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Exponential sums associated with algebraic number fields

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§1. Let $\zeta_K(s)$ denote the Dedekind zeta-function of an algebraic number field K of degree n. For $\operatorname{Re} s>1$, $\zeta_K(s)=\sum_{k=1}^{\infty} a_k k^{-s}$, where a_k stands for the number of integral ideals in K with norm k. If r_1 is the number of real conjugates of K, and $2r_2$ the number of imaginary conjugates, and D the discriminant, $\zeta_K(s)$ satisfies the functional equation [1] $\xi(s)=\xi(1-s)$, where

$$\xi(s)=\Gamma^{r_1}(\tfrac{1}{2}s)\cdot\Gamma^{r_2}(s)\cdot B^{-s}\zeta_K(s),$$

with $B=2^{r_2}\pi^{n/2}|D|^{-1/2}$, $r_1+2r_2=n$. It is known that $a_k=O(k^{\varepsilon})$, for every $\varepsilon>0$, and

$$\sum_{k\leq x} a_k=\lambda x+O(x^{(n-1)/(n+1)}),\tag{1.1}$$

where λ stands for the residue of $\zeta_K(s)$ at $s=1$.

Our purpose is to prove the following

THEOREM 1. *If η is real, $\eta\neq 0$, $\alpha>1/n$, $\lambda_k=B\cdot k$ for integral $k\geq 1$, then*

$$\sum_{\lambda_k\leq x} a_k\exp(2\pi i\eta\lambda_k^{\alpha})=c_2\sum_{\lambda_k\leq c_0x^{n\alpha-1}} a_k\cdot\lambda_k^{(2-\alpha n)/[2(n\alpha-1)]}\cdot\exp\left\{-iq\left(\frac{\lambda_k^{\alpha}}{2\pi\eta}\right)^{1/(n\alpha-1)}\right\}$$

$$+O(x^{[(n-1)/2]\alpha+\varepsilon})+O(x^{1-\alpha_1}),\tag{1.2}$$

for $n\geq 3$, and every $\varepsilon>0$, where $\alpha_1=\alpha$ if $\alpha\leq 2/(n+1)$, while $\alpha_1=\alpha-\varepsilon<\alpha$, if $\alpha>2/(n+1)$; $c_2=c_2(\alpha,\eta,K)$ is a constant that can be explicitly determined, $c_0=(2\pi\eta n\alpha/h)^n$, where $h=n\cdot 2^{r_1/n}$, and $q=(\alpha n-1)(2^{r_1}\cdot\alpha^{-n})^{\alpha/(n\alpha-1)}$.

If $n=2$, (1.2) holds with the error-term

$$O(x^{(\alpha/2)+\varepsilon})+O(x^{1-\alpha+\varepsilon}).$$

The case $\alpha = 2/n$ of Theorem 1 gives the approximate reciprocity formula, which is stated as

THEOREM 2. *Under the same conditions as in Theorem* 1, *we have, for* $n \geq 2$,

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^{2/n}) = c_2' \sum_{\lambda_k \leq c_0' x} a_k \cdot \exp\left\{\frac{-iq'\lambda_k^{2/n}}{2\pi\eta}\right\} + O(x^{1-(1/n)+\varepsilon}), \tag{1.3}$$

for every $\varepsilon > 0$. *Here* c_0', c_2', *and* q' *denote respectively the values of* c_0, c_2, *and* q, *for* $\alpha = 2/n$.

Theorem 2 yields as a special case the approximate reciprocity formula for quadratic fields previously obtained by us by a different method [3], though the error-term here is somewhat less sharp, in that we have x^ε instead of $\log x$.

If we choose $4\pi\eta = h$ in Theorem 2, so that $c_0' = 1$, we get the

COROLLARY.

$$\sum_{k \leq x} a_k \exp(i\pi n \cdot |D|^{-1/n} k^{2/n}) = e^{i\pi[(1/2)-(r_1/4)]} \sum_{k \leq x} a_k \exp(-i\pi n\, |D|^{-1/n} k^{2/n}) + O(x^{1-(1/n)+\varepsilon}), \tag{1.4}$$

for $n \geq 2$. *This can also be written as*

$$\sum_{k \leq x} a_k \sin\{\pi n\, |D|^{-1/n} k^{2/n} + \tfrac{1}{2}\pi(r_1 - 2)/4\} = O(x^{1-(1/n)+\varepsilon}). \tag{1.5}$$

Theorem 1, combined with the known estimate (1.1), also yields the following

THEOREM 3. *If* $1/n < \alpha < 2/n$, *then*

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^{\alpha}) = O(x^{n\alpha/2}) + O(x^{1-\alpha_1}), \quad \textit{for} \quad n \geq 2. \tag{1.6}$$

In particular, if $n \geq 3$, *and we take* $\alpha = 1/(n-1)$, *we obtain the result:*

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^{1/(n-1)}) = \begin{cases} O(x^{3/4}), & \textit{if} \quad n = 3; \\ O(x^{1-1/(n-1)}), & \textit{if} \quad n \geq 4. \end{cases} \tag{1.7}$$

This is sharper than the estimate recently obtained by us [4], namely $O(x^{1-1/2(n-1)})\log(1+x)$, for all $n\geq 3$.

If $1/n<\alpha\leq 2/(n+2)$, Theorem 3 implies that

$$\sum_{\lambda_k\leq x} a_k\cdot\exp(2\pi i\eta\lambda_k^\alpha)=O(x^{1-\alpha}),\quad \text{for}\quad n\geq 3. \tag{1.8}$$

The case $0<\alpha\leq 1/n$ is covered by the next two theorems.

THEOREM 4. *If $n\geq 3$, and $0<\alpha\leq 1/n$, then*

$$\sum_{\lambda_k\leq x} a_k\cdot\exp(2\pi i\eta\lambda_k^\alpha)=O(x^{1-\alpha}). \tag{1.9}$$

THEOREM 5. *If $n=2$, and $0<\alpha<\frac{1}{2}$, then*

$$\sum_{\lambda_k\leq x} a_k\cdot\exp(2\pi i\eta\lambda_k^\alpha)=O(x^{1-\alpha}). \tag{1.10}$$

If $n=2$, $\alpha=\frac{1}{2}$, then

$$\sum_{\lambda_k\leq x} a_k\cdot\exp(2\pi i\eta\lambda_k^{1/2})=O(x^{1/2}), \tag{1.11}$$

provided that $\eta\neq h\lambda_k^{1/2}$ for all k, while

$$\sum_{\lambda_k\leq x} a_k\cdot\exp(2\pi i\eta\cdot\lambda_k^{1/2})=c_{k_0}\cdot a_{k_0}\cdot x^{3/4}+O(x^{1/2}),\qquad c_{k_0}\neq 0, \tag{1.12}$$

if $\eta=h\lambda_{k_0}^{1/2}$ for some k_0.

The general method of attack is similar to that of [4], though a number of additional difficulties caused by the introduction of the parameter α have to be overcome. The estimates for the wider class of exponential sums considered here should find their use in the study of the critical zeros of the Dedekind zeta-function of an ideal class in K, in case $n\geq 3$.

§2. The basic lemmas

An indispensable tool in the following analysis is the identity:

$$\frac{1}{\Gamma(\rho+1)}\sum_{\lambda_k\le x} a_k(x-\lambda_k)^\rho = Q_\rho(x)+\sum_{k=1}^{\infty} a_k\cdot\lambda_k^{-1-\rho}I_\rho(\lambda_k x), \tag{2.1}$$

which holds for $x>0$, $\rho>\frac{1}{2}(n-1)$, *ρ integral,* and which is implied by the functional equation of $\zeta_K(s)$ [1]. Here $\lambda_k=B\cdot k$, where B is defined as in §1, and

$$Q_\rho(x)=\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{B^{-s}\zeta_K(s)\cdot\Gamma(s)}{\Gamma(s+\rho+1)}\cdot x^{s+\rho}\,ds, \tag{2.2}$$

where $\mathscr{C}$ is a curve which encloses all the singularities of the integrand, and

$$I_\rho(x)=\frac{1}{2\pi i}\int_{\mathscr{C}'_\rho}\frac{\Gamma(1-s)\Delta(s)}{\Gamma(\rho+2-s)\Delta(1-s)}\cdot x^{1+\rho-s}\,ds, \tag{2.3}$$

where $\Delta(s)=\Gamma^{r_1}(\frac{1}{2}s)\Gamma^{r_2}(s)$, and $\mathscr{C}'_\rho$ denotes the path of integration extending from $c_\rho-i\infty$ to $c_\rho-iR$, thence to $c_\rho+r-iR$, $c_\rho+r+iR$, $c_\rho+iR$, and $c_\rho+i\infty$, with r and R chosen suitably large, and with $c_\rho=\frac{1}{2}+(\rho/n)-\varepsilon$, $0<\varepsilon<(1/2n)$.

The following asymptotic formula [2] is crucial to the proof of our main theorem:

$$I_\rho(x)=\sum_{\nu=0}^{m} e'_\nu x^{\omega_\rho-\nu/n}\cos(hx^{1/n}+\pi_\nu)+O(x^{\omega_\rho-(m+1)/n}), \tag{2.4}$$

where $\omega_\rho=\frac{1}{2}-(1/2n)+\rho(1-1/n)$, $h=n\cdot 2^{r_1/n}$, $\pi_\nu=\pi_\nu(\rho)=\pi\{(n/2)+(\rho/2)+\frac{1}{4}(r_1+3)-(\nu/2)\}$, for all integers ρ, positive or negative. [It may be noted that in (2.3) of [4], the exponent $(m+1/n)$ should be $(m+1)/n$, and in the expression for h one should have $2r_2$ in place of r_2].

We define, as usual, $a_0=0=\lambda_0$, $A(x)=\sum_{\lambda_k\le x}a_k$, for $x>0$, $A(x)=0$ for $0\le x<\lambda_1$, and $A^{r-1}(x)=(d/dx)(A^r(x))$, almost everywhere, for $r\ge 1$.

Let

$$x>\lambda_1,\qquad x_1=x+x^{1-\alpha},\qquad \alpha>\frac{1}{n}, \tag{2.5}$$

and let $u(t)$ be an infinitely differentiable function in $(-\infty<t<\infty)$, such that $u(t)=0$, for $t\le c<\frac{1}{2}\lambda_1$; $u(t)=1$, in a neighbourhood of $\lambda_1\le t\le x$; $u(t)=0$, for

$t \geqslant x_1$; $0 \leqslant u(t) \leqslant 1$, for $-\infty < t < \infty$; and

$$|u^{(k)}(t)| \leqslant c_k(1+t)^{k(\alpha-1)}, \quad \text{for} \quad t \geqslant 0, \qquad k \geqslant 0, \tag{2.6}$$

where $u^{(k)}$ denotes the k^{th} derivative of u, and c_k is a constant depending only on k.

