J R/pR.$$

Comparing the ranks of the free R/pR -modules R/pR and $\bigoplus_J \bigoplus_J R/pR$, we find that J is a singleton, as required.

(b) If $\alpha, \alpha' \in I$, $g \in G$ are such that $G_{\alpha'} \cap gG_\alpha g^{-1}$ is not R -torsion-free, then $\alpha = \alpha'$ and $g \in G_\alpha$.

Otherwise, choose an R -torsion subgroup S of $G_{\alpha'} \cap gG_\alpha g^{-1}$ of prime order, and apply (a).

(c) The general case follows by induction on the order of S . For the inductive step, we apply an argument of Wall ([16], Proposition 8). Here (a) is the initial case of the induction and (b) plays the rôle of Wall's Lemma 7. Note that if S is an R -torsion group, so is any subgroup of S .

COROLLARY 6.1. *Suppose $R = \mathbb{Z}$ and $G, \{G_\alpha\}_I$ are as in Theorem 6. Then*

(i) *The set $\{G_\alpha\}_I$ is a complete set of representatives of conjugacy classes of maximal finite subgroups of G .*

(ii) *If $G \neq 1$ is finite, then I is a singleton, say $I = \{0\}$, and $G_0 = G$.*

Thus, in the particular case $R = \mathbb{Z}$, the subgroups G_α are determined up to conjugacy by G itself independently of the choice of a $\mathbb{Z}G$ -quasi-projective

resolution. In this sense we may speak about “a set of subgroups $\{G_\alpha\}_I$ associated to G .”

In the case $R = \mathbf{Z}_{(p)}$, the localisation of \mathbf{Z} at a prime p , a weaker form of Corollary 6.1 holds. We state it as a second corollary, since we refer to it in the proof of the next theorem.

COROLLARY 6.2. *Suppose $R = \mathbf{Z}_{(p)}$ and $G, \{G_\alpha\}_I$ are as in Theorem 6. For each $\alpha \in I$, choose a p -Sylow subgroup S_α in G_α . Then*

(i) *The set $\{S_\alpha\}$ is a complete set of representatives of conjugacy classes of maximal finite p -subgroups of G .*

(ii) *If G is finite and not p -torsion-free, then I is a singleton, say $I = \{0\}$, and S_0 is a p -Sylow subgroup of G .*

3.2. Periodicity

Recall ([4] Ch. XII) that a finite group G has p -period $k > 0$ if the p -component of its Tate cohomology satisfies $\hat{H}^i(G, -)_{(p)} \cong \hat{H}^{i+k}(G, -)_{(p)}$ and k is minimal with this property. This is equivalent to $\hat{H}^*(G, -)$ having period k on the category of $\mathbf{Z}_{(p)}G$ -modules.

If π is a non-empty set of primes, then G is π -periodic if and only if it is p -periodic for each $p \in \pi$. The π -period of G is the least common multiple of the p -periods for all $p \in \pi$.

THEOREM 7. *Suppose $R = \mathbf{Z}_{(\pi)}$, where π is a set of primes. Let G be a finite group which is not R -torsion-free. Then G has π -period k if and only if $\text{qpd}_R G = k$.*

Proof. Suppose that G has finite π -period k . By a theorem of Swan [15], there exists an RG -quasi-projective resolution of the form

$$0 \rightarrow R \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_0 \rightarrow R \rightarrow 0.$$

Therefore, $\text{qpd}_R G \leq k$.

Conversely, suppose $\text{qpd}_R G = k < \infty$ and let $p \in \pi$. If G is p -torsion-free, then G has p -period $1 < k$, so we may assume that G has p -torsion.

Choose an RG -quasi-projective resolution $\mathfrak{Q} \rightarrow R$ of length k and apply the exact functor $\mathbf{Z}_{(p)} \otimes_R -$ to \mathfrak{Q} to obtain a $\mathbf{Z}_{(p)}G$ -quasi-projective resolution $\mathfrak{Q}_{(p)} \rightarrow \mathbf{Z}_{(p)}$ of length k .

