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Exponential sums associated with algebraic number fields

by K. CHANDRASEKHARAN and RAGHAVAN NARASIMHAN

§1. Let $\zeta_K(s)$ denote the Dedekind zeta-function of an algebraic number field K of degree n . For $\text{Re } s > 1$, $\zeta_K(s) = \sum_{k=1}^{\infty} a_k k^{-s}$, where a_k stands for the number of integral ideals in K with norm k . If r_1 is the number of real conjugates of K , and $2r_2$ the number of imaginary conjugates, and D the discriminant, $\zeta_K(s)$ satisfies the functional equation [1] $\xi(s) = \xi(1-s)$, where

$$\xi(s) = \Gamma^{r_1}(\frac{1}{2}s) \cdot \Gamma^{r_2}(s) \cdot B^{-s} \zeta_K(s),$$

with $B = 2^{r_2} \pi^{n/2} |D|^{-1/2}$, $r_1 + 2r_2 = n$. It is known that $a_k = O(k^\epsilon)$, for every $\epsilon > 0$, and

$$\sum_{k \leq x} a_k = \lambda x + O(x^{(n-1)/(n+1)}), \tag{1.1}$$

where λ stands for the residue of $\zeta_K(s)$ at $s = 1$.

Our purpose is to prove the following

THEOREM 1. *If η is real, $\eta \neq 0$, $\alpha > 1/n$, $\lambda_k = B \cdot k$ for integral $k \geq 1$, then*

$$\begin{aligned} \sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^\alpha) &= c_2 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{(2-\alpha n)/[2(n\alpha-1)]} \cdot \exp\left\{-iq \left(\frac{\lambda_k^\alpha}{2\pi\eta}\right)^{1/(n\alpha-1)}\right\} \\ &+ O(x^{[(n-1)/2]\alpha+\epsilon}) + O(x^{1-\alpha_1}), \end{aligned} \tag{1.2}$$

for $n \geq 3$, and every $\epsilon > 0$, where $\alpha_1 = \alpha$ if $\alpha \leq 2/(n+1)$, while $\alpha_1 = \alpha - \epsilon < \alpha$, if $\alpha > 2/(n+1)$; $c_2 = c_2(\alpha, \eta, K)$ is a constant that can be explicitly determined, $c_0 = (2\pi\eta\alpha/h)^n$, where $h = n \cdot 2^{r_1/n}$, and $q = (\alpha n - 1)(2^{r_1} \cdot \alpha^{-n})^{\alpha/(n\alpha-1)}$.

If $n = 2$, (1.2) holds with the error-term

$$O(x^{(\alpha/2)+\epsilon}) + O(x^{1-\alpha+\epsilon}).$$

The case $\alpha = 2/n$ of Theorem 1 gives the approximate reciprocity formula, which is stated as

THEOREM 2. *Under the same conditions as in Theorem 1, we have, for $n \geq 2$,*

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^{2/n}) = c'_2 \sum_{\lambda_k \leq c'_0 x} a_k \cdot \exp\left\{\frac{-iq' \lambda_k^{2/n}}{2\pi \eta}\right\} + O(x^{1-(1/n)+\varepsilon}), \quad (1.3)$$

for every $\varepsilon > 0$. Here c'_0 , c'_2 , and q' denote respectively the values of c_0 , c_2 , and q , for $\alpha = 2/n$.

Theorem 2 yields as a special case the approximate reciprocity formula for quadratic fields previously obtained by us by a different method [3], though the error-term here is somewhat less sharp, in that we have x^ε instead of $\log x$.

If we choose $4\pi\eta = h$ in Theorem 2, so that $c'_0 = 1$, we get the

COROLLARY.

$$\sum_{k \leq x} a_k \exp(i\pi n \cdot |D|^{-1/n} k^{2/n}) = e^{i\pi[(1/2)-(r_1/4)]} \sum_{k \leq x} a_k \exp(-i\pi n |D|^{-1/n} k^{2/n}) + O(x^{1-(1/n)+\varepsilon}), \quad (1.4)$$

for $n \geq 2$. This can also be written as

$$\sum_{k \leq x} a_k \sin\left\{\pi n |D|^{-1/n} k^{2/n} + \frac{1}{2}\pi(r_1 - 2)/4\right\} = O(x^{1-(1/n)+\varepsilon}). \quad (1.5)$$

Theorem 1, combined with the known estimate (1.1), also yields the following

THEOREM 3. *If $1/n < \alpha < 2/n$, then*

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^\alpha) = O(x^{n\alpha/2}) + O(x^{1-\alpha}), \quad \text{for } n \geq 2. \quad (1.6)$$

In particular, if $n \geq 3$, and we take $\alpha = 1/(n-1)$, we obtain the result:

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^{1/(n-1)}) = \begin{cases} O(x^{3/4}), & \text{if } n = 3; \\ O(x^{1-1/(n-1)}), & \text{if } n \geq 4. \end{cases} \quad (1.7)$$

This is sharper than the estimate recently obtained by us [4], namely $O(x^{1-1/2(n-1)}) \log(1+x)$, for all $n \geq 3$.

If $1/n < \alpha \leq 2/(n+2)$, Theorem 3 implies that

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^\alpha) = O(x^{1-\alpha}), \quad \text{for } n \geq 3. \quad (1.8)$$

The case $0 < \alpha \leq 1/n$ is covered by the next two theorems.