Further let

$$u_1(t) = \begin{cases} u(t), & \text{for} \quad t \leqslant x, \\ 1, & \text{for} \quad t \geqslant x. \end{cases} \tag{2.7}$$

Let $f(t) = \exp(2\pi i \eta t^{\alpha})$, where $\eta > 0$, $t \geqslant 0$, $\alpha > 1/n$; let $0 \leqslant \gamma < 1$, and

$$F(t) = t^{-\gamma} \cdot u(t) \cdot f(t). \tag{2.8}$$

If $F^{(r)}$ denotes the r^{th} derivative of F, we have

$$F^{(r+1)}(t) \ll (1+t)^{(r+1)(\alpha-1)-\gamma}, \tag{2.9}$$

since $f^{(k)}(t) \ll (1+t)^{k(\alpha-1)}$, and $u^{(k)}(t) \ll (1+t)^{k(\alpha-1)}$. We have

$$\sum_{\lambda_k \leqslant x_1} a_k F(\lambda_k) = \sum_{\lambda_k \leqslant x} a_k F(\lambda_k) + O\Big\{ \sum_{x < \lambda_k \leqslant x_1} a_k \lambda_k^{-\gamma} \Big\}. \tag{2.10}$$

Since $a_k = O(k^{\varepsilon})$, for every $\varepsilon > 0$, we have

$$\sum_{x < \lambda_k \leqslant x_1} a_k \lambda_k^{-\gamma} \leqslant x^{-\gamma} \sum_{x < \lambda_k \leqslant x_1} a_k = \begin{cases} x^{-\gamma} \cdot O(x^{1-\alpha+\varepsilon}), & \text{if} \quad \alpha \leqslant 1; \\ x^{-\gamma} \cdot O(x^{\varepsilon}), & \text{if} \quad \alpha > 1. \end{cases} \tag{2.11}$$

Because of (1.1) we have also

$$\sum_{x < \lambda_k \leqslant x_1} a_k \lambda_k^{-\gamma} \ll x^{-\gamma}(x^{1-\alpha} + x^{1-2/(n+1)}). \tag{2.11$'$}$$

We shall express the first sum on the right-hand side of (2.10) as an integral, and estimate it in different ranges of λ_k. Clearly

$$\sum_{k=0}^{\infty} a_k F(\lambda_k) = \int_0^{\infty} F(t)\, dA(t) = (-1)^{r+1} \int_0^{\infty} A^r(t) \cdot F^{(r+1)}(t)\, dt. \tag{2.12}$$

[It may be noted that in (3.3) of [4] we should have $(-1)^{r+1}$ in place of $(-1)^r$ with the consequent changes in sign.] We choose the integer r so large that the infinite series in (2.1) converges absolutely, and uniformly, for $x \geq c$ and $\rho \geq r > 0$, so that we may substitute for $A^r(t)$ the corresponding series in (2.1) plus $Q_r(t)$. We then seek to estimate, for a suitably chosen y,

$$\sum_{\lambda_k > y} a_k \cdot \lambda_k^{-1-r} \int_0^\infty F^{(r+1)}(t) \cdot I_r(\lambda_k t)\, dt,$$

which equals

$$\sum_{\lambda_k > y} a_k \cdot \lambda_k^{-1-r} \int_c^{x_1} F^{(r+1)}(t) \cdot I_r(\lambda_k t)\, dt$$

$$\ll \sum_{\lambda_k > y} a_k \cdot \lambda_k^{-1-r} \cdot \lambda_k^{(1/2)-(1/2n)+r(1-1/n)} \int_c^{x_1} (1+t)^{(r+1)(\alpha-1)+(1/2)-(1/2n)+r(1-1/n)-\gamma}\, dt$$

$$\ll \sum_{\lambda_k > y} a_k \cdot \lambda_k^{-(1/2)-(1/2n)-(r/n)} \cdot x^{r(\alpha-1/n)+\alpha+(1/2)-(1/2n)-\gamma}. \tag{2.13}$$

If we choose

$$y = c_0 \cdot x^\delta, \qquad \delta = n\alpha - 1 + \varepsilon_0 > 0, \qquad \varepsilon_0 > 0, \qquad c_0 > 0, \tag{2.14}$$

then (2.13) is

$$\ll x^{\delta[(1/2)-(1/2n)-(r/n)]+r(\alpha-1/n)+\alpha+(1/2)-(1/2n)-\gamma}$$

$$\ll x^{r[\alpha(1/n)-(\delta/n)]+(\delta+1)[(1/2)-(1/2n)]+\alpha-\gamma}$$

$$\ll x^{-q}, \tag{2.15}$$

for any given $q > 0$, provided that r is large enough.

Next we have the section

$$(-1)^{r+1} \sum_{\lambda_k \leq y} a_k \cdot \lambda_k^{-1-r} \int_0^\infty F^{(r+1)}(t) \cdot I_r(\lambda_k t)\, dt = \sum_{\lambda_k \leq y} a_k \int_0^\infty F(t) \cdot I_{-1}(\lambda_k t)\, dt \tag{2.16}$$

as well as the term

$$(-1)^{r+1} \int_0^\infty F^{(r+1)}(t) \cdot Q_r(t)\, dt = \int_0^\infty F(t) \cdot Q_{-1}(t)\, dt = \left\{ \int_0^x + \int_x^{x_1} \right\} F(t) \cdot Q_{-1}(t)\, dt. \tag{2.17}$$

Since $Q_{-1}(t)$ is a constant (see [4, p. 82]), we have

$$\int_x^{x_1} F(t)Q_{-1}(t)\,dt \ll x^{1-\gamma-\alpha}, \tag{2.18}$$

while

$$\begin{aligned} c\int_0^x F(t)Q_{-1}(t)\,dt &= \alpha^{-1}\int_0^{x^\alpha} u_1(t^{1/\alpha})\cdot t^{\{(1-\gamma)/\alpha\}-1}\cdot \exp(2\pi i\eta t)\,dt \\ &= \alpha^{-1}\int_0^{x^\alpha} v(t)\cdot \exp(2\pi i\eta t)\,dt, \qquad v(t)=u_1(t^{1/\alpha})\cdot t^{\{(1-\gamma)/\alpha\}-1} \\ &= (2\pi i\eta\alpha)^{-1}\left(v(x^\alpha)\cdot\exp(2\pi i\eta x^\alpha)-\int_0^{x^\alpha} v'(t)\cdot\exp(2\pi i\eta t)\,dt\right), \end{aligned}$$

since $v(t)\in C^\infty(-\infty<t<\infty)$, with $v(0)=0$, and $v(x^\alpha)=x^{1-\gamma-\alpha}$. Hence

$$\begin{aligned} \int_0^x F(t)Q_{-1}(t)\,dt &= c_1x^{1-\gamma-\alpha}\cdot\exp(2\pi i\eta x^\alpha)-c_1\int_0^{x^\alpha} v'(t)\cdot\exp(2\pi i\eta t)\,dt \\ &= \sum_{\nu=1}^{k} c_\nu x^{1-\gamma-\nu\alpha}\cdot\exp(2\pi i\eta x^\alpha)-c_k\int_0^{x^\alpha} v^{(k)}(t)\cdot\exp(2\pi i\eta t)\,dt. \end{aligned}$$

Since $v^{(k)}(t)=O((1+t)^{(1-\gamma)/\alpha-k-1})$, for large t, we see that

$$\int_0^\infty v^{(k)}(t)\cdot\exp(2\pi i\eta t)\,dt$$

converges. Therefore

$$\begin{aligned} \int_0^{x^\alpha} v^{(k)}(t)\cdot\exp(2\pi i\eta t)\,dt &= \left(\int_0^\infty-\int_{x^\alpha}^\infty\right)v^{(k)}(t)\cdot\exp(2\pi i\eta t)\,dt \\ &= C+O(x^{1-\gamma-k\alpha}). \end{aligned}$$

Hence

$$\int_0^x F(t)\cdot Q_{-1}(t)\,dt = C+O(x^{1-\gamma-\alpha}). \tag{2.19}$$

Now (2.19), (2.18), and (2.17) lead to the estimate

$$\int_0^\infty F(t)\cdot Q_{-1}(t)\,dt = C + O(x^{1-\gamma-\alpha}). \tag{2.20}$$

This, together with (2.16), (2.15), (2.12), (2.10), (2.11), and (2.11)′, lead to the estimate

$$\begin{aligned}\sum_{\lambda_k\le x} a_k F(\lambda_k) &= \sum_{\lambda_k\le x_1} a_k F(\lambda_k) + O\Big\{\sum_{x<\lambda_k\le x_1} a_k\lambda_k^{-\gamma}\Big\}\\ &= \sum_{\lambda_k\le c_0 x^{n\alpha-1+\varepsilon_0}} a_k\int_0^\infty F(t)\cdot I_{-1}(\lambda_k t)\,dt + O(x^{1-\alpha_1-\gamma}+x^{\varepsilon}),\end{aligned} \tag{2.21}$$

where $\alpha_1=\alpha$ if $\alpha\le 2/(n+1)$, and $\alpha_1=\alpha-\varepsilon<\alpha$, if $\alpha>2/(n+1)$. Now let

$$c_0=\left(\frac{2\pi\eta n\alpha}{h}\right)^n, \tag{2.22}$$

where h is defined as in (2.4). Then the last sum in (2.21) equals

$$\begin{aligned}&\sum_{\lambda_k\le c_0 x^{n\alpha-1+\varepsilon_0}} a_k\int_0^\infty F(t)\cdot I_{-1}(\lambda_k t)\,dt\\ &= \sum_{\lambda_k\le c_0 x^{n\alpha-1+\varepsilon_0}} a_k\int_0^\infty u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt\\ &= \sum_{\lambda_k\le c_0 x^{n\alpha-1}} a_k\int_0^\infty u_1(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt\\ &\quad+\sum_{\lambda_k\le c_0 x^{n\alpha-1}} a_k\int_0^\infty \{u(t)-u_1(t)\}t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt\\ &\quad+\sum_{c_0x^{n\alpha-1}<\lambda_k\le c_0 x^{n\alpha-1+\varepsilon_0}} a_k\int_0^{x_1} u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt.\end{aligned} \tag{2.23}$$

Now define

$$H(x,\lambda_k)=\begin{cases}\displaystyle\int_x^\infty t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt, & \text{if}\quad \lambda_k\le c_0x^{n\alpha-1},\\[2ex] \displaystyle\int_0^x u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt, & \text{if}\quad \lambda_k> c_0x^{n\alpha-1};\end{cases} \tag{2.24}$$

and

$$H_1(x, \lambda_k) = \begin{cases} \displaystyle\int_x^{\infty} \{u_1(t) - u(t)\} \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^{\alpha}) \cdot I_{-1}(\lambda_k t)\, dt, & \text{if } \lambda_k \leqslant c_0 x^{n\alpha-1} \\ \displaystyle\int_0^{x_1} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^{\alpha}) \cdot I_{-1}(\lambda_k t)\, dt, & \text{if } \lambda_k > c_0 x^{n\alpha-1}; \end{cases} \tag{2.25}$$

provided that the integrals converge (see Lemma 3). If we combine (2.25), (2.24), (2.23), and (2.21), we get the following

LEMMA 1. *We have*

$$\sum_{\lambda_k \leqslant x} a_k F(\lambda_k) = \sum_{\lambda_k \leqslant c_0 x^{n\alpha-1}} a_k \int_0^{\infty} u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^{\alpha}) \cdot I_{-1}(\lambda_k t)\, dt$$

$$- \sum_{\lambda_k \leqslant c_0 x^{n\alpha-1}} a_k H_1(x, \lambda_k) + \sum_{c_0 x^{n\alpha-1} < \lambda_k \leqslant c_0 x^{n\alpha-1+\varepsilon_0}} a_k H_1(x, \lambda_k)$$

$$+ O\{x^{1-\alpha_1-\gamma} + x^{\varepsilon}\},$$

where F is defined as in (2.8), *u_1 as in* (2.7), *H_1 as in* (2.25), *$\alpha > 1/n$, $\frac{1}{2} + (1/2n) - \alpha < \gamma < 1$, $\gamma \geqslant 0$, in which case the integral $\int_0^{\infty}$ converges* (*as proved in Lemma* 3). *Here $\alpha_1 = \alpha$ if $\alpha \leqslant 2/(n+1)$, and $\alpha_1 = \alpha - \varepsilon < \alpha$ if $\alpha > 2/(n+1)$.*

LEMMA 2. *We have, for $x > 0$,*

$$H_1(x, \lambda_k) \ll \sup_{x \leqslant t \leqslant x_1} |H(t, \lambda_k)|,$$

where, as before, $x_1 = x + x^{1-\alpha}$, $\alpha > 1/n$.

Proof. If $\lambda_k \leq c_0 x^{n\alpha-1}$, then

$$H_1(x, \lambda_k) = \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt + \int_x^{x_1} u(t) \cdot \left\{\frac{d}{dt} H(t, \lambda_k)\right\} \cdot dt$$

$$= H(x, \lambda_k) + [H(t, \lambda_k) \cdot u(t)]_{t=x}^{t=x_1} - \int_x^{x_1} u'(t) \cdot H(t, \lambda_k)\, dt$$

$$= H(x, \lambda_k) - H(x, \lambda_k) - \int_x^{x_1} u'(t) \cdot H(t, \lambda_k)\, dt$$

$$= -\int_x^{x_1} u'(t) \cdot H(t, \lambda_k)\, dt \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|.$$

If $\lambda_k > c_0 x^{n\alpha-1}$, then

$$H_1(x, \lambda_k) = \int_0^{x_1} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt = \int_0^x + \int_x^{x_1}.$$

Since

$$\int_x^{x+x^{1-\alpha}} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|,$$

as before, the result follows.

LEMMA 3. *If $\eta > 0$, then the integral*

$$\int_0^\infty t^{-\gamma} \cdot \exp(i\eta t^\alpha) \cdot I_{-1}(t)\, dt$$

converges for all α, such that $\frac{1}{2} + (1/2n) - \alpha < \gamma < 1$ and $\alpha > 1/n$.