Since G has p -torsion, it follows from Corollary 6.2. that any set of subgroups associated to $\mathfrak{Q}_{(p)}$ consists of a single subgroup G_0 whose index in G is prime to p . By Proposition 2, we have natural isomorphisms

$$(1) \ H^{2k+1}(G; -) \cong H^{k+1}(G_0; -) \quad (2) \ H^{k+1}(G; -) \cong H^1(G_0; -).$$

By Theorem 1 and Corollary 6.2, the set $\{G_0\}$ is also associated to $\mathfrak{Q}_{(p)}$ regarded as a $\mathbf{Z}_{(p)}G_0$ -quasi-projective resolution. By another application of Proposition 2, we have a natural isomorphism

$$(3) \quad H^{k+1}(G_0; -) \cong H^1(G_0; -)$$

Combining (1), (2) and (3) and interpreting the result as Tate cohomology, we have a natural isomorphism

$$\hat{H}^{2k+1}(G, -) \cong \hat{H}^{k+1}(G, -).$$

Now dimension shifts may be used to establish, for all $i \in \mathbf{Z}$, the natural isomorphisms

$$\hat{H}^{i+k}(G, -) \cong \hat{H}^i(G, -)$$

of functors on $\mathbf{Z}_{(p)}G$ -modules. Hence the p -period of G divides k for all $p \in \pi$.

COROLLARY 7.1. *If $G \leq 2$, then every finite subgroup of G is cyclic.*

Proof. Let F be a finite subgroup of G . Then F has period 1 or 2 and hence $F_{ab} \cong \hat{H}^{-2}(F, \mathbf{Z}) \cong \hat{H}^0(F, \mathbf{Z}) \cong \mathbf{Z}/|F|\mathbf{Z}$

COROLLARY 7.2. *Suppose $\text{qpd } G < \infty$ and $\{G_\alpha\}_I$ is a full set of representatives of conjugacy classes of maximal finite subgroups of G . Then for all $q > \text{qpd } G$, there are natural isomorphisms*

$$H_q(G, -) \cong \bigoplus_I H_q(G_\alpha, -) \quad H^q(G, -) \cong \prod_I H^q(G_\alpha, -)$$

of functors from $\mathbf{Z}G$ -modules to \mathbf{Z} -modules.

COROLLARY 7.3. *Suppose $G, \{G_\alpha\}$ are as in Corollary 7.2. Then for each $\alpha \in I$, $\text{qpd } G_\alpha$ divides $\text{qpd } G$.*

4. Applications, Examples and Comments

4.1. Groups of finite qpd with torsion

Our theory provides some insight into the structure of groups with non-trivial torsion and finite qpd. The following results include some known facts about one-relator groups with torsion. No new results about torsion-free groups are to be expected, since in this case $\text{qpd } G = \text{cd } G$.

PROPOSITION 8. *Suppose $\text{qpd } G < \infty$ and N is the normaliser in G of a non-trivial finite subgroup. Then N is finite.*

Proof. By hypothesis, N contains a non-trivial finite normal subgroup S . By Corollary 1.1, $\text{qpd } N < \infty$. Let $\{N_\alpha\}_I$ be set of subgroups of N associated to some $\mathbf{Z}N$ -quasi-projective resolution of \mathbf{Z} . By Theorem 6, there exist a unique $\alpha \in I$ and a unique left coset nN_α of N_α in N such that $S \subset nN_\alpha n^{-1}$. Since S is normal in N , we have $S \subset xN_\alpha x^{-1}$ for all x in N . It follows that $N = N_\alpha$, so by Corollary 1.2, N is finite.

COROLLARY 8.1. *Suppose $\text{qpd } G < \infty$ and G has non-trivial torsion. Then either G is finite or G has trivial centre.*

COROLLARY 8.2. *Let G be an abelian group satisfying the hypotheses of Corollary 8.1. Then G is finite cyclic.*

Proof. Since $H_{2n}(\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}; \mathbf{Z})$ has non-trivial torsion, it follows from Corollary 3.1 that $\text{qpd } (\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}) = \infty$. Hence any finite abelian group of finite qpd is cyclic. (cf. [4], p. 262).