THEOREM 4. *If $n \geq 3$, and $0 < \alpha \leq 1/n$, then*

$$\sum_{\lambda_k \leq x} \alpha_k \cdot \exp(2\pi i \eta \lambda_k^\alpha) = O(x^{1-\alpha}). \quad (1.9)$$

THEOREM 5. *If $n = 2$, and $0 < \alpha < \frac{1}{2}$, then*

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^\alpha) = O(x^{1-\alpha}). \quad (1.10)$$

If $n = 2$, $\alpha = \frac{1}{2}$, then

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^{1/2}) = O(x^{1/2}), \quad (1.11)$$

provided that $\eta \neq h\lambda_k^{1/2}$ for all k , while

$$\sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \cdot \lambda_k^{1/2}) = c_{k_0} \cdot a_{k_0} \cdot x^{3/4} + O(x^{1/2}), \quad c_{k_0} \neq 0, \quad (1.12)$$

if $\eta = h\lambda_{k_0}^{1/2}$ for some k_0 .

The general method of attack is similar to that of [4], though a number of additional difficulties caused by the introduction of the parameter α have to be overcome. The estimates for the wider class of exponential sums considered here should find their use in the study of the critical zeros of the Dedekind zeta-function of an ideal class in K , in case $n \geq 3$.

§2. The basic lemmas

An indispensable tool in the following analysis is the identity:

$$\frac{1}{\Gamma(\rho+1)} \sum_{\lambda_k \leq x} a_k (x - \lambda_k)^\rho = Q_\rho(x) + \sum_{k=1}^{\infty} a_k \cdot \lambda_k^{-1-\rho} I_\rho(\lambda_k x), \quad (2.1)$$

which holds for $x > 0$, $\rho > \frac{1}{2}(n-1)$, ρ integral, and which is implied by the functional equation of $\zeta_K(s)$ [1]. Here $\lambda_k = B \cdot k$, where B is defined as in §1, and

$$Q_\rho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{B^{-s} \zeta_K(s) \cdot \Gamma(s)}{\Gamma(s+\rho+1)} \cdot x^{s+\rho} ds, \quad (2.2)$$

where \mathcal{C} is a curve which encloses all the singularities of the integrand, and

$$I_\rho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}'_p} \frac{\Gamma(1-s)\Delta(s)}{\Gamma(\rho+2-s)\Delta(1-s)} \cdot x^{1+\rho-s} ds, \quad (2.3)$$

where $\Delta(s) = \Gamma^{r_1}(\frac{1}{2}s)\Gamma^{r_2}(s)$, and \mathcal{C}'_p denotes the path of integration extending from $c_p - i\infty$ to $c_p - iR$, thence to $c_p + r - iR$, $c_p + r + iR$, $c_p + iR$, and $c_p + i\infty$, with r and R chosen suitably large, and with $c_p = \frac{1}{2} + (\rho/n) - \varepsilon$, $0 < \varepsilon < (1/2n)$.

The following asymptotic formula [2] is crucial to the proof of our main theorem:

$$I_\rho(x) = \sum_{\nu=0}^m e'_\nu x^{\omega_\nu - \nu/n} \cos(hx^{1/n} + \pi_\nu) + O(x^{\omega_\nu - (m+1)/n}), \quad (2.4)$$

where $\omega_\rho = \frac{1}{2} - (1/2n) + \rho(1 - 1/n)$, $h = n \cdot 2^{r_1/n}$, $\pi_\nu = \pi_\nu(\rho) = \pi\{(n/2) + (\rho/2) + \frac{1}{4}(r_1 + 3) - (\nu/2)\}$, for all integers ρ , positive or negative. [It may be noted that in (2.3) of [4], the exponent $(m+1/n)$ should be $(m+1)/n$, and in the expression for h one should have $2r_2$ in place of r_2].

We define, as usual, $a_0 = 0 = \lambda_0$, $A(x) = \sum_{\lambda_k \leq x} a_k$, for $x > 0$, $A(x) = 0$ for $0 \leq x < \lambda_1$, and $A^{r-1}(x) = (d/dx)(A^r(x))$, almost everywhere, for $r \geq 1$.

Let

$$x > \lambda_1, \quad x_1 = x + x^{1-\alpha}, \quad \alpha > \frac{1}{n}, \quad (2.5)$$

and let $u(t)$ be an infinitely differentiable function in $(-\infty < t < \infty)$, such that $u(t) = 0$, for $t \leq c < \frac{1}{2}\lambda_1$; $u(t) = 1$, in a neighbourhood of $\lambda_1 \leq t \leq x$; $u(t) = 0$, for

$t \geq x_1$; $0 \leq u(t) \leq 1$, for $-\infty < t < \infty$; and

$$|u^{(k)}(t)| \leq c_k(1+t)^{k(\alpha-1)}, \quad \text{for } t \geq 0, \quad k \geq 0, \tag{2.6}$$

where $u^{(k)}$ denotes the k^{th} derivative of u , and c_k is a constant depending only on k .

Further let

$$u_1(t) = \begin{cases} u(t), & \text{for } t \leq x, \\ 1, & \text{for } t \geq x. \end{cases} \tag{2.7}$$

Let $f(t) = \exp(2\pi i \eta t^\alpha)$, where $\eta > 0$, $t \geq 0$, $\alpha > 1/n$; let $0 \leq \gamma < 1$, and

$$F(t) = t^{-\gamma} \cdot u(t) \cdot f(t). \tag{2.8}$$

If $F^{(r)}$ denotes the r^{th} derivative of F , we have

$$F^{(r+1)}(t) \ll (1+t)^{(r+1)(\alpha-1)-\gamma}, \tag{2.9}$$

since $f^{(k)}(t) \ll (1+t)^{k(\alpha-1)}$, and $u^{(k)}(t) \ll (1+t)^{k(\alpha-1)}$. We have

$$\sum_{\lambda_k \leq x_1} a_k F(\lambda_k) = \sum_{\lambda_k \leq x} a_k F(\lambda_k) + O\left\{ \sum_{x < \lambda_k \leq x_1} a_k \lambda_k^{-\gamma} \right\}. \tag{2.10}$$