Proof. We have, from (2.4),

$$I_{-1}(t) = \sum_{\nu=0}^{m} e_\nu t^{\omega_{-1} - (\nu/n)} \cos(ht^{1/n} + \pi_\nu) + O(t^{\omega_{-1} - (m+1)/n}),$$

for $t > 0$, where $\omega_{-1} = (1/2n) - \frac{1}{2}$, so that $\omega_{-1} - (\nu/n) < 0$, for $\nu \geq 0$. This leads us to consider integrals of the form

$$\int_1^\infty t^{-\gamma-(1/2)+(1/2n)-(\nu/n)} \cdot \exp\{i(\eta t^\alpha \mp h t^{1/n})\}\, dt$$

$$= \alpha^{-1} \int_1^\infty t^{-(\gamma'/\alpha)+(1/\alpha)-1} \cdot \exp\{i(\eta t \mp h t^{1/n\alpha})\}\, dt,$$

with $-\gamma' = -\gamma - (\frac{1}{2}) + (1/2n) - (\nu/n)$. If $w(t) = (\eta t \mp h t^{1/n\alpha})$, then $|dw/dt| \geqslant \frac{1}{2}\eta$, for $t \geqslant \{2h/(n\alpha\eta)\}^{n\alpha/(n\alpha-1)}$. Hence the above integral converges if $-\gamma' + (1/\alpha) - 1 < 0$, that is if $1 - \gamma' < \alpha$, or $(\frac{1}{2}) + (1/2n) - \alpha < \gamma$. (The reasoning is the same as in Lemma 2 of [4]).

The integral arising from the error-term in I_{-1} is

$$\int_1^\infty t^{-\gamma-\omega_{-1}-(m+1)/n}\, dt,$$

which converges, if m is chosen sufficiently large.

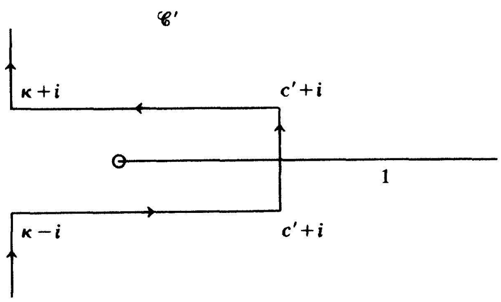
Next let us consider

$$\int_0^1 t^{-\gamma} \cdot \exp(i\eta t^\alpha) \cdot I_{-1}(t)\, dt,$$

where

$$I_{-1}(t) = \frac{1}{2\pi i} \int_{\mathscr{C}'} \frac{\Delta(s)}{\Delta(1-s)} \cdot t^{-s}\, ds,$$

with $0 < \operatorname{Re} s \leqslant c' < 1$ on $\mathscr{C}'$.



If $0 < t < 1$, then

$$|t^{-s}| = t^{-\operatorname{Re} s} = (1/t)^{\operatorname{Re} s} \leqslant t^{-c'},$$

so that $I_{-1}(t)=O(t^{-c'})$, for all c' such that $0<c'<1$. Hence

$$\int_0^1 t^{-\gamma}\cdot\exp(i\eta t^{\alpha})\cdot I_{-1}(t)\,dt$$

converges *absolutely*, provided that $\gamma<1$.

§3. Some asymptotic expansions

LEMMA 4. *We have, for $\alpha>1/n$, and $0\leq\gamma<1$, the asymptotic expansion*

$$\int_0^\infty t^{-\gamma}\cdot\exp(2\pi i\eta t^{\alpha})\cdot I_{-1}(\lambda_k t)\,dt$$

$$=(i)^{(1-\gamma-\alpha)/\alpha}\cdot(2\pi\eta)^{(\gamma-1)/\alpha}\cdot\alpha^{-1}\cdot\pi\sum_{\nu=0}^{m} m_k^{\theta-\nu/(n\alpha-1)}$$

$$\times\{C_\nu\cos(qm_k^{1/(n\alpha-1)}+k_\nu\pi)-iD_\nu\sin(qm_k^{1/(n\alpha-1)}+k_\nu\pi)\}$$

$$+\sum_{0\leq p\leq\{\alpha(n+2m)-2(1-\gamma)\}/\{2(n\alpha-1)\}}\frac{\Delta(1-\gamma+\alpha p)}{\Delta(\gamma-\alpha p)}\cdot\frac{(2\pi i\eta)^p}{p!}\cdot\frac{1}{\lambda_k^{1-\gamma+\alpha p}}$$

$$+O(m_k^{\theta-(m+1)/(n\alpha-1)}),$$

where

$$\eta>0,\qquad m_k=\lambda_k^{\alpha}/(2\pi\eta),\qquad \theta=\frac{2(1-\gamma)-\alpha n}{2\alpha(n\alpha-1)},$$

$$q=(\alpha n-1)\cdot(2^{r_1}\alpha^{-n})^{\alpha/(n\alpha-1)},$$

$$k_\nu=\omega'+\tfrac{1}{2}\nu,\qquad \omega'=\tfrac{1}{4}r_1+\{(1-\gamma)/2\alpha\}-1,\qquad p\text{ integral},$$

$$C_0=D_0=\pi^{-1}(\alpha n-1)^{-1/2}\cdot\alpha^{1+\{n(2\gamma-1)\}/\{2(n\alpha-1)\}}\cdot 2^{\{r_1(1-2\gamma)\}/\{2(n\alpha-1)\}}.$$

The first term in the expansion is given by

$$c_0^{\gamma/(n\alpha-1)}\cdot\eta^{-n/\{2(n\alpha-1)\}}\cdot\lambda_k^{\{-\gamma/(n\alpha-1)\}+\{2-\alpha n\}/\{2(n\alpha-1)\}}\cdot e^{i\pi[(1/2)-(r_1/4)]}$$

$$\times(2\pi)^{-n/\{2(n\alpha-1)\}}\cdot(\alpha n-1)^{-1/2}\cdot\alpha^{-n/\{2(n\alpha-1)\}}\cdot 2^{r_1/\{2(n\alpha-1)\}}\cdot e^{-iqm_k^{1/(n\alpha-1)}},$$

where $c_0=(2\pi\eta n\alpha/h)^n$, $h=n\cdot 2^{r_1/n}$.

Proof. Let

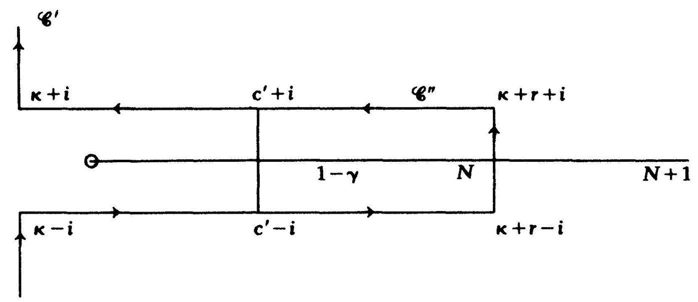
$$J=\int_0^\infty t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt=\lambda_k^{\gamma-1}\int_0^\infty t^{-\gamma}\cdot\exp(it^\alpha m_k^{-1})\cdot I_{-1}(t)\,dt.$$

As in Lemma 3 of [4], we have, for $\xi>0$,

$$\int_0^\infty t^{-\gamma}\cdot\exp(i\xi t^\alpha)\cdot I_{-1}(t)\,dt=\frac{\alpha^{-1}}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)}{\Delta(1-s)}\cdot\Gamma\left(\frac{1-s-\gamma}{\alpha}\right)\cdot\left(\frac{\xi}{i}\right)^{(s+\gamma-1)/\alpha}ds.$$

The path of integration $\mathscr{C}'$ is as shown in the diagram, with $\operatorname{Re}s\leqslant 1-\gamma-\varepsilon<1-\gamma$, and κ is sufficiently large and negative. Putting $\xi=(m_k)^{-1}$, we obtain

$$J=\alpha^{-1}\cdot\lambda_k^{\gamma-1}\cdot\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)}{\Delta(1-s)}\cdot\Gamma\left(\frac{1-s-\gamma}{\alpha}\right)\cdot\left(\frac{1}{m_k i}\right)^{(s+\gamma-1)/\alpha}ds.$$



Now deform the path of integration $\mathscr{C}'$ into $\mathscr{C}''$, by choosing N to be a sufficiently large integer, and $N<\kappa+r<N+1$, as indicated in the diagram. We then have

$$J=\alpha^{-1}\cdot\lambda_k^{\gamma-1}\cdot\frac{1}{2\pi i}\int_{\mathscr{C}''}\frac{\Delta(s)}{\Delta(1-s)}\cdot\Gamma\left(\frac{1-s-\gamma}{\alpha}\right)\cdot\left(\frac{1}{m_k i}\right)^{(s+\gamma-1)/\alpha}ds$$

$$+\sum_{l=0}^{N}\frac{\Delta(1-\gamma+\alpha l)}{\Delta(\gamma-\alpha l)}\cdot\frac{(2\pi i\eta)^l}{l!}\cdot\frac{1}{\lambda_k^{1-\gamma+l\alpha}}\tag{3.1}$$

[We may note here that the residual term in Lemma 4 of [4] should carry the sign + instead of −].

We seek an expansion for

$$J' = \alpha^{-1} \cdot \lambda_k^{\gamma-1} \cdot \frac{1}{2\pi i} \int_{\mathscr{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \cdot \left(\frac{1}{m_k i}\right)^{(s+\gamma-1)/\alpha} ds$$

$$= \left(\frac{1}{2\pi\eta}\right)^{(1-\gamma)/\alpha} \cdot \alpha^{-1} \cdot \frac{1}{2\pi i} \int_{\mathscr{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) m_k^{-s/\alpha} \cdot (-i)^{s/\alpha} \, ds$$

$$= \left(\frac{i}{2\pi\eta}\right)^{(1-\gamma)/\alpha} \cdot \alpha^{-1} \cdot \frac{1}{2\pi i} \int_{\mathscr{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \times m_k^{-s/\alpha}\left(\cos\frac{\pi s}{2\alpha} - i \sin\frac{\pi s}{2\alpha}\right) ds.$$

Now

$$\cos\left(\frac{\pi s}{2\alpha}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2\alpha}+\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}-\frac{s}{2\alpha}\right)}, \qquad \sin\left(\frac{\pi s}{2\alpha}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2\alpha}\right) \cdot \Gamma\left(1-\frac{s}{2\alpha}\right)}.$$

Therefore

$$J' = \left(\frac{i}{2\pi\eta}\right)^{(1-\gamma)/\alpha} \cdot \alpha^{-1} \cdot \pi \cdot \frac{1}{2\pi i} \int_{\mathscr{C}''} \{V_0(s) - iV_1(s)\} m_k^{-s/\alpha} \, ds, \tag{3.2}$$

where

$$V_0(s) = \frac{\Delta(s)}{\Delta(1-s)} \cdot \frac{\Gamma\left(\frac{1-\gamma-s}{\alpha}\right)}{\Gamma\left(\frac{s}{2\alpha}+\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}-\frac{s}{2\alpha}\right)},$$

$$V_1(s) = \frac{\Delta(s)}{\Delta(1-s)} \cdot \frac{\Gamma\left(\frac{1-\gamma-s}{\alpha}\right)}{\Gamma\left(\frac{s}{2\alpha}\right) \cdot \Gamma\left(1-\frac{s}{2\alpha}\right)}.$$

Now choose

$$U_0(s) = \frac{b^{-s}\Gamma(S)}{\Gamma\ (-\frac{1}{2}S-\omega') \cdot \Gamma(1+\frac{1}{2}S+\omega')},$$

$$U_1(s) = \frac{b^{-s}\Gamma(S)}{\Gamma(\frac{1}{2}-\frac{1}{2}S-\omega') \cdot \Gamma(\frac{1}{2}+\frac{1}{2}S+\omega')},$$

where

$$S=\left(n-\frac{1}{\alpha}\right)s-\frac{n}{2}+\left(\frac{1-\gamma}{\alpha}\right),\qquad \omega'=\frac{r_1}{4}+\left(\frac{1-\gamma}{2\alpha}\right)-1,$$

$$b=\left(n-\frac{1}{\alpha}\right)^{(n-1/\alpha)}\cdot 2^{r_1}\cdot\alpha^{-1/\alpha}.$$

Then we have

$$U_0(s)=-\frac{b^{-s}}{\pi}\,\Gamma(S)\cdot\sin\pi\{\tfrac{1}{2}S+\omega'\},\qquad U_1(s)=\frac{b^{-s}}{\pi}\cdot\Gamma(S)\cdot\cos\pi\{\tfrac{1}{2}S+\omega'\}.$$

By choosing

$$a=\left(n-\frac{1}{\alpha}\right)^{\{(n+1)/2-(1-\gamma)/2\}}\cdot 2^{r_1/2}\cdot\alpha^{1/2-(1-\gamma)/\alpha},$$

and comparing the expansions of V_0, U_0, on the one hand, and of V_1, U_1 on the other, we get, as before,

$$V_1(s)=aU_1(s)\left\{1+\sum_{\nu=1}^{m}\frac{e_\nu}{s^\nu}+O(|s|^{-m-1})\right\},$$

$$V_0(s)=aU_0(s)\left\{1+\sum_{\nu=1}^{m}\frac{e_\nu}{s^\nu}+O(|s|^{-m-1})\right\}.$$