COROLLARY 8.3. *Suppose $G \cong H \times K$ with $H \neq 1 \neq K$ and G satisfies the hypotheses of Corollary 8.1. Then H and K are finite of coprime order.*

4.2. Conjugacy classes of finite subgroups

PROPOSITION 9. *Suppose $\text{qpd } G = n < \infty$. Then the following are equivalent:*

- (i) *There are only finitely many conjugacy classes of finite subgroups in G .*
- (ii) *There exists an integer j such that for all $i > j$ the groups $H^i(G, \mathbf{Z})$ are finite.*
- (iii) *$H^{2n}(G, \mathbf{Z})$ is finite.*

Proof. Use Theorems 6, 7 and Corollary 7.1 together with the isomorphisms

$$H^{2n}(G, \mathbf{Z}) \cong \prod_I H^{2n}(G_\alpha, \mathbf{Z}) \cong \prod_I \hat{H}^0(G_\alpha, \mathbf{Z}) \cong \prod_I \mathbf{Z}/|G_\alpha| \mathbf{Z}.$$

Remark. Proposition 9 is a partial answer to Wall's question F7 in [17]. It implies, for example, that a group of type FP_∞ and of finite qpd admits only finitely many conjugacy classes of finite subgroups.

4.3. Virtually torsion-free groups of finite qpd

(a) Virtual cohomological dimension

PROPOSITION 10. *Let G be a one-relator group and R a ring. Then*

- (i) $\text{vcd}_R G \leq 2$.
- (ii) *For any R -torsion-free subgroup H of G , $\text{cd}_R H \leq 2$ holds.*

Proof. Corollary 1.5 applies since G is virtually torsion-free [7]. Hence $\text{vcd}_R G \leq \text{qpd}_R G \leq 2$. Statement (ii) follows from Corollary 1.4. A proof based on combinatorial arguments is given in ([2], Theorem 7.7).

The inequality $\text{vcd}_R G \leq \text{qpd}_R G$ holds whenever $\text{vcd}_R G$ is finite. In the case $R = \mathbf{Z}$, this is the most one can say about the relationship between $\text{vcd } G$ and $\text{qpd } G$. We refer to the examples 1, 2 below and also Example 3 in 4.4.

EXAMPLE 1. Let G and H be finite groups. Then $\text{vcd}(G * H) = 1$ and $\text{qpd}(G * H)$ equals the least common multiple of $\text{qpd } G$ and $\text{qpd } H$.

EXAMPLE 2. The matrix group $SL(2, \mathbf{Z})$ is infinite, has torsion and non-trivial centre. Hence $\text{qpd } SL(2, \mathbf{Z}) = \infty$ by Corollary 8.1. However $\text{vcd}(SL(2, \mathbf{Z})) = 1$.

There is a decomposition for $SL(2, \mathbf{Z})$:

$$SL(2, \mathbf{Z}) \cong \mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$$

This shows that some care is needed if one wishes to weaken the hypothesis of Theorem 5. However, the condition set there on the edge-groups is probably stronger than is necessary for the conclusions to hold.

(b) Farrell–Tate cohomology

Recall that the Farrell–Tate cohomology \hat{H}^* is defined in [6] for any group of finite vcd . Now suppose that G is virtually torsion-free and $\text{qpd } G$ is finite, then by Corollary 1.5 we have $\text{vcd } G \leq \text{qpd } G$. Since any finite subgroup of G has periodic Tate cohomology, the Farrell–Tate functors $\hat{H}^*(G, -)$ are periodic ([3], § 14). For all $q > \text{vcd } G$, there are natural isomorphisms $\hat{H}^q(G, -) = H^q(G, -)$. Hence the next proposition is a consequence of our results in § 3.