Since $a_k = O(k^\epsilon)$, for every $\epsilon > 0$, we have

$$\sum_{x < \lambda_k \leq x_1} a_k \lambda_k^{-\gamma} \leq x^{-\gamma} \sum_{x < \lambda_k \leq x_1} a_k = \begin{cases} x^{-\gamma} \cdot O(x^{1-\alpha+\epsilon}), & \text{if } \alpha \leq 1; \\ x^{-\gamma} \cdot O(x^\epsilon), & \text{if } \alpha > 1. \end{cases} \tag{2.11}$$

Because of (1.1) we have also

$$\sum_{x < \lambda_k \leq x_1} a_k \lambda_k^{-\gamma} \ll x^{-\gamma} (x^{1-\alpha} + x^{1-2/(n+1)}). \tag{2.11}'$$

We shall express the first sum on the right-hand side of (2.10) as an integral, and estimate it in different ranges of λ_k . Clearly

$$\sum_{k=0}^{\infty} a_k F(\lambda_k) = \int_0^{\infty} F(t) dA(t) = (-1)^{r+1} \int_0^{\infty} A^r(t) \cdot F^{(r+1)}(t) dt. \tag{2.12}$$

[It may be noted that in (3.3) of [4] we should have $(-1)^{r+1}$ in place of $(-1)^r$ with the consequent changes in sign.] We choose the integer r so large that the infinite series in (2.1) converges absolutely, and uniformly, for $x \geq c$ and $\rho \geq r > 0$, so that we may substitute for $A^r(t)$ the corresponding series in (2.1) plus $Q_r(t)$. We then seek to estimate, for a suitably chosen y ,

$$\sum_{\lambda_k > y} a_k \cdot \lambda_k^{-1-r} \int_0^\infty F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt,$$

which equals

$$\begin{aligned} & \sum_{\lambda_k > y} a_k \cdot \lambda_k^{-1-r} \int_c^{x_1} F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt \\ & \ll \sum_{\lambda_k > y} a_k \cdot \lambda_k^{-1-r} \cdot \lambda_k^{(1/2)-(1/2n)+r(1-1/n)} \int_c^{x_1} (1+t)^{(r+1)(\alpha-1)+(1/2)-(1/2n)+r(1-1/n)-\gamma} dt \\ & \ll \sum_{\lambda_k > y} a_k \cdot \lambda_k^{-(1/2)-(1/2n)-(r/n)} \cdot x^{r(\alpha-1/n)+\alpha+(1/2)-(1/2n)-\gamma}. \end{aligned} \quad (2.13)$$

If we choose

$$y = c_0 \cdot x^\delta, \quad \delta = n\alpha - 1 + \varepsilon_0 > 0, \quad \varepsilon_0 > 0, \quad c_0 > 0, \quad (2.14)$$

then (2.13) is

$$\begin{aligned} & \ll x^{\delta[(1/2)-(1/2n)-(r/n)]+r(\alpha-1/n)+\alpha+(1/2)-(1/2n)-\gamma} \\ & \ll x^{r[\alpha(1/n)-(\delta/n)]+(\delta+1)[(1/2)-(1/2n)]+\alpha-\gamma} \\ & \ll x^{-q}, \end{aligned} \quad (2.15)$$

for any given $q > 0$, provided that r is large enough.

Next we have the section

$$(-1)^{r+1} \sum_{\lambda_k \leq y} a_k \cdot \lambda_k^{-1-r} \int_0^\infty F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt = \sum_{\lambda_k \leq y} a_k \int_0^\infty F(t) \cdot I_{-1}(\lambda_k t) dt \quad (2.16)$$

as well as the term

$$(-1)^{r+1} \int_0^\infty F^{(r+1)}(t) \cdot Q_r(t) dt = \int_0^\infty F(t) \cdot Q_{-1}(t) dt = \left\{ \int_0^x + \int_x^{x_1} \right\} F(t) \cdot Q_{-1}(t) dt. \quad (2.17)$$

Since $Q_{-1}(t)$ is a constant (see [4, p. 82]), we have

$$\int_x^{x_1} F(t)Q_{-1}(t) dt \ll x^{1-\gamma-\alpha}, \tag{2.18}$$

while

$$\begin{aligned} c \int_0^x F(t)Q_{-1}(t) dt &= \alpha^{-1} \int_0^{x^\alpha} u_1(t^{1/\alpha}) \cdot t^{\{(1-\gamma)/\alpha\}-1} \cdot \exp(2\pi i \eta t) dt \\ &= \alpha^{-1} \int_0^{x^\alpha} v(t) \cdot \exp(2\pi i \eta t) dt, \quad v(t) = u_1(t^{1/\alpha}) \cdot t^{\{(1-\gamma)/\alpha\}-1} \\ &= (2\pi i \eta \alpha)^{-1} \left(v(x^\alpha) \cdot \exp(2\pi i \eta x^\alpha) - \int_0^{x^\alpha} v'(t) \cdot \exp(2\pi i \eta t) dt \right), \end{aligned}$$

since $v(t) \in C^\infty(-\infty < t < \infty)$, with $v(0) = 0$, and $v(x^\alpha) = x^{1-\gamma-\alpha}$. Hence

$$\begin{aligned} \int_0^x F(t)Q_{-1}(t) dt &= c_1 x^{1-\gamma-\alpha} \cdot \exp(2\pi i \eta x^\alpha) - c_1 \int_0^{x^\alpha} v'(t) \cdot \exp(2\pi i \eta t) dt \\ &= \sum_{\nu=1}^k c_\nu x^{1-\gamma-\nu\alpha} \cdot \exp(2\pi i \eta x^\alpha) - c_k \int_0^{x^\alpha} v^{(k)}(t) \cdot \exp(2\pi i \eta t) dt. \end{aligned}$$