Now if we follow the same procedure as in the asymptotic expansion of I_ρ [2], we get

$$\frac{1}{2\pi i}\int_{\mathscr{C}''}V_1(s)x^{-s}\,ds=\sum_{\nu=0}^{m}C_\nu x^{\alpha\theta-[\alpha\nu/(n\alpha-1)]}\cos\{qx^{\alpha/(n\alpha-1)}+k_\nu\pi\}$$
$$+O(x^{\alpha\theta-\{[\alpha(m+1)/(n\alpha-1)]\}}),\tag{3.3}$$

while

$$\frac{1}{2\pi i}\int_{\mathscr{C}''}V_0(s)x^{-s}\,ds=-\sum_{\nu=0}^{m}D_\nu x^{\alpha\theta-[\alpha\nu/(n\alpha-1)]}\sin\{qx^{\alpha/(n\alpha-1)}+k_\nu\pi\}$$
$$+O(x^{\alpha\theta-[\alpha(m+1)/(n\alpha-1)]}),\tag{3.4}$$

where $q=b^{\alpha/(n\alpha-1)}$, and $k_\nu=\omega'+\frac{1}{2}\nu$, provided that

$$N\geqslant\frac{m+\frac{1}{2}n-(1-\gamma)/\alpha}{n-1/\alpha}=\frac{2m\alpha+n\alpha-2(1-\gamma)}{2(n\alpha-1)}.$$

We find by calculation that

$$C_0=D_0=\frac{a}{\pi(n-1/\alpha)}\cdot q^{-\{(n/2)-[(1-\gamma)/\alpha]\}}.$$

From (3.1) and (3.2) we have

$$J=(i)^{(1-\gamma-\alpha)/\alpha}\cdot(2\pi\eta)^{(\gamma-1)/\alpha}\cdot\alpha^{-1}\cdot\pi\cdot\frac{1}{2\pi i}\int_{\mathscr{C}''}\{V_1(s)+iV_0(s)\}m_k^{-s/\alpha}\,ds$$
$$+\sum_{l=0}^{N}\frac{\Delta(1-\gamma+\alpha l)}{\Delta(\gamma-\alpha l)}\cdot\frac{(2\pi i\eta)^l}{l!}\cdot\frac{1}{\lambda_k^{1-\gamma+l\alpha}}.$$

Now (3.3) and (3.4) lead to the lemma.

LEMMA 5. *If* $\xi>0$, $\alpha>0$, $0\leqslant\gamma<1$, *and*

$$J'(\xi)=\int_0^\xi t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt,$$

then $J'(\xi)$ *has the asymptotic expansion*

$$J'(\xi)=\sum_{p=0}^{l}\frac{a_p(\xi)\cdot I_p(\lambda_k\xi)}{(\lambda_k\xi)^{p+1}}+\sum_{p=0}^{l}\frac{b_p}{\lambda_k^{1-\gamma+\alpha p}}+O(\lambda_k^{-l/n}),$$

uniformly for $0<\xi_0\leqslant\xi\leqslant\xi_1<\infty$, *where the coefficients* $a_p(\xi)$ *are continuous in* ξ, *and*

$$b_p=\frac{(2\pi i\eta)^p}{p!}\cdot\frac{\Delta(1-\gamma+\alpha p)}{\Delta(\gamma-\alpha p)}.$$

Proof. For $x>0$, we have

$$I_{-1}(x)=\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)x^{-s}}{\Delta(1-s)}\,ds,$$

where $\mathscr{C}'$ is the same as in Lemma 4, so that $\operatorname{Re} s \leq 1-\gamma-\varepsilon < 1-\gamma$ on $\mathscr{C}'$. Therefore

$$J'(\xi)=\int_0^\xi t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot dt\cdot\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)}{\Delta(1-s)}\cdot(\lambda_k t)^{-s}\,ds,$$

and the repeated integral remains finite, if we replace the integrands by their absolute values. Hence

$$J'(\xi)=\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Delta(s)}{\Delta(1-s)}\cdot\lambda_k^{-s}\cdot j(s,\xi)\,ds, \tag{3.5}$$

where

$$j(s,\xi)=\int_0^\xi t^{-\gamma-s}\cdot\exp(2\pi i\eta t^\alpha)\cdot dt.$$

By a change of variable, we have

$$j(s,\xi)=\alpha^{-1}\int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}-1}\cdot\exp(2\pi i\eta t)\,dt.$$

An integration by parts gives

$$j(s,\xi)=\frac{\xi^{1-\gamma-s}}{(1-\gamma-s)}\cdot\exp(2\pi i\eta\xi^\alpha)-\frac{2\pi i\eta}{(1-\gamma-s)}\int_0^{\xi^\alpha} t^{(1-\gamma-s)/\alpha}\cdot\exp(2\pi i\eta t)\,dt,$$

since $\operatorname{Re} s<1-\gamma$, and $\alpha>0$. Repeating this process, we obtain

$$\begin{aligned} j(s,\xi)=\xi^{-s}\Big\{&\frac{a_0'(\xi)}{(1-\gamma-s)}+\frac{a_1'(\xi)}{(1-\gamma-s)(1-\gamma+\alpha-s)}+\cdots\\ &+\frac{a_l'(\xi)}{(1-\gamma-s)\cdots(1-\gamma+l\alpha-s)}\Big\}\\ &+\frac{c(\alpha)}{(1-\gamma-s)(1-\gamma+\alpha-s)\cdots(1-\gamma+l\alpha-s)}\int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}+l}\\ &\times\exp(2\pi i\eta t)\,dt. \end{aligned} \tag{3.6}$$

We may write

$$\frac{1}{(1-\gamma-s)(1-\gamma+\alpha-s)\cdots(1-\gamma+\nu\alpha-s)} = \sum_{p=0}^{l} \frac{c_p}{(1-s)(2-s)\cdots(p+1-s)} + \varphi_{\nu,l},$$

where $\varphi_{\nu,l} = O(|s|^{-l-1})$, in any vertical strip, and has simple poles at the points $s=1, 2, \ldots, l+1$, as well as $1-\gamma$, $1-\gamma+\alpha$, $1-\gamma+2\alpha, \ldots, 1-\gamma+\alpha\nu$, and the '$O$' depends on ξ. Further the integral

$$\int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}+l} \exp(2\pi i\eta t)\, dt$$

converges absolutely, and is holomorphic for $\operatorname{Re} s < \alpha(l+1)+1-\gamma$, and bounded in the half-plane

$$\operatorname{Re}\left\{\frac{(1-\gamma-s)}{\alpha}+l\right\} \geqslant -1+\varepsilon > -1.$$

Hence

$$j(s, \xi) = \xi^{-s} \sum_{p=0}^{l} \frac{a_p(\xi)}{(1-s)(2-s)\cdots(1+p-s)} + \Phi_l(s, \xi), \tag{3.7}$$

where

$$\Phi_l(s, \xi) = \xi^{-s} \sum_{p=0}^{l} d_p(\xi)\varphi_{p,l}(s) + \frac{c(\alpha, \gamma)}{(1-\gamma-s)\cdots(1-\gamma+l\alpha-s)} \times \int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}+l} \exp(2\pi i\eta t)\, dt$$

and $\Phi_l(s, \xi)$ is meromorphic for $\operatorname{Re} s < \alpha(l+1)+1-\gamma$, with simple poles at $s = 1, 2, \ldots, l+1$, as well as $1-\gamma$, $1-\gamma+\alpha, \ldots, 1-\gamma+l\alpha$. Further $\Phi_l(s, \xi) = O(|s|^{-l-1})$, in any closed vertical strip contained in *that* half-plane, uniformly in ξ for ξ in any compact set.

Now $j(s, \xi)$ can also be written as

$$j(s, \xi) = \int_0^{\xi} t^{\gamma-s} \sum_{\nu=0}^{l} \frac{(2\pi i\eta)^\nu}{\nu!} \cdot t^{\alpha\nu}\, dt + \int_0^{\xi} t^{-\gamma-s} \tilde{f}_l(t)\, dt,$$

say, where $\tilde{f}_l(t)=O(t^{\alpha(l+1)})$, so that the second integral is holomorphic in s for $\operatorname{Re}(-\gamma-s)>-\alpha(l+1)-1$, or for $\operatorname{Re} s<1-\gamma+\alpha(l+1)$, for *any* integer $l>0$. It follows that $j(s,\xi)$ is a meromorphic function, whose only poles are at $s=1-\gamma+\alpha\nu$, $\nu=0,1,2,\ldots$, with the corresponding residues $-(2\pi i\eta)^\nu/(\nu!)$.

Now if $\mathscr{C}'''$ is the path obtained by deforming $\mathscr{C}'$ so as to have the corners at $\kappa-i$, $c'+N-i$, $c'+N+i$, $\kappa+i$ (the infinite half-lines being left as they are), with N sufficiently large, then we have, from (3.5),

$$J'(\xi)=\frac{1}{2\pi i}\int_{\mathscr{C}'''}\frac{\Delta(s)}{\Delta(1-s)}\cdot\lambda_k^{-s}\cdot j(s,\xi)\,ds$$
$$+\sum_{p=0}^{l}\frac{(2\pi i\eta)^p}{p!}\cdot\frac{\Delta(1-\gamma+\alpha p)}{\Delta(\gamma-\alpha p)}\cdot\frac{1}{\lambda_k^{1-\gamma+\alpha p}}. \tag{3.8}$$

(This requires that $c'+N<1-\gamma+\alpha(l+1)$, and $c'+N>1-\gamma+\alpha l$)

The integral here can be considered as a sum of two integrals J'' and J''', because of the expression for $j(s,\xi)$ given in (3.7), where

$$J''(\xi)=\sum_{p=0}^{l}\frac{a_p(\xi)}{2\pi i}\int_{\mathscr{C}'''}\frac{\Delta(s)}{\Delta(1-s)}\cdot\frac{\Gamma(1-s)}{\Gamma(1+p+1-s)}\cdot(\lambda_k\xi)^{-s}\,ds$$
$$=\sum_{p=0}^{l}\frac{a_p(\xi)\cdot I_p(\lambda_k\xi)}{(\lambda_k\xi)^{p+1}}, \tag{3.9}$$

provided that $c'+N>l+1$, while

$$J'''(\xi)=\frac{1}{2\pi i}\int_{\mathscr{C}'''}\frac{\Delta(s)}{\Delta(1-s)}\cdot\Phi_l(s,\xi)\cdot\lambda_k^{-s}\,ds.$$

If $\mathscr{C}_0'''$ is the path obtained by deforming $\mathscr{C}'''$, so that the two infinite half-lines in $\mathscr{C}'''$ are moved to the right, then

$$J'''(\xi)=\frac{1}{2\pi i}\int_{\mathscr{C}_0'''}\frac{\Delta(s)}{\Delta(1-s)}\cdot\Phi_l(s,\xi)\cdot\lambda_k^{-s}\,ds,$$

and this integral converges absolutely for $n(\sigma-\frac{1}{2})-l-1<-1$, or $\sigma<(l/n)+\frac{1}{2}$, where $\sigma=\operatorname{Re} s$. Note that $\Phi_l(s)$ has *no* poles off the real axis. If we take $\sigma=l/n+\frac{1}{2}-\varepsilon$, with $0<\varepsilon<\frac{1}{2}$, then

$$J'''(\xi)=O(\lambda_k^{-(l/n)-(1/2)+\varepsilon})=O(\lambda_k^{-l/n}). \tag{3.10}$$

Now (3.8)–(3.10) give the required result.

LEMMA 6. *Let a be a real number,* $a \neq 0$, $\alpha > 1/n$, $\mu = (h/2\pi\eta) \cdot \lambda_k^{1/n}$, $\eta > 0$, *and h defined as in* (2.4).

Let $\varphi(t) \equiv \varphi(t, \mu) = t^\alpha - \mu t^{1/n}$, $F_0(t) \equiv F_0(t, \mu) = t^a/\varphi'(t)$, $F_{l+1}(t) \equiv F_{l+1}(t, \mu) = 1/\varphi'(t) \cdot (d/dt)F_l(t)$, *for* $l = 0, 1, 2, \ldots$. *Then*

$$F_l(t) = \frac{t^{a-l}}{(\varphi'(t))^{l+1}} \cdot \sum_{p,q=0;\, q \leq p}^{l} c_{p,q,l} \frac{\left(\dfrac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^q}{\left(1 - \dfrac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^p},$$

where the $c_{p,q,l}$ *are suitable constants.*

Analogously, let $\psi(t) \equiv \psi(t, \mu) = t^\alpha + \mu t^{1/n}$, $G_0(t) \equiv G_0(t, \mu) = t^a/\psi'(t)$, and $G_{l+1}(t) = G_{l+1}(t, \mu) = 1/\psi'(t) \cdot (d/dt)G_l(t)$, *for* $l = 0, 1, 2, \ldots$. *Then*

$$G_l(t) = \frac{t^{a-l}}{(\psi'(t))^{l+1}} \cdot \sum_{p,q=0;\, q \leq p}^{l} d_{p,q,l} \frac{\left(\dfrac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^q}{\left(1 + \dfrac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^p}.$$

The proof follows by induction on l.