PROPOSITION 11. Suppose G is virtually torsion-free, $\text{qpd } G < \infty$ and $\{G_\alpha\}_I$ is a set of associated subgroups of G . Let k be the least common multiple of $\{\text{qpd } G_\alpha\}_I$ and m the least common multiple of $\{|G_\alpha|\}_I$. Then

- (i) The Farrell–Tate cohomology $\hat{H}^*(G, -)$ has period k dividing $\text{qpd } G$.
- (ii) For any $\mathbf{Z}G$ -module M , the groups $\hat{H}^*(G, M)$ are annihilated by m .

Remarks. (1) Part (ii) of Proposition 11 answers a question of ([3], § 11) in the special case of groups of finite qpd .

(2) The group $SL(2, \mathbf{Z})$ has periodic Farrell–Tate cohomology with period 2, but $\text{qpd } SL(2, \mathbf{Z}) = \infty$.

4.4 Groups with the Identity Property

Subgroups of one-relator groups have certain properties not shared by groups of finite qpd in general. We give two examples of groups of finite qpd, each of which violates a general property of subgroups of one-relator groups. In each case, the group in question has the Identity Property.

EXAMPLE 3. Let H denote Higman's group

$$\langle a, b, c, d \mid a^2 = a^b, b^2 = b^c, c^2 = c^d, d^2 = d^a \rangle,$$

and let S denote the one-relator group

$$\langle x, y \mid (x^{-1}y)^r = 1 \rangle, \quad (r > 1).$$

Choose non-trivial elements $h, h' \in H$ and define

$$G = H \underset{h=x}{*} S \underset{y=h'}{*} H$$

The two canonical embeddings of H into G extend to an epimorphism of $H * H$ onto G . Since H has no proper subgroups of finite index [8], neither has G . But the element $x^{-1}y$ of G has finite order $r > 1$, so G is not virtually torsion-free and $\text{vcd } G = \infty$.

However $\text{cd } H = 2$ ([2], p. 167) and $\text{qpd } S = 2$ since S is a one-relator group. Now the proof of Theorem 5 gives a $\mathbf{Z}G$ -quasi-projective resolution of \mathbf{Z} of length 2, so $\text{qpd } G = 2$.

EXAMPLE 4. Let D denote the infinite dihedral group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ and let C denote the infinite cyclic subgroup of index 2 in D . Define $G = D *_C D$. Then $\text{qpd } G$ is finite, by Theorem 5. In fact, the mapping cone construction in the proof of Theorem 5 yields a $\mathbf{Z}G$ -quasi-projective resolution of \mathbf{Z} of length 2. It has four associated subgroups, each of order 2, and together they generate G . By Corollary 6.1, these four subgroups of order 2 form a complete set of representatives of conjugacy classes of finite subgroups.

Now suppose G is isomorphic to a free product $*_I G_\alpha$ of finite groups $G_\alpha \neq 1$ ($\alpha \in I$). From the above discussion, we must have $\text{card}(I) = 4$ and, for each $\alpha \in I$, $|G_\alpha| = 2$. In other words, $G \cong *_4 \mathbf{Z}/2\mathbf{Z}$, and so $G_{ab} \cong \bigoplus_4 \mathbf{Z}/2\mathbf{Z}$. However, it follows from the decomposition $G \cong D *_C D$ that $G_{ab} \cong \bigoplus_3 \mathbf{Z}/2\mathbf{Z}$. Thus we have obtained a contradiction, and so G cannot be expressed as a free product of finite groups.

This example contrasts with the following general fact about one-relator groups. Let H be a one-relator group and let N be the subgroup of H generated

by all the torsion elements of H . Then N is a free product of finite cyclic subgroups of H . ([7], Theorem 1). More generally, if S is any subgroup of H , generated by torsion elements, then $S \subset N$. Applying the Kurosh subgroup theorem, we deduce that S is a free product of finite cyclic groups.

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Note added in proof: Recently, K. W. Gruenberg has informed us of a version of Serre's Theorem stronger than the form presented in [9]. For groups of finite qpd over \mathbb{Z} it contains our Theorem 6.

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