Since $v^{(k)}(t) = O((1+t)^{(1-\gamma)/\alpha-k-1})$, for large t , we see that

$$\int_0^\infty v^{(k)}(t) \cdot \exp(2\pi i \eta t) dt$$

converges. Therefore

$$\begin{aligned} \int_0^{x^\alpha} v^{(k)}(t) \cdot \exp(2\pi i \eta t) dt &= \left(\int_0^\infty - \int_{x^\alpha}^\infty \right) v^{(k)}(t) \cdot \exp(2\pi i \eta t) dt \\ &= C + O(x^{1-\gamma-k\alpha}). \end{aligned}$$

Hence

$$\int_0^x F(t) \cdot Q_{-1}(t) dt = C + O(x^{1-\gamma-\alpha}). \tag{2.19}$$

Now (2.19), (2.18), and (2.17) lead to the estimate

$$\int_0^\infty F(t) \cdot Q_{-1}(t) dt = C + O(x^{1-\gamma-\alpha}). \tag{2.20}$$

This, together with (2.16), (2.15), (2.12), (2.10), (2.11), and (2.11)', lead to the estimate

$$\begin{aligned} \sum_{\lambda_k \leq x} a_k F(\lambda_k) &= \sum_{\lambda_k \leq x_1} a_k F(\lambda_k) + O\left\{ \sum_{x < \lambda_k \leq x_1} a_k \lambda_k^{-\gamma} \right\} \\ &= \sum_{\lambda_k \leq c_0 x^{n\alpha-1+\varepsilon_0}} a_k \int_0^\infty F(t) \cdot I_{-1}(\lambda_k t) dt + O(x^{1-\alpha_1-\gamma} + x^\varepsilon), \end{aligned} \tag{2.21}$$

where $\alpha_1 = \alpha$ if $\alpha \leq 2/(n+1)$, and $\alpha_1 = \alpha - \varepsilon < \alpha$, if $\alpha > 2/(n+1)$. Now let

$$c_0 = \left(\frac{2\pi\eta n\alpha}{h} \right)^n, \tag{2.22}$$

where h is defined as in (2.4). Then the last sum in (2.21) equals

$$\begin{aligned} &\sum_{\lambda_k \leq c_0 x^{n\alpha-1+\varepsilon_0}} a_k \int_0^\infty F(t) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\lambda_k \leq c_0 x^{n\alpha-1+\varepsilon_0}} a_k \int_0^\infty u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &\quad + \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \int_0^\infty \{u(t) - u_1(t)\} t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &\quad + \sum_{c_0 x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1+\varepsilon_0}} a_k \int_0^{x_1} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt. \end{aligned} \tag{2.23}$$

Now define

$$H(x, \lambda_k) = \begin{cases} \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt, & \text{if } \lambda_k \leq c_0 x^{n\alpha-1}, \\ \int_0^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt, & \text{if } \lambda_k > c_0 x^{n\alpha-1}; \end{cases} \tag{2.24}$$

and

$$H_1(x, \lambda_k) = \begin{cases} \int_x^\infty \{u_1(t) - u(t)\} \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt, & \text{if } \lambda_k \leq c_0 x^{n\alpha-1} \\ \int_0^{x_1} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt, & \text{if } \lambda_k > c_0 x^{n\alpha-1}; \end{cases} \tag{2.25}$$

provided that the integrals converge (see Lemma 3). If we combine (2.25), (2.24), (2.23), and (2.21), we get the following

LEMMA 1. *We have*

$$\begin{aligned} \sum_{\lambda_k \leq x} a_k F(\lambda_k) &= \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &\quad - \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k H_1(x, \lambda_k) + \sum_{c_0 x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1+\varepsilon_0}} a_k H_1(x, \lambda_k) \\ &\quad + O\{x^{1-\alpha_1-\gamma} + x^\varepsilon\}, \end{aligned}$$

where F is defined as in (2.8), u_1 as in (2.7), H_1 as in (2.25), $\alpha > 1/n$, $\frac{1}{2} + (1/2n) - \alpha < \gamma < 1$, $\gamma \geq 0$, in which case the integral \int_0^∞ converges (as proved in Lemma 3). Here $\alpha_1 = \alpha$ if $\alpha \leq 2/(n+1)$, and $\alpha_1 = \alpha - \varepsilon < \alpha$ if $\alpha > 2/(n+1)$.

LEMMA 2. *We have, for $x > 0$,*

$$H_1(x, \lambda_k) \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|,$$

where, as before, $x_1 = x + x^{1-\alpha}$, $\alpha > 1/n$.

Proof. If $\lambda_k \leq c_0 x^{n\alpha-1}$, then

$$\begin{aligned} H_1(x, \lambda_k) &= \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt + \int_x^{x_1} u(t) \cdot \left\{ \frac{d}{dt} H(t, \lambda_k) \right\} \cdot dt \\ &= H(x, \lambda_k) + [H(t, \lambda_k) \cdot u(t)]_{t=x}^{t=x_1} - \int_x^{x_1} u'(t) \cdot H(t, \lambda_k) dt \\ &= H(x, \lambda_k) - H(x, \lambda_k) - \int_x^{x_1} u'(t) \cdot H(t, \lambda_k) dt \\ &= - \int_x^{x_1} u'(t) \cdot H(t, \lambda_k) dt \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|. \end{aligned}$$

If $\lambda_k > c_0 x^{n\alpha-1}$, then

$$H_1(x, \lambda_k) = \int_0^{x_1} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt = \int_0^x + \int_x^{x_1}.$$

Since

$$\int_x^{x+x^{1-\alpha}} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|,$$

as before, the result follows.