LEMMA 7. *For a fixed* ξ, *such that* $0 < \xi_0 \leq \xi \leq \xi_1 < \infty$, *we have as* $\mu \to \infty$, *the following asymptotic expansion in decreasing powers of* μ:

$$F_m(\xi, \mu) = \frac{1}{\mu^{m+1}} \left(\sum_{l=0}^{L} \frac{d_{l,m}}{\mu^l} + o(\mu^{-L}) \right).$$

For the proof we have only to use the expression for $F_m(\xi, \mu)$ given by Lemma 6 together with the Binomial Theorem.

In what follows we shall frequently use the notation $F_{l,\nu}(t) = F_l(t)$, and $G_{l,\nu}(t) = G_l(t)$, *with* $a = a(\nu) - \gamma = \omega_{-1} - (\nu/n) - \gamma = (1/2n) - (\frac{1}{2}) - (\nu/n) - \gamma$, for $l = 0, 1, 2, \ldots$, $\nu = 0, 1, 2, \ldots$, and $0 \leq \gamma < 1$.

LEMMA 8. *Let* $a(\nu) = \omega_{-1} - \nu/n = (1/2n) - (\frac{1}{2}) - (\nu/n)$, *for* $\nu = 0, 1, 2, \ldots$, $0 \leq \gamma < 1$. *Let* $\delta > 0$, *sufficiently small, and* $0 < \eta_0 < \eta$, $\alpha > 1/n$. *Let* $\lambda_k \leq (c_0 - \delta)x^{n\alpha - 1}$.

Then we have

$$\int_x^{\infty} t^{-\gamma} \cdot \exp(2\pi i \eta t^{\alpha}) \cdot I_{-1}(\lambda_k t)\, dt$$

$$= \sum_{\nu=0}^{n(m+1)} \lambda_k^{a(\nu)} \sum_{0 \leq l \leq \{(m+1)/\alpha\}+1}$$

$$\times \{\exp(2\pi i \eta \varphi(x)) \cdot b_{l,\nu} F_{l,\nu} + \exp(2\pi i \eta \psi(x)) \cdot b'_{l,\nu} G_{l,\nu}(x)\} + O(x^{\omega_{-1} - \nu - m}).$$

Proof. The asymptotic expansion of $I_{-1}(\lambda_k t)$ leads us to consider integrals of the form

$$\int_x^{\infty} t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t))\, dt, \qquad \varphi(t) = t^{\alpha} - \mu t^{1/n}.$$

As in Lemma 6 of [4], we prove that

$$F_{l,\nu}(t) = O(t^{a(\nu)-\gamma+1-(l+1)\alpha}),$$

for $t \geq x$, $\lambda_k \leq (c_0 - \delta) x^{n\alpha - 1}$, where the '$O$' depends on $a(\nu)$, l, and δ, but *not* on t, x, or μ. Repeated integration by parts then gives

$$\int_x^{\infty} t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t))\, dt = e^{2\pi i \eta \varphi(x)} \sum_{l=0}^{L} \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{a(\nu)-\gamma+1-(L+1)\alpha}),$$

and analogously

$$\int_x^{\infty} t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \psi(t))\, dt = e^{2\pi i \eta \psi(x)} \sum_{l=0}^{L} \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}}$$
$$+ O(x^{a(\nu)-\gamma+1-(L+1)\alpha}).$$

These expansions together with the asymptotic expansion of $I_{-1}(\lambda_k t)$ therefore yield

$$\int_x^\infty t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$=\sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\left(b_\nu\int_x^\infty t^{a(\nu)-\gamma}e^{2\pi i\eta\varphi(t)}\,dt+b'_\nu\int_x^\infty t^{a(\nu)-\gamma}e^{2\pi i\eta\psi(t)}\,dt\right)$$

$$+O_m(\lambda_k^{-1}(\lambda_k x)^{a(m+1)-\gamma+1})$$

$$=\sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\left\{b_\nu e^{2\pi i\eta\varphi(x)}\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}+b'_\nu e^{2\pi i\eta\psi(x)}\sum_{l=0}^{L}\frac{(-1)^{l+1}G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}\right.$$

$$\left.+O(x^{a(\nu)-\gamma-(L+1)\alpha})\right\}+O_m(x^{a(m+1)-\gamma+1}).$$

If $L=[(m+1)/n\alpha]$, then $L+1>(m+1)/n\alpha$, and we have

$$\int_x^\infty t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$=\sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\left\{b_\nu e^{2\pi i\eta\varphi(x)}\sum_{0\le l\le(m+1)/n\alpha}\frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}\right.$$

$$\left.+b'_\nu e^{2\pi i\eta\psi(x)}\sum_{0\le l\le(m+1)/n\alpha}\frac{(-1)^{l+1}G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}\right\}+O_m(x^{a(m+1)-\gamma+1})$$

The lemma follows upon replacing m by $n(m+1)$.

LEMMA 9. *Let* $a(\nu)=\omega_{-1}-\nu/n=(1/2n)-\frac{1}{2}-(\nu/n)$, $\nu=0,1,2,\ldots$, $0\le\gamma<1$, $\delta>0$, δ *sufficiently small,* $0<2\varepsilon<\alpha$, $\alpha>1/n$. *Then we have for* $(c_0-\delta)x^{n\alpha-1}<\lambda_k\le c_0x^{n\alpha-1}-x^{n\alpha-1-\varepsilon}$,

$$\int_x^\infty t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$=\sum_{\nu=0}^{n(m+1)}\lambda_k^{a(\nu)}\left(b_\nu e^{2\pi i\eta\varphi(x)}\sum_{0\le l\le(m+1)/(n\alpha-2n\varepsilon)}\frac{(-1)^{l+1}F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\right.$$

$$\left.+b'_\nu e^{2\pi i\eta\psi(x)}\sum_{0\le l\le(m+1)/(n\alpha-2n\varepsilon)}\frac{(-1)^{l+1}G_{l,\nu}(x,\mu)}{(2\pi i_\eta)^{l+1}}\right)+O(x^{\omega_{-1}-\gamma-m}).$$

Proof. The pattern of proof is similar to that of Lemmas 11 and 12 in [4]. We first prove that

$$\int_x^\infty t^{a(\nu)-\gamma}\cdot\exp(2\pi i\eta\varphi(t))\,dt = e^{2\pi i\eta\varphi(x)}\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} + O_L(x^{a(\nu)-\gamma-\alpha L+(1-\alpha)+(L+1)2\varepsilon}).$$

An analogous expansion is valid with ψ in place of φ, and $G_{l,\nu}$ in place of $F_{l,\nu}$. As in Lemma 8, we then have

$$\int_x^\infty t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\left(b_\nu e^{2\pi i\eta\varphi(x,\mu)}\sum_{l=0}^{L}\frac{(-1)^{l+1}F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} + b'_\nu e^{2\pi i\eta\psi(x,\mu)}\sum_{l=0}^{L}\frac{(-1)^{l+1}G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\right) + O_m(x^{a(m+1)-\gamma+1}),$$

provided that $L=[(m+1)/(n\alpha-2n\varepsilon)]$, $0<2\varepsilon<\alpha$. The lemma follows upon replacing m by $n(m+1)$.

LEMMA 10. *If $\delta>0$, and sufficiently small, and $0<\eta_0\leq\eta$, and $\alpha>1/n$, then for $\lambda_k\geq(c_0+\delta)x^{n\alpha-1}$, we have the asymptotic expansion*

$$\int_0^x u(t)\cdot t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\left(b_\nu e^{2\pi i\eta\varphi(x)}\sum_{l=0}^{L}\frac{(-1)^{l}F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} + b'_\nu e^{2\pi i\eta\psi(x)}\sum_{l=0}^{L}\frac{(-1)^{l}G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\right) + O(\mu^{-L-1}) + O(\lambda_k^{-m/n}).$$

If $L=m$, the term $O(\mu^{-L-1})$ can be dropped.

Proof. Choose ξ such that $\lambda_1<\xi<x$, and consider the given integral as the

sum of $\int_0^\xi$ and $\int_\xi^x$. We then have

$$\int_\xi^x u(t)\cdot t^{a(\nu)-\gamma}\cdot e^{2\pi i\eta\varphi(t)}\,dt$$

$$=e^{2\pi i\eta\varphi(x)}\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}-e^{2\pi i\eta\varphi(\xi)}\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}+O(\mu^{-L-1}),$$

where the 'O' does *not* depend on ξ. An analogous expansion is valid with ψ in place of φ, and $G_{l,\nu}$ in place of $F_{l,\nu}$ (see the proof of Lemma 10 in [4]). It follows that

$$\int_\xi^x u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$=\sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\Bigg(b''_\nu \exp(2\pi i\eta\varphi(x,\mu))\cdot\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$

$$-b''_\nu \exp(2\pi i\eta\varphi(\xi,\mu))\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}\Bigg)$$

$$+\sum_{\nu=0}^{m}\lambda_k^{a(\nu)}\Bigg(b'''_\nu \exp(2\pi i\eta\psi(x,\mu))\cdot\sum_{l=0}^{L}\frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}$$

$$-b'''_\nu \exp(2\pi i\eta\varphi(\xi,\mu)\sum_{l=0}^{L}\frac{(-1)^l G_{l,\nu}(\xi,\mu)}{(2\pi i\eta)^{l+1}}\Bigg)+O(\mu^{-L-1})+O(\lambda_k^{a(m+1)-\gamma}). \quad (3.11)$$

On the other hand, if $F(t)=u(t)\cdot t^{-\gamma}\cdot\exp(2\pi i\eta t^\alpha)$, as in the proof of Lemma 1, we have

$$\int_0^\xi F(t)I_{-1}(\lambda_k t)\,dt=\sum_{p=0}^{L}\frac{(-1)^p F^{(p)}(\xi)\cdot I_p(\lambda_k\xi)}{\lambda_k^{1+p}}$$

$$+(-1)^{L+1}\cdot\lambda_k^{-L-1}\int_0^\xi F^{(L+1)}(t)\cdot I_L(\lambda_k t)\,dt, \quad (3.12)$$

and the last term is $O(\lambda_k^{-L/n})$. Since I_p has the asymptotic expansion (2.4), we can combine (3.12) and (3.11), and apply Lemma 9 of [4] to obtain the stated result.

LEMMA 11. *If* $c_0x^{n\alpha-1}+x^{n\alpha-1-\varepsilon}\leqslant\lambda_k\leqslant(c_0+\delta)x^{n\alpha-1}$, $\delta>0$, *and sufficiently* *small, and* $0<2\varepsilon<\alpha$, $\alpha>1/n$, *then*

$$\int_0^x u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{m} \lambda_k^{a(\nu)}\left(b_\nu e^{2\pi i\eta\varphi(x)} \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x)}{(2\pi i\eta)^{l+1}} + b'_\nu e^{2\pi i\eta\psi(x)} \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x)}{(2\pi i\eta)^{l+1}}\right)$$
$$+ O(x^{a(m+1)-\gamma+1}) + O(\lambda_k^{-m/n}),$$

provided that $L=[(m+1)/(n\alpha-2n\varepsilon)]$. *As before* $a(\nu)=\omega_{-1}-(\nu/n)$.