LEMMA 3. If $\eta > 0$, then the integral

$$\int_0^\infty t^{-\gamma} \cdot \exp(i\eta t^\alpha) \cdot I_{-1}(t) dt$$

converges for all α , such that $\frac{1}{2} + (1/2n) - \alpha < \gamma < 1$ and $\alpha > 1/n$.

Proof. We have, from (2.4),

$$I_{-1}(t) = \sum_{\nu=0}^m e_\nu t^{\omega_{-1} - (\nu/n)} \cos(ht^{1/n} + \pi_\nu) + O(t^{\omega_{-1} - (m+1)/n}),$$

for $t > 0$, where $\omega_{-1} = (1/2n) - \frac{1}{2}$, so that $\omega_{-1} - (\nu/n) < 0$, for $\nu \geq 0$. This leads us to

consider integrals of the form

$$\int_1^\infty t^{-\gamma-(1/2)+(1/2n)-(v/n)} \cdot \exp \{i(\eta t^\alpha \mp ht^{1/n})\} dt$$

$$= \alpha^{-1} \int_1^\infty t^{-(\gamma'/\alpha)+(1/\alpha)-1} \cdot \exp \{i(\eta t \mp ht^{1/n\alpha})\} dt,$$

with $-\gamma' = -\gamma - (\frac{1}{2}) + (1/2n) - (v/n)$. If $w(t) = (\eta t \mp ht^{1/n\alpha})$, then $|dw/dt| \geq \frac{1}{2}\eta$, for $t \geq \{2h/(n\alpha\eta)\}^{n\alpha/(n\alpha-1)}$. Hence the above integral converges if $-\gamma' + (1/\alpha) - 1 < 0$, that is if $1 - \gamma' < \alpha$, or $(\frac{1}{2}) + (1/2n) - \alpha < \gamma$. (The reasoning is the same as in Lemma 2 of [4]).

The integral arising from the error-term in I_{-1} is

$$\int_1^\infty t^{-\gamma-\omega_{-1}-(m+1)/n} dt,$$

which converges, if m is chosen sufficiently large.

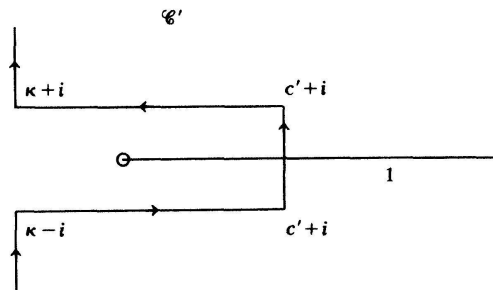
Next let us consider

$$\int_0^1 t^{-\gamma} \cdot \exp(i\eta t^\alpha) \cdot I_{-1}(t) dt,$$

where

$$I_{-1}(t) = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\Delta(s)}{\Delta(1-s)} \cdot t^{-s} ds,$$

with $0 < \text{Re } s \leq c' < 1$ on \mathcal{C}' .



If $0 < t < 1$, then

$$|t^{-s}| = t^{-\text{Re } s} = (1/t)^{\text{Re } s} \leq t^{-c'},$$

so that $I_{-1}(t) = O(t^{-c'})$, for all c' such that $0 < c' < 1$. Hence

$$\int_0^1 t^{-\gamma} \cdot \exp(i\eta t^\alpha) \cdot I_{-1}(t) dt$$

converges *absolutely*, provided that $\gamma < 1$.

§3. Some asymptotic expansions

LEMMA 4. We have, for $\alpha > 1/n$, and $0 \leq \gamma < 1$, the asymptotic expansion

$$\begin{aligned} & \int_0^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= (i)^{(1-\gamma-\alpha)/\alpha} \cdot (2\pi\eta)^{(\gamma-1)/\alpha} \cdot \alpha^{-1} \cdot \pi \sum_{\nu=0}^m m_k^{\theta-\nu/(\alpha\alpha-1)} \\ & \quad \times \{C_\nu \cos(qm_k^{1/(\alpha\alpha-1)} + k_\nu \pi) - iD_\nu \sin(qm_k^{1/(\alpha\alpha-1)} + k_\nu \pi)\} \\ & \quad + \sum_{0 \leq p \leq \{\alpha(n+2m)-2(1-\gamma)\}/\{2(\alpha\alpha-1)\}} \frac{\Delta(1-\gamma+\alpha p)}{\Delta(\gamma-\alpha p)} \cdot \frac{(2\pi i \eta)^p}{p!} \cdot \frac{1}{\lambda_k^{1-\gamma+\alpha p}} \\ & \quad + O(m_k^{\theta-(m+1)/(\alpha\alpha-1)}), \end{aligned}$$

where

$$\eta > 0, \quad m_k = \lambda_k^\alpha / (2\pi\eta), \quad \theta = \frac{2(1-\gamma) - \alpha n}{2\alpha(\alpha\alpha-1)},$$

$$q = (\alpha n - 1) \cdot (2^{r_1} \alpha^{-n})^{\alpha/(\alpha\alpha-1)},$$

$$k_\nu = \omega' + \frac{1}{2}\nu, \quad \omega' = \frac{1}{4}r_1 + \{(1-\gamma)/2\alpha\} - 1, \quad p \text{ integral,}$$

$$C_0 = D_0 = \pi^{-1} (\alpha n - 1)^{-1/2} \cdot \alpha^{1+\{n(2\gamma-1)\}/\{2(\alpha\alpha-1)\}} \cdot 2^{\{r_1(1-2\gamma)\}/\{2(\alpha\alpha-1)\}}.$$

The first term in the expansion is given by

$$\begin{aligned} & c_0^{\gamma/(\alpha\alpha-1)} \cdot \eta^{-n/\{2(\alpha\alpha-1)\}} \cdot \lambda_k^{\{-\gamma/(\alpha\alpha-1)\} + \{2-\alpha n\}/\{2(\alpha\alpha-1)\}} \cdot e^{i\pi[(1/2)-(r_1/4)]} \\ & \quad \times (2\pi)^{-n/\{2(\alpha\alpha-1)\}} \cdot (\alpha n - 1)^{-1/2} \cdot \alpha^{-n/\{2(\alpha\alpha-1)\}} \cdot 2^{r_1/\{2(\alpha\alpha-1)\}} \cdot e^{-iqm_k^{1/(\alpha\alpha-1)}}, \end{aligned}$$

where $c_0 = (2\pi\eta\alpha/h)^n$, $h = n \cdot 2^{r_1/n}$.

Proof. Let

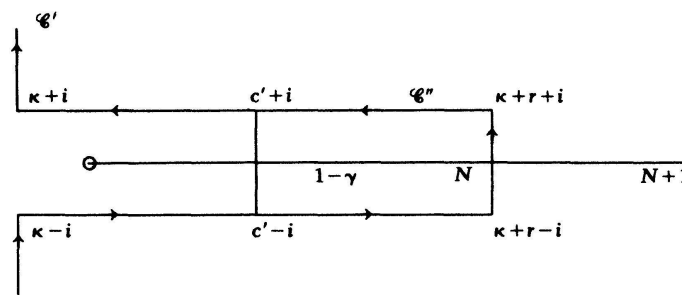
$$J = \int_0^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt = \lambda_k^{\gamma-1} \int_0^\infty t^{-\gamma} \cdot \exp(it^\alpha m_k^{-1}) \cdot I_{-1}(t) dt.$$

As in Lemma 3 of [4], we have, for $\xi > 0$,

$$\int_0^\infty t^{-\gamma} \cdot \exp(i\xi t^\alpha) \cdot I_{-1}(t) dt = \frac{\alpha^{-1}}{2\pi i} \int_{\mathcal{C}'} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \cdot \left(\frac{\xi}{i}\right)^{(s+\gamma-1)/\alpha} ds.$$

The path of integration \mathcal{C}' is as shown in the diagram, with $\text{Re } s \leq 1 - \gamma - \varepsilon < 1 - \gamma$, and κ is sufficiently large and negative. Putting $\xi = (m_k)^{-1}$, we obtain

$$J = \alpha^{-1} \cdot \lambda_k^{\gamma-1} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \cdot \left(\frac{1}{m_k i}\right)^{(s+\gamma-1)/\alpha} ds.$$



Now deform the path of integration \mathcal{C}' into \mathcal{C}'' , by choosing N to be a sufficiently large integer, and $N < \kappa + r < N + 1$, as indicated in the diagram. We then have

$$J = \alpha^{-1} \cdot \lambda_k^{\gamma-1} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \cdot \left(\frac{1}{m_k i}\right)^{(s+\gamma-1)/\alpha} ds + \sum_{l=0}^N \frac{\Delta(1-\gamma+\alpha l)}{\Delta(\gamma-\alpha l)} \cdot \frac{(2\pi i \eta)^l}{l!} \cdot \frac{1}{\lambda_k^{1-\gamma+l\alpha}} \tag{3.1}$$

[We may note here that the residual term in Lemma 4 of [4] should carry the sign + instead of -].

We seek an expansion for

$$\begin{aligned}
 J' &= \alpha^{-1} \cdot \lambda_k^{\gamma-1} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \cdot \left(\frac{1}{m_k i}\right)^{(s+\gamma-1)/\alpha} ds \\
 &= \left(\frac{1}{2\pi\eta}\right)^{(1-\gamma)/\alpha} \cdot \alpha^{-1} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) m_k^{-s/\alpha} \cdot (-i)^{s/\alpha} ds \\
 &= \left(\frac{i}{2\pi\eta}\right)^{(1-\gamma)/\alpha} \cdot \alpha^{-1} \cdot \frac{1}{2\pi i} \int_{\mathcal{C}''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Gamma\left(\frac{1-s-\gamma}{\alpha}\right) \\
 &\qquad \qquad \qquad \times m_k^{-s/\alpha} \left(\cos \frac{\pi s}{2\alpha} - i \sin \frac{\pi s}{2\alpha}\right) ds.
 \end{aligned}$$

Now

$$\cos\left(\frac{\pi s}{2\alpha}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2\alpha} + \frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2} - \frac{s}{2\alpha}\right)}, \quad \sin\left(\frac{\pi s}{2\alpha}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2\alpha}\right) \cdot \Gamma\left(1 - \frac{s}{2\alpha}\right)}.$$

Therefore

$$J' = \left(\frac{i}{2\pi\eta}\right)^{(1-\gamma)/\alpha} \cdot \alpha^{-1} \cdot \pi \cdot \frac{1}{2\pi i} \int_{\mathcal{C}''} \{V_0(s) - iV_1(s)\} m_k^{-s/\alpha} ds, \tag{3.2}$$

where

$$\begin{aligned}
 V_0(s) &= \frac{\Delta(s)}{\Delta(1-s)} \cdot \frac{\Gamma\left(\frac{1-\gamma-s}{\alpha}\right)}{\Gamma\left(\frac{s}{2\alpha} + \frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2} - \frac{s}{2\alpha}\right)}, \\
 V_1(s) &= \frac{\Delta(s)}{\Delta(1-s)} \cdot \frac{\Gamma\left(\frac{1-\gamma-s}{\alpha}\right)}{\Gamma\left(\frac{s}{2\alpha}\right) \cdot \Gamma\left(1 - \frac{s}{2\alpha}\right)}.
 \end{aligned}$$