Proof. Choose λ such that $0<\lambda<1$, and consider the given integral as the sum of the integrals $\int_0^{x(1-\lambda)}$ and $\int_{x(1-\lambda)}^x$. Following the same pattern of proof as in Lemma 13 of [4], we obtain

$$\int_{x(1-\lambda)}^x u(t)\cdot t^{a(\nu)-\gamma}\cdot \exp(2\pi i\eta\varphi(t))\,dt = \int_{x(1-\lambda)}^x t^{a(\nu)-\gamma}\cdot \exp(2\pi i\eta\varphi(t))\,dt$$

$$= e^{2\pi i\eta\varphi(x)} \sum_{l=0}^{L} \frac{(-1)^{l+1} F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} - e^{2\pi i\eta\varphi(x(1-\lambda))} \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x(1-\lambda),\mu)}{(2\pi i\eta)^{l+1}}$$
$$+ O(x^{a(\nu)-\gamma-\alpha L+(1-\alpha)+(L+1)2\varepsilon}),$$

as well as the analogue with ψ in place of φ, and $G_{l,\nu}$ in place of $F_{l,\nu}$. It follows that

$$\int_{x(1-\lambda)}^x u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{m} \lambda_k^{a(\nu)}\left(b_\nu e^{2\pi i\eta\varphi(x)} \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}} + b'_\nu e^{2\pi i\eta\psi(x)} \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i\eta)^{l+1}}\right)$$
$$- \sum_{\nu=0}^{m} \lambda_k^{a(\nu)}\left(b_\nu e^{2\pi i\eta\varphi(x(1-\lambda))} \sum_{l=0}^{L} \frac{(-1)^l F_{l,\nu}(x(1-\lambda),\mu)}{(2\pi i\eta)^{l+1}}\right.$$
$$\left.+ b'_\nu e^{2\pi i\eta\psi(x(1-\lambda))} \sum_{l=0}^{L} \frac{(-1)^l G_{l,\nu}(x(1-\lambda),\mu)}{(2\pi i\eta)^{l+1}}\right) + O(x^{a(m+1)-\gamma}), \tag{3.13}$$

provided that $L=[(m+1)/(n\alpha-2n\varepsilon)]$.

On the other hand, by Lemma 10, (since $\lambda_k > (c_0+\delta_1)\{x(1-\lambda)\}^{n\alpha-1}$), we have

$$\int_0^{x(1-\lambda)} u(t)\cdot t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{m} \lambda_k^{a(\nu)}\Bigg(b_\nu e^{2\pi i\eta\varphi(x(1-\lambda))}\sum_{l=0}^{L}\frac{(-1)^l F_{l,\nu}(x(1-\lambda),\mu)}{(2\pi i\eta)^{l+1}}$$

$$+ b'_\nu e^{2\pi i\eta\psi(x(1-\lambda),\mu)}\sum_{l=0}^{L}\frac{(-1)^l G_{l,\nu}(x(1-\lambda),\mu)}{(2\pi i\eta)^{l+1}}\Bigg) + O(\lambda_k^{a(m+1)-\gamma}) + O(\lambda_k^{-m/n}). \tag{3.14}$$

Combining (3.14) with (3.13), and applying Lemma 9 of [4], we obtain the stated result.

LEMMA 12. *If* $c_0x^{n\alpha-1} - x^{n\alpha-1-\varepsilon} < \lambda_k \leq c_0 x^{n\alpha-1}$, $\varepsilon > 0$, $\alpha > 1/n$, *and, as hitherto,* $a(\nu) = \omega_{-1} - \nu/n$, *then*

$$\int_x^{\infty} t^{-\gamma}\cdot \exp(2\pi i\eta t^\alpha)\cdot I_{-1}(\lambda_k t)\,dt$$

$$= \sum_{\nu=0}^{nN}\sum_{m=0}^{N-1} u_{\nu,m}\cdot x^{a(\nu)-\gamma+1}\cdot \lambda_k^{a(\nu)}\cdot \frac{e^{2\pi i\eta\varphi(x)}}{(2\pi x^\alpha\eta)^{m+1}}$$

$$+ \sum_{\nu=0}^{nN}\sum_{m=0}^{N-1} u'_{\nu,m}\cdot x^{a(\nu)-\gamma+1}\cdot \lambda_k^{a(\nu)}\cdot \frac{e^{2\pi i\eta x^\alpha P(v_0)}}{(2\pi x^\alpha\eta)^{m+(1/2)}}$$

$$+ \sum_{\nu=0}^{nN}\sum_{m=0}^{N-1} u''_{\nu,m} x^{a(\nu)-\gamma+1}\cdot \lambda_k^{a(\nu)}\cdot \frac{e^{2\pi i\eta x^\alpha P(v_0)}}{(2\pi x^\alpha\eta)^{m}}\cdot \int_0^{\tau} s^{-1/2}\cdot e^{2\pi i x^\alpha\eta s}\,ds$$

$$+ \sum_{\nu=0}^{nN}\sum_{m=0}^{N-1}\frac{(-1)^{m+1}G_{m,\nu}(x)\cdot e^{2\pi i\eta\psi(x)}}{(2\pi i\eta)^{m+1}}\cdot \lambda_k^{a(\nu)}\cdot + O(x^{a(\nu)-\gamma+1-\alpha(N+1)}),$$

where $P(v) = v^{n\alpha} - \beta v$, $\beta = \mu x^{(1/n)-\alpha}$, $\tau = P(1) - v_0$, $v_0 = (\beta/n\alpha)^{1/(n\alpha-1)}$, *and where the coefficients* $u_{\nu,m}$, $u'_{\nu,m}$, $u''_{\nu,m}$ *are continuous functions of* β *for fixed* ν *and* m, *while*

$$\int_0^{\tau} s^{-1/2}\exp(2\pi i\eta x^\alpha s)\,ds = O(x^{-\alpha/2}),$$

uniformly in τ.

Proof. The pattern of proof here is the same as in Lemmas 16, 17 and 18 of

[4], with the difference that $P(v)$ is no longer a polynomial, but holomorphic in v for $\operatorname{Re} v > 0$, and v_0 is a zero of $P'(v)$.

If $0 < \rho < 1$, then $\mu/\{x(1+\rho)\}^{\alpha-(1/n)} \leq n\alpha/(1+\rho)^{\alpha-(1/n)} \leq n\alpha - \delta_1$, say, so that, as in the proof of Lemma 8, we get

$$\int_{x(1+\rho)}^{\infty} t^{a(\nu)-\gamma} \exp(2\pi i\eta\varphi(t))\,dt = e^{2\pi i\eta\varphi(x(1+\rho))} \sum_{l=0}^{L} \frac{(-1)^{l+1} F_{l,\nu}(x(1+\rho))}{(2\pi i\eta)^{l+1}} + O(x^{a(\nu)-\gamma+1-(L+1)\alpha}), \tag{3.15}$$

and a similar result holds with ψ in place of φ, and $G_{l,\nu}$ in place of $F_{l,\nu}$ for the entire integral $\int_x^\infty$.

The integral $\int_x^{x(1+\rho)}$ is then dealt with as in Lemma 17 of [4], and gives

$$\begin{aligned}\int_x^{x(1+\rho)} & t^{a(\nu)-\gamma} \exp(2\pi i\eta\varphi(t))\,dt \\ &= \tfrac{1}{2} n x^{a(\nu)-\gamma+1} \Bigg\{ \sum_{p=0}^{N} \frac{A_p(1+\rho)\exp(2\pi i\eta\varphi(x(1+\rho))) - B_p(1)\exp(2\pi i\eta\varphi(x))}{(2\pi x^\alpha \eta)^{p+1}} \\ &\quad + \sum_{m=0}^{N} \frac{\alpha_m' \exp(2\pi_c x^\alpha \eta P(\nu_0))}{(2\pi x^\alpha \eta)^{m+(1/2)}} - \sum_{m=0}^{N} \frac{\alpha_m''}{(2\pi x^\alpha \eta)^m} \int_0^T s^{-1/2} e^{2\pi i\eta x^\alpha s}\,ds \Bigg\} \\ &\quad + O(x^{a(\nu)-\gamma+1-\alpha(N+1)}),\end{aligned} \tag{3.16}$$

where A_p, B_p, α_m', α_m'' depend on β, and, for fixed p, m, are continuous functions of β in a neighbourhood of the point $\beta = n\alpha$.

We now combine (3.15) and (3.16), and apply Lemma 9 of [4] to get rid of the ρ from the expansions.

LEMMA 13. *If* $c_0 x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1} + x^{n\alpha-1-\varepsilon}$, $\varepsilon > 0$, *then*

$$\int_0^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i\eta t^\alpha) \cdot I_{-1}(\lambda_k t)\,dt$$

has the same expansion as in Lemma 12.

Proof. We consider the integrals $\int_0^{x(1-\rho)}$ and $\int_{x(1-\rho)}^x$ separately, where $0 < \rho < 1$.

The first integral is dealt with as in Lemma 10, while the second is handled as in Lemma 12, and Lemma 9 of [4] is then called into play.

LEMMA 14. *For $\alpha > 1/n$, and $0 \leq \gamma < 1$, the integral*

$$\int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

has the asymptotic expansion

$$c_1(\alpha, \gamma, \eta, K) \sum_{\nu=0}^{m} m_k^{\theta-[\nu/(n\alpha-1)]}\{C_\nu \cos(q m_k^{1/(n\alpha-1)} + k_\nu \pi)$$

$$- iD_\nu \sin(q m_k^{1/(n\alpha-1)} + k_\nu \pi\} + O(m_k^{\theta-(m+1)/(n\alpha-1)}),$$

where c_1 is a constant determined by Lemma 4, *and* $\theta = [2(1-\gamma) - \alpha n]/[2\alpha(n\alpha - 1)]$.

Proof. We have

$$\int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

$$= \int_0^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

$$+ \int_0^{\lambda_1} (u_1(t) - 1) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

The first integral on the right has an asymptotic expansion given by Lemma 4.

If we choose a ρ, such that $\rho < c$ (defined along with the function u at the beginning), then

$$\int_0^{\lambda_1} (u_1(t) - 1) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

$$= -\int_0^{\rho} t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

$$+ \int_\rho^{\lambda_1} (u_1(t) - 1) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

Lemma 5 gives an asymptotic expansion of the first integral on the right-hand side:

$$\sum_{p=0}^{l} \frac{a_p(\rho)\cdot I_p(\lambda_k\rho)}{(\lambda_k\rho)^{p+1}}+\sum_{p=0}^{l}\frac{b_p}{\lambda_k^{1-\gamma+\alpha p}}+O(\lambda_k^{-l/n}),$$

where $b_p=(2\pi i\eta)^p/p!\cdot\Delta(1-\gamma+\alpha p)/\Delta(\gamma-\alpha p)$. For the second integral, we apply repeated integration by parts, together with the fact that u_1-1 (and all its derivatives) vanish at $t=\lambda_1$, to get an asymptotic expansion as in (3.12). By proper choice of l, the term $\sum_{p=0}^{l} b_p\cdot\lambda_k^{-1+\gamma-\alpha p}$ cancels out with the residual term given by Lemma 4, and the application of Lemma 9 of [4] leads to the stated result.

§4. Proofs of the theorems

Proof of Theorem 1. From Lemma 1 we have

$$\begin{aligned}\sum_{\lambda_k\le x} a_kF(\lambda_k)&=\sum_{\lambda_k\le x} a_k\cdot\lambda_k^{-\gamma}\cdot u(\lambda_k)\cdot\exp(2\pi i\eta\lambda_k^{\alpha})\\&=\sum_{\lambda_k\le c_0x^{n\alpha-1}} a_k\int_0^\infty u_1(t)\cdot t^{-\gamma}\cdot\exp(2\pi i\eta t^{\alpha})\cdot I_{-1}(\lambda_kt)\,dt\\&\quad-\sum_{\lambda_k\le c_0x^{n\alpha-1}} a_kH_1(x,\lambda_k)+\sum_{c_0x^{n\alpha-1}<\lambda_k\le c_0x^{n\alpha-1+\varepsilon_0}} a_kH_1(x,\lambda_k)\\&\quad+O\{x^{1-\alpha_1-\gamma}+x^{\varepsilon}\},\end{aligned}\tag{4.1}$$

for $\alpha>1/n$, $\frac{1}{2}+(1/2n)-\alpha<\gamma<1$, $\gamma\ge0$, $c_0=(2\pi\eta n\alpha/h)^n$, $h=n\cdot2^{r_1/n}$. We shall estimate the first three terms on the right-hand side of (4.1). By Lemma 14, we have

$$\begin{aligned}&\int_0^\infty u_1(t)\cdot t^{-\gamma}\cdot\exp(2\pi i\eta t^{\alpha})\cdot I_{-1}(\lambda_kt)\,dt\\&=c_1(\alpha,\gamma,\eta,K)\cdot\lambda_k^{\alpha\theta}\exp\{-iqm_k^{1/(n\alpha-1)}\}+O(m_k^{\theta-1/(n\alpha-1)}),\end{aligned}$$

where $\theta=\{2(1-\gamma)-\alpha n\}/\{2\alpha(n\alpha-1)\}$. If $\gamma<\frac{1}{2}n\alpha$, then $\alpha\theta>-1$, and we obtain

$$\begin{aligned}&\sum_{\lambda_k\le c_0x^{n\alpha-1}} a_k\int_0^\infty u_1(t)\cdot t^{-\gamma}\cdot\exp(2\pi i\eta t^{\alpha})\cdot I_{-1}(\lambda_kt)\,dt\\&=c_1(\alpha,\gamma,\eta,K)\sum_{\lambda_k\le c_0x^{n\alpha-1}} a_k\cdot\lambda_k^{\alpha\theta}\cdot\exp\{-iqm_k^{1/(n\alpha-1)}\}+O(x^{[(n-1)/2]\alpha-\gamma}).\end{aligned}\tag{4.2}$$

For a $\delta > 0$, chosen sufficiently small, we then write

$$\sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k H_1(x, \lambda_k)$$

$$= \sum_{\lambda_k \leq (c_0-\delta)x^{n\alpha-1}} + \sum_{(c_0-\delta)x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon}} + \sum_{c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon} < \lambda_k \leq c_0 x^{n\alpha-1}}$$

$$= V_1 + V_2 + V_3, \quad \text{say.}$$

In V_1 we use Lemma 8, and note that $\lambda_k \leq (c_0 - \delta)t^{n\alpha-1}$ for $0 < x \leq t \leq x_1$, so that

$$H(t, \lambda_k) \ll \lambda_k^{\omega-1} \cdot |F_{0,0}(t, \lambda_k)| \ll \lambda_k^{\omega-1} \cdot t^{\omega_{-1}-\gamma+1-\alpha},$$

hence

$$V_1 = \sum_{\lambda_k \leq (c_0-\delta)x^{n\alpha-1}} a_k H_1(x, \lambda_k) \ll x^{\omega_{-1}-\gamma+1-\alpha} \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \lambda_k^{\omega_{-1}}$$

$$\ll x^{\omega_{-1}-\gamma+1-\alpha} \cdot x^{(n\alpha-1)(\omega_{-1}+1)}$$

$$\ll x^{[(n-1)/2]\alpha-\gamma}. \tag{4.3}$$

(cf. the estimate of $W_{5,1}$ in [4]).