Now choose

$$\begin{aligned}
 U_0(s) &= \frac{b^{-s}\Gamma(S)}{\Gamma\left(-\frac{1}{2}S - \omega'\right) \cdot \Gamma\left(1 + \frac{1}{2}S + \omega'\right)}, \\
 U_1(s) &= \frac{b^{-s}\Gamma(S)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}S - \omega'\right) \cdot \Gamma\left(\frac{1}{2} + \frac{1}{2}S + \omega'\right)},
 \end{aligned}$$

where

$$S = \left(n - \frac{1}{\alpha}\right)s - \frac{n}{2} + \left(\frac{1-\gamma}{\alpha}\right), \quad \omega' = \frac{r_1}{4} + \left(\frac{1-\gamma}{2\alpha}\right) - 1,$$

$$b = \left(n - \frac{1}{\alpha}\right)^{(n-1/\alpha)} \cdot 2^{r_1} \cdot \alpha^{-1/\alpha}.$$

Then we have

$$U_0(s) = -\frac{b^{-s}}{\pi} \Gamma(S) \cdot \sin \pi \left\{\frac{1}{2}S + \omega'\right\}, \quad U_1(s) = \frac{b^{-s}}{\pi} \cdot \Gamma(S) \cdot \cos \pi \left\{\frac{1}{2}S + \omega'\right\}.$$

By choosing

$$a = \left(n - \frac{1}{\alpha}\right)^{\{(n+1)/2 - (1-\gamma)/2\}} \cdot 2^{r_1/2} \cdot \alpha^{1/2 - (1-\gamma)/\alpha},$$

and comparing the expansions of V_0, U_0 , on the one hand, and of V_1, U_1 on the other, we get, as before,

$$V_1(s) = aU_1(s) \left\{1 + \sum_{\nu=1}^m \frac{e_\nu}{s^\nu} + O(|s|^{-m-1})\right\},$$

$$V_0(s) = aU_0(s) \left\{1 + \sum_{\nu=1}^m \frac{e_\nu}{s^\nu} + O(|s|^{-m-1})\right\}.$$

Now if we follow the same procedure as in the asymptotic expansion of I_ρ [2], we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}''} V_1(s)x^{-s} ds &= \sum_{\nu=0}^m C_\nu x^{\alpha\theta - [\alpha\nu/(\alpha-1)]} \cos \{qx^{\alpha/(\alpha-1)} + k_\nu\pi\} \\ &\quad + O(x^{\alpha\theta - [\alpha(m+1)/(\alpha-1)]}), \end{aligned} \tag{3.3}$$

while

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}''} V_0(s)x^{-s} ds &= -\sum_{\nu=0}^m D_\nu x^{\alpha\theta - [\alpha\nu/(\alpha-1)]} \sin \{qx^{\alpha/(\alpha-1)} + k_\nu\pi\} \\ &\quad + O(x^{\alpha\theta - [\alpha(m+1)/(\alpha-1)]}), \end{aligned} \tag{3.4}$$

where $q = b^{\alpha/(n\alpha-1)}$, and $k_\nu = \omega' + \frac{1}{2}\nu$, provided that

$$N \geq \frac{m + \frac{1}{2}n - (1-\gamma)/\alpha}{n - 1/\alpha} = \frac{2m\alpha + n\alpha - 2(1-\gamma)}{2(n\alpha - 1)}.$$

We find by calculation that

$$C_0 = D_0 = \frac{a}{\pi(n - 1/\alpha)} \cdot q^{-\{(n/2) - [(1-\gamma)/\alpha]\}}.$$

From (3.1) and (3.2) we have

$$J = (i)^{(1-\gamma-\alpha)/\alpha} \cdot (2\pi\eta)^{(\gamma-1)/\alpha} \cdot \alpha^{-1} \cdot \pi \cdot \frac{1}{2\pi i} \int_{\mathcal{C}''} \{V_1(s) + iV_0(s)\} m_k^{-s/\alpha} ds \\ + \sum_{l=0}^N \frac{\Delta(1-\gamma+\alpha l)}{\Delta(\gamma-\alpha l)} \cdot \frac{(2\pi i \eta)^l}{l!} \cdot \frac{1}{\lambda_k^{1-\gamma+\alpha l}}.$$

Now (3.3) and (3.4) lead to the lemma.

LEMMA 5. If $\xi > 0$, $\alpha > 0$, $0 \leq \gamma < 1$, and

$$J'(\xi) = \int_0^\xi t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt,$$

then $J'(\xi)$ has the asymptotic expansion

$$J'(\xi) = \sum_{p=0}^l \frac{a_p(\xi) \cdot I_p(\lambda_k \xi)}{(\lambda_k \xi)^{p+1}} + \sum_{p=0}^l \frac{b_p}{\lambda_k^{1-\gamma+\alpha p}} + O(\lambda_k^{-l/n}),$$

uniformly for $0 < \xi_0 \leq \xi \leq \xi_1 < \infty$, where the coefficients $a_p(\xi)$ are continuous in ξ , and

$$b_p = \frac{(2\pi i \eta)^p}{p!} \cdot \frac{\Delta(1-\gamma+\alpha p)}{\Delta(\gamma-\alpha p)}.$$

Proof. For $x > 0$, we have

$$I_{-1}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\Delta(s)x^{-s}}{\Delta(1-s)} ds,$$

where \mathcal{C}' is the same as in Lemma 4, so that $\operatorname{Re} s \leq 1 - \gamma - \varepsilon < 1 - \gamma$ on \mathcal{C}' . Therefore

$$J'(\xi) = \int_0^\xi t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot dt \cdot \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\Delta(s)}{\Delta(1-s)} \cdot (\lambda_k t)^{-s} ds,$$

and the repeated integral remains finite, if we replace the integrands by their absolute values. Hence

$$J'(\xi) = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\Delta(s)}{\Delta(1-s)} \cdot \lambda_k^{-s} \cdot j(s, \xi) ds, \tag{3.5}$$

where

$$j(s, \xi) = \int_0^\xi t^{-\gamma-s} \cdot \exp(2\pi i \eta t^\alpha) \cdot dt.$$