In V_2 we observe that $(c_0 - 2\delta)t^{n\alpha-1} < \lambda_k \leq c_0 t^{n\alpha-1} - t^{n\alpha-1-\varepsilon}$, and use Lemma 8, so that

$$H(t, \lambda_k) \ll \frac{\lambda_k^{\omega_{-1}} \cdot t^{\omega_{-1}-\gamma}}{|\varphi'(t)|}, \quad \text{for} \quad 0 < x \leq t \leq x_1.$$

We have $\varphi'(t) = \alpha t^{\alpha-1}(1 - c_0^{-1/n}\lambda_k^{1/n} t^{(1/n)-\alpha})$, and in the given range

$$|1 - c_0^{-1/n}\lambda_k^{1/n} x^{(1/n)-\alpha}| \gg x^{-\varepsilon}.$$

However,

$$1 - c_0^{-1/n}\lambda_k^{1/n} t^{(1/n)-\alpha} = 1 - (c_0^{-1/n}\lambda_k^{1/n}) x^{(1/n)-\alpha} + O(x^{-\alpha}),$$

so that

$$\frac{1 - c_0^{-1/n} \cdot \lambda_k^{1/n} t^{(1/n)-\alpha}}{1 - c_0^{-1/n} \cdot \lambda_k^{1/n} x^{(1/n)-\alpha}} = 1 + O(x^{\varepsilon-\alpha}), \qquad (\varepsilon < \tfrac{1}{2}\alpha).$$

Hence

$$H_1(x, \lambda_k) \ll \frac{\lambda_k^{\omega_{-1}} \cdot x^{\omega_{-1}-\gamma+1-\alpha}}{|1-c_0^{-1/n}\lambda_k^{1/n} x^{(1/n)-\alpha}|},$$

so that

$$V_2 \ll x^{\omega_{-1}-\gamma+1-\alpha} \sum_{(c_0-\delta)x^{n\alpha-1}<\lambda_k\leqslant c_0x^{n\alpha-1}-x^{n\alpha-1-\varepsilon}} \frac{a_k \cdot \lambda_k^{\omega_{-1}}}{|1-c_0^{-1/n}\lambda_k^{1/n} \cdot x^{(1/n)-\alpha}|}.$$

The last sum is estimated in the same way as $W_{5,2}$ in [4], so as to yield

$$V_2 \ll x^{[(n-1)/2]\alpha-\gamma} \cdot \log(1+x). \tag{4.4}$$

In V_3 we observe that $c_0t^{n\alpha-1}-2t^{n\alpha-1-\varepsilon}<\lambda_k\leqslant c_0t^{n\alpha-1}$, for $0<x\leqslant t\leqslant x_1$, so that Lemma 12 applies, and we get

$$H_1(x, \lambda_k) \ll \lambda_k^{\omega_{-1}} \cdot x^{\omega_{-1}-\gamma+1-(\alpha/2)},$$

so that

$$\begin{aligned} V_3 &\ll x^{\omega_{-1}+1-\gamma-\alpha/2} \sum_{c_0x^{n\alpha-1}-x^{n\alpha-1-\varepsilon}<\lambda_k\leqslant c_0x^{n\alpha-1}} a_k\lambda_k^{\omega_{-1}}, \qquad (\omega_{-1}<0) \\ &\ll x^{\omega_{-1}+1-\gamma-\alpha/2} \cdot x^{\omega_{-1}(n\alpha-1)} \sum a_k, \qquad a_k = O(k^{\varepsilon'}), \quad \text{say.} \\ &\ll x^{\varepsilon'+(n\alpha-1-\varepsilon)+\omega_{-1}+1-\gamma-(\alpha/2)+\omega_{-1}(n\alpha-1)} \\ &\ll x^{(\omega_{-1}+1)n\alpha-\gamma-(\alpha/2)-\varepsilon+\varepsilon'} \\ &\ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1}, \end{aligned} \tag{4.5}$$

where $\varepsilon_1 = \varepsilon - \varepsilon'$, for any positive ε, such that $\varepsilon < \frac{1}{2}\alpha$, $\varepsilon < n\alpha - 1$, and all $\varepsilon' > 0$, so that (4.5) holds for *any* $\varepsilon_1 < \alpha/2$, $\varepsilon_1 < n\alpha - 1$. From (4.3), (4.4), and (4.5) we obtain

$$\sum_{\lambda_k\leqslant c_0x^{n\alpha-1}} a_kH_1(x, \lambda_k) \ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1} + x^{[(n-1)/2]\alpha-\gamma} \log(1+x). \tag{4.6}$$

Next we consider

$$\sum_{c_0x^{n\alpha-1}<\lambda_k\leqslant c_0x^{n\alpha-1-\varepsilon_0}} a_kH_1(x,\lambda_k)$$

$$= \sum_{c_0x^{n\alpha-1}<\lambda_k\leqslant c_0x^{n\alpha-1}+x^{n\alpha-1-\varepsilon}} + \sum_{c_0x^{n\alpha-1}+x^{n\alpha-1-\varepsilon}<\lambda_k\leqslant(c_0+\delta)x^{n\alpha-1}}$$

$$+ \sum_{(c_0+\delta)x^{n\alpha-1}<\lambda_k\leqslant x^{n\alpha-1+\varepsilon_0}}$$

$$= V_4+V_5+V_6, \quad \text{say.}$$

In V_6 we note that $t^{n\alpha-1+\varepsilon_0}\geqslant\lambda_k\geqslant(c_0+\delta_1)t^{n\alpha-1}$, for $1<x\leqslant t\leqslant x_1$, with $\delta_1=\frac{1}{2}\delta$, say, so that $H(t,\lambda_k)$ can be estimated with the help of Lemma 10, leading to an estimate of $H_1(x,\lambda_k)$, and thence

$$V_6\ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1}, \qquad \varepsilon_1<\tfrac{1}{2}\alpha, \quad \varepsilon_1<n\alpha-1. \tag{4.7}$$

In V_5 we note that $c_0t^{n\alpha-1}=c_0x^{n\alpha-1}+O(x^{n\alpha-1-\alpha})$, for $1<x\leqslant t\leqslant x_1$, so that $c_0t^{n\alpha-1}+\frac{1}{2}t^{n\alpha-1-\varepsilon}<\lambda_k\leqslant(c_0+\delta)t^{n\alpha-1}$, and Lemma 11 can be used. We further note that, as in the case of V_2, we can replace $\varphi'(t)$ by $\varphi'(x)$, and obtain in the same way

$$V_5\ll x^{[(n-1)/2]\alpha-\gamma}\log(1+x). \tag{4.8}$$

In V_4 we consider the integral for $H(t)$ as the sum $\int_0^{t(1-\rho)}+\int_{t(1-\rho)}^t$, for a sufficiently small ρ, such that $0<\rho<1$. In the first integral, we have, for $x\leq t\leq x_1$,

$$c_0t^{n\alpha-1}(1-\rho)^{n\alpha-1}\leqslant c_0(x_1(1-\rho))^{n\alpha-1}=c_0(x^{n\alpha-1}(1+O(x^{-\alpha}))\cdot(1-\rho)^{n\alpha-1}$$
$$\ll c_0x^{n\alpha-1}<\lambda_k,$$

so that Lemma 10 can be applied to yield

$$H(t,\lambda_k)\ll\lambda_k^{\omega_{-1}}\cdot t^{\omega_{-1}+1-\gamma-\alpha},$$

while in the second integral, we have

$$\frac{\lambda_k}{c_0t^{n\alpha-1}}\geqslant\frac{\lambda_k}{c_0x^{n\alpha-1}(1+O(x^{-\alpha}))}, \qquad \frac{\lambda_k}{c_0t^{n\alpha-1}}\leqslant\frac{\lambda_k}{c_0x^{n\alpha-1}}\leqslant1+O(x^{-\varepsilon}).$$

Here Lemma 12 can be applied to obtain (as in V_3)

$$H(t, \lambda_k) \ll \lambda_k^{\omega_{-1}} t^{\omega_{-1}+1-\gamma-(\alpha/2)}.$$

Altogether, we therefore have

$$H_1(x, \lambda_k) \ll \lambda_k^{\omega_{-1}} \cdot x^{\omega_{-1}+1-\gamma-(\alpha/2)},$$

and

$$V_4 \ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we obtain

$$\sum_{c_0 x^{n\alpha-1} < \lambda_k < c_0 x^{n\alpha-1+\varepsilon_0}} a_k H_1(x, \lambda_k) \ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1} + x^{[(n-1)/2]\alpha-\gamma} \cdot \log(1+x). \tag{4.10}$$

If we use (4.10), (4.6), and (4.2) in (4.1), we obtain:

$$\sum_{\lambda_k \leq x} a_k \cdot \lambda_k^{-\gamma} \cdot \exp(2\pi i \eta \lambda_k^{\alpha})$$
$$= c_1(\alpha, \gamma, \eta, K) \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{\alpha\theta} \cdot \exp\{-iqm_k^{1/(n\alpha-1)}\} + O(x^{[(n\alpha)/2]-\gamma-\alpha})$$
$$+ O(x^{[(n-1)/2]\alpha-\gamma} \log(1+x)) + O(x^{[(n\alpha)/2]-\gamma-\varepsilon_1}) + O(x^{1-\alpha_1-\gamma}) + O(x^{\varepsilon}). \tag{4.11}$$

Since $\varepsilon_1 < \frac{1}{2}\alpha$, the exponent $\frac{1}{2}(n\alpha) - \gamma - \varepsilon_1 > 0$, *if* $\frac{1}{2}(n\alpha) - \gamma - \frac{1}{2}\alpha > 0$, that is, *if* $\gamma < \frac{1}{2}(n-1)\alpha$, in which case the term x^{ε} can be dropped. Such a choice of γ can indeed be made. If $\alpha > \frac{1}{2} + (1/2n)$, we choose $\gamma = 0$. Otherwise $\frac{1}{2} + (1/2n) - \alpha < \frac{1}{2}(n-1)\alpha$ for $\alpha > 1/n$, so that if γ is greater than $\frac{1}{2} + (1/2n) - \alpha$ and close enough to it, we have $\frac{1}{2}(n-1)\alpha > \gamma$. Clearly the only O-terms that remain in (4.11) are:

$$O(x^{[(n\alpha)/2]\alpha-\gamma} \log(1+x)) + O(x^{[(n\alpha)/2]-\gamma-\varepsilon_1}) + O(x^{1-\alpha_1-\gamma}). \tag{4.12}$$

Thus (4.11) can be rewritten as

$$\sum_{\lambda_k \leq x} a_k \cdot \lambda_k^{-\gamma} \cdot \exp(2\pi i \eta \lambda_k^{\alpha})$$

$$= c_1 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{-[\gamma/(n\alpha-1)]+[(2-\alpha n)/2(n\alpha-1)]} \exp\{-iqm_k^{1/(n\alpha-1)}\}$$

$$+ O(x^{[(n\alpha)/2]-\gamma-\varepsilon_1}) + O(x^{[(n\alpha)/2]\alpha-\gamma} \log(1+x)) + O(x^{1-\alpha_1-\gamma}). \tag{4.13}$$

Since $\gamma < \frac{1}{2}(n\alpha)$, the exponent of λ_k is greater than -1, and we can apply partial summation, and obtain

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^{\alpha})$$

$$= c_2 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{(2-\alpha n)/2(n\alpha-1)} \cdot \exp\{-iqm_k^{1/(n\alpha-1)}\}$$

$$+ O(x^{[(n\alpha)/2]-\varepsilon_1}) + O(x^{[(n-1)/2]\alpha+\varepsilon}) + O(x^{1-\alpha_1}), \tag{4.14}$$

where $c_2 = c_1 \cdot c_0^{-\gamma/(n\alpha-1)}$ is independent of γ, for all $\varepsilon_1 > 0$, such that $\varepsilon_1 < \frac{1}{2}\alpha$, $\varepsilon_1 < n\alpha - 1$. These conditions on ε_1 imply that

$$x^{[(n\alpha)/2]-\varepsilon_1} \ll x^{[(n-1)/2]\alpha+\varepsilon} + x^{1-(n\alpha/2)+\varepsilon}$$

Hence (4.14) leads to

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^{\alpha})$$

$$= c_2 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{(2-\alpha n)/2(n\alpha-1)} \exp\{-iqm_k^{1/(n\alpha-1)}\}$$

$$+ O(x^{[(n-1)/2]\alpha+\varepsilon}) + O(x^{1-(n\alpha/2)+\varepsilon}) + O(x^{1-\alpha_1}). \tag{4.15}$$

If $n \geq 3$, then $\frac{1}{2}(n\alpha) > \alpha \geq \alpha_1$, so that the term $O(x^{1-(n\alpha/2)+\varepsilon})$ may be dropped. If $n = 2$, the only O-terms in (4.15) are $O(x^{(\alpha/2)+\varepsilon}) + O(x^{1-\alpha+\varepsilon})$, and that completes the proof of Theorem 1.

Proof of Theorem 2. If $\alpha = 2/n$, (4.15) gives the approximate reciprocity formula:

$$\sum_{\lambda_k \le x} a_k \exp(2\pi i\eta\lambda_k^{2/n}) = c_3 \sum_{\lambda_k \le c_0 x} a_k \exp\left(\frac{-iq\lambda_k^{2/n}}{2\pi\eta}\right) + O(x^{1-(1/n)+\varepsilon}), \tag{4.16}$$

for every $\varepsilon > 0$.

In this special case, we have

$$c_0 = \left(\frac{4\pi\eta}{h}\right)^n, \qquad h = n2^{r_1/n}, \qquad c_3 = \left(\frac{h}{4\pi\eta}\right)^{n/2} e^{i\pi[(1/2)-(r_1/4)]}, \qquad q = \frac{h^2}{4},$$

while, in general, $\lambda_k = B \cdot k$, $B = 2^{r_2}\pi^{n/2} |D|^{-1/2} = (2\pi n/h)^{n/2} \cdot |D|^{-1/2}$.

If we choose $4\pi\eta = h$, then $c_0 = 1$, $c_3 = e^{i\pi[(1/2)-(r_1/4)]}$. Setting $Y = x \cdot B^{-1}$, we get:

$$\sum_{k \le Y} a_k \cdot e^{i\pi n |D|^{-1/n} k^{2/n}} = e^{i\pi[(1/2)-(r_1/4)]} \sum_{k \le Y} a_k \cdot e^{-i\pi n |D|^{-1/n} k^{2/n}} + O(Y^{1-(1/n)+\varepsilon}),$$

which is (1.4).

Similarly if we choose $4\pi\eta = h\,|D|^{-1/n} \cdot m$, where m is an integer, $m \neq 0$, then we get a formula which, in the case $n = 2$, gives the Corollary to Theorem 1 in [3] with Y^ε in place of $\log Y$.

If we take $n = 2$ in (4.16), so that $\alpha = 1$, and $r_1 = 2$ (this is the case when K is a real quadratic field), then $c_0 = (\pi\eta)^2$, $c_3 = (\pi\eta)^{-1}$, $q = 4$; if $n = 2$, $\alpha = 1$, $r_1 = 0$ (the case of an imaginary quadratic field), then $c_0 = (2\pi\eta)^2$, $c_3 = i(2\pi\eta)^{-1}$, $q = 1$. Formula (4.16) then reduces to the one which we proved sometime ago [3] by a different method, but with $\log x$ taking the place of x^ε in the error-term.

Proof of Theorem 3. The sum on the right-hand side of (4.15) is $O(x^{(n\alpha)/2})$, for $\alpha < 2/n$. We can therefore conclude that *if* $1/n < \alpha < 2/n$, and $n \geq 3$, *then*

$$\sum_{\lambda_k \le x} a_k \exp(2\pi i\eta\lambda_k^{\alpha}) = O(x^{(n\alpha)/2}) + O(x^{1-\alpha_1}), \tag{4.17}$$

with $\alpha_1 = \alpha$ if $\alpha \leq 2/(n+1)$, and $\alpha_1 = \alpha - \varepsilon < \alpha$, if $\alpha > 2/(n+1)$. If $n = 2$, and $\frac{1}{2} < \alpha < 1$, the sum is $O(x^\alpha)$, since $1 - \alpha < \alpha$. If $n \geq 3$, and we take $\alpha = 1/(n-1)$ in

(4.17), we get

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i\eta\lambda_k^{1/(n-1)}) = O(x^{n/2(n-1)}) + O(x^{1-1/(n-1)})$$

$$= \begin{cases} O(x^{3/4}), & \text{if} \quad n=3; \\ O(x^{1-1/(n-1)}), & \text{if} \quad n \geq 4. \end{cases} \tag{4.18}$$

We note that if $n \geq 3$, (4.18) gives a stronger result than the one we obtained before [4].

Proof of Theorems 4 *and* 5. If $0<\alpha<1/n$, the proof is much simpler than if $\alpha>1/n$. Proceeding as in Lemma 1, we consider here the infinite series

$$(-1)^{r+1} \sum_{k=1}^{\infty} a_k \cdot \lambda_k^{-1-r} \int_c^{x_1} F^{(r+1)}(t) \cdot I_r(\lambda_k t)\, dt, \qquad \gamma=0, \qquad F(t)=u(t) \cdot e^{2\pi i \eta t^\alpha}.$$

$$\ll \sum_{k=1}^{\infty} a_k \cdot \lambda_k^{-(1/2)-(1/2n)-(r/n)} = O(1),$$

if r is chosen sufficiently large. On the other hand,

$$\int_0^\infty F^{(r+1)}(t) \cdot Q_r(t)\, dt = O(x^{1-\alpha}), \tag{4.18$'$}$$

as before, while $\sum_{x<\lambda_k \leq x_1} a_k = O(x^{1-\alpha})$. Hence

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i\eta\lambda_k^\alpha) = O(x^{1-\alpha}), \quad \text{for} \quad 0<\alpha<\frac{1}{n}.$$

If $\alpha = 1/n$, and $\varepsilon>0$, we first obtain the estimate

$$(-1)^{r+1} \sum_{\lambda_k > x^\varepsilon} a_k \cdot \lambda_k^{-1-r} \int_c^{x_1} F^{(r+1)}(t) \cdot I_r(\lambda_k t)\, dt$$

$$= O(x^{\varepsilon[(1/2)-(1/2n)-(r/n)]+(1/2)+(1/2n)}) = O(x^{-q}) \tag{4.19}$$

for any $q>0$, provided that r is chosen sufficiently large. We next consider

$$\sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} F(t) \cdot I_{-1}(\lambda_k t)\, dt, \qquad F(t) = u(t) \cdot \exp(2\pi i \eta t^\alpha)$$

$$= \sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} u(t) \cdot dH(t, \lambda_k), \qquad H(t, \lambda_k) = \int_c^t \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t)\, dt$$

$$= - \sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} H(t, \lambda_k) \cdot u'(t)\, dt,$$

since $H(t, \lambda_k)$ vanishes for $t=c$, while $u(t)$ vanishes for $t=x_1$. Now

$$\int_c^{x_1} H(t, \lambda_k) u'(t)\, dt = \left(\int_c^{\lambda_1} + \int_{\lambda_1}^{x_1} \right) H(t, \lambda_k) u'(t)\, dt,$$

and the first integral on the right-hand side is $\ll \lambda_k^{\omega-1} = O(1)$, since $I_{-1}(\lambda_k t) \ll \lambda_k^{\omega-1}$, for $c \leq t \leq \lambda_1$, while

$$\int_x^{x_1} H(t, \lambda_k) u'(t)\, dt \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|.$$

We shall estimate the order of magnitude of $H(t, \lambda_k)$ by using the full asymptotic expansion of $I_{-1}(\lambda_k t)$, (cf. Lemma 3). We then have to consider integrals of the form

$$\int_c^y t^{-(1/2)+(1/2n)-(\nu/n)} \exp\{2\pi i t^{1/n}(\eta - \lambda_k^{1/n} \cdot h)\}\, dt, \qquad \nu = 0, 1, 2, \ldots,$$

together with an O-term of the order $O(\lambda_k^{-(1/2)+(1/2n)-[(\nu+1)/n]})$, if $-(1/2)+(1/2n)-[(\nu+1)/n] < -1$. It is sufficient to consider the case $\nu=0$. If $\eta = \lambda_{k_0}^{1/n} \cdot h$ for some $k_0 \geq 1$, we have

$$H(y, \lambda_{k_0}) = \lambda_{k_0}^{\omega-1} \left\{ c_{k_0} \int_c^y t^{-(1/2)+(1/2n)}\, dt + O\left(\left(\int_c^y t^{-[(1/2)+(1/2n)]}\, dt \right) \right) \right\}$$

$$= c'_{k_0} y^{(n+1)/2n} + O(y^{(n-1)/2n}),$$

while

$$H(y, \lambda_k) \ll \lambda_k^{\omega-1} y^{(n-1)/2n}, \quad \text{for} \quad k \neq k_0,$$

(which follows by replacing $t^{1/n}$ by t and integrating by parts). Hence, if $n \geq 3$,

$$-\sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} H(t, \lambda_k) \cdot u'(t)\, dt \ll x^{(n+1)/2n} + \sum_{\lambda_k \leq x^\varepsilon, k \neq k_0} a_k \cdot \lambda_k^{\omega-1} x^{(n-1)/2n}$$

$$\ll x^{(n+1)/2n} \ll x^{1-(1/n)}. \tag{4.20}$$

If $n=2$, and $\eta = h \cdot \lambda_{k_0}^{1/2}$, we have

$$\int_c^{x_1} H(t, \lambda_{k_0}) u'(t)\, dt = c_{k_0} \lambda_{k_0}^{\omega-1} \int_c^{x_1} t^{3/4} u'(t)\, dt + O(x^{1/4})$$

$$= c_{k_0} \lambda_{k_0}^{\omega-1} \int_x^{x_1} t^{3/4} u'(t)\, dt + O(x^{1/4})$$

$$= -c_{k_0} \lambda_{k_0}^{\omega-1} \left\{ x^{3/4} + \tfrac{3}{4} \int_x^{x_1} u(t) \cdot t^{-1/4}\, dt \right\} + O(x^{1/4})$$

$$= -c_{k_0} \lambda_{k_0}^{\omega-1} x^{3/4} + O(x^{1/4}).$$

Hence, in this case,

$$-\sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} H(t, \lambda_k) \cdot u'(t)\, dt = c_{k_0} \cdot a_{k_0} \cdot \lambda_{k_0}^{\omega-1} \cdot x^{3/4} + O(x^{(1/4)+\varepsilon}), \tag{4.21}$$

while, if $\eta \neq h\lambda_{k}^{1/2}$, for all $k \geq 1$, the sum is $O(x^{1/2})$. The result follows from this and (4.18)′.

Remark. The method of proof adopted here makes it possible to prove corresponding results for the coefficient-sums of Dirichlet series satisfying a functional equation of the type studied in [1, 2]. Particular cases are the zeta-function of an ideal class and Hecke's zeta-function with Grössencharacters. Since no new ideas are required, we do not go into the details.

Our estimates prove that the sequence $\{\eta(N\mathfrak{A})^\alpha\}$, for $0<\alpha<2/n$, η real, $\eta \neq 0$, and or an integral ideal in K with norm $N\mathfrak{A}$, when arranged in order of increasing norms, is uniformly distributed modulo 1. It is likely that this is true for $0<\alpha<1$.

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