By a change of variable, we have

$$j(s, \xi) = \alpha^{-1} \int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}-1} \cdot \exp(2\pi i \eta t) dt.$$

An integration by parts gives

$$j(s, \xi) = \frac{\xi^{1-\gamma-s}}{(1-\gamma-s)} \cdot \exp(2\pi i \eta \xi^\alpha) - \frac{2\pi i \eta}{(1-\gamma-s)} \int_0^{\xi^\alpha} t^{(1-\gamma-s)/\alpha} \cdot \exp(2\pi i \eta t) dt,$$

since $\operatorname{Re} s < 1 - \gamma$, and $\alpha > 0$. Repeating this process, we obtain

$$\begin{aligned} j(s, \xi) = & \xi^{-s} \left\{ \frac{a'_0(\xi)}{(1-\gamma-s)} + \frac{a'_1(\xi)}{(1-\gamma-s)(1-\gamma+\alpha-s)} + \dots \right. \\ & \left. + \frac{a'_l(\xi)}{(1-\gamma-s) \cdots (1-\gamma+l\alpha-s)} \right\} \\ & + \frac{c(\alpha)}{(1-\gamma-s)(1-\gamma+\alpha-s) \cdots (1-\gamma+l\alpha-s)} \int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}+l} \\ & \times \exp(2\pi i \eta t) dt. \end{aligned} \tag{3.6}$$

We may write

$$\frac{1}{(1-\gamma-s)(1-\gamma+\alpha-s)\cdots(1-\gamma+\nu\alpha-s)} = \sum_{p=0}^l \frac{c_p}{(1-s)(2-s)\cdots(p+1-s)} + \varphi_{\nu,l},$$

where $\varphi_{\nu,l} = O(|s|^{-l-1})$, in any vertical strip, and has simple poles at the points $s = 1, 2, \dots, l+1$, as well as $1-\gamma, 1-\gamma+\alpha, 1-\gamma+2\alpha, \dots, 1-\gamma+\alpha\nu$, and the ‘O’ depends on ξ . Further the integral

$$\int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}+l} \exp(2\pi i \eta t) dt$$

converges absolutely, and is holomorphic for $\text{Re } s < \alpha(l+1) + 1 - \gamma$, and bounded in the half-plane

$$\text{Re} \left\{ \frac{(1-\gamma-s)}{\alpha} + l \right\} \geq -1 + \epsilon > -1.$$

Hence

$$j(s, \xi) = \xi^{-s} \sum_{p=0}^l \frac{a_p(\xi)}{(1-s)(2-s)\cdots(1+p-s)} + \Phi_l(s, \xi), \tag{3.7}$$

where

$$\begin{aligned} \Phi_l(s, \xi) &= \xi^{-s} \sum_{p=0}^l d_p(\xi) \varphi_{p,l}(s) + \frac{c(\alpha, \gamma)}{(1-\gamma-s)\cdots(1-\gamma+l\alpha-s)} \\ &\quad \times \int_0^{\xi^\alpha} t^{\{(1-\gamma-s)/\alpha\}+l} \exp(2\pi i \eta t) dt \end{aligned}$$

and $\Phi_l(s, \xi)$ is meromorphic for $\text{Re } s < \alpha(l+1) + 1 - \gamma$, with simple poles at $s = 1, 2, \dots, l+1$, as well as $1-\gamma, 1-\gamma+\alpha, \dots, 1-\gamma+l\alpha$. Further $\Phi_l(s, \xi) = O(|s|^{-l-1})$, in any closed vertical strip contained in *that* half-plane, uniformly in ξ for ξ in any compact set.

Now $j(s, \xi)$ can also be written as

$$j(s, \xi) = \int_0^\xi t^{\gamma-s} \sum_{\nu=0}^l \frac{(2\pi i \eta)^\nu}{\nu!} \cdot t^{\alpha\nu} dt + \int_0^\xi t^{-\gamma-s} \tilde{f}_l(t) dt,$$

say, where $\tilde{f}_l(t) = O(t^{\alpha(l+1)})$, so that the second integral is holomorphic in s for $\text{Re}(-\gamma - s) > -\alpha(l+1) - 1$, or for $\text{Re} s < 1 - \gamma + \alpha(l+1)$, for any integer $l > 0$. It follows that $j(s, \xi)$ is a meromorphic function, whose only poles are at $s = 1 - \gamma + \alpha\nu$, $\nu = 0, 1, 2, \dots$, with the corresponding residues $-(2\pi i \eta)^\nu / (\nu!)$.

Now if \mathcal{C}''' is the path obtained by deforming \mathcal{C}' so as to have the corners at $\kappa - i$, $c' + N - i$, $c' + N + i$, $\kappa + i$ (the infinite half-lines being left as they are), with N sufficiently large, then we have, from (3.5),

$$J'(\xi) = \frac{1}{2\pi i} \int_{\mathcal{C}'''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \lambda_k^{-s} \cdot j(s, \xi) ds + \sum_{p=0}^l \frac{(2\pi i \eta)^p}{p!} \cdot \frac{\Delta(1-\gamma+\alpha p)}{\Delta(\gamma-\alpha p)} \cdot \frac{1}{\lambda_k^{1-\gamma+\alpha p}}. \tag{3.8}$$

(This requires that $c' + N < 1 - \gamma + \alpha(l+1)$, and $c' + N > 1 - \gamma + \alpha l$)

The integral here can be considered as a sum of two integrals J'' and J''' , because of the expression for $j(s, \xi)$ given in (3.7), where

$$J''(\xi) = \sum_{p=0}^l \frac{a_p(\xi)}{2\pi i} \int_{\mathcal{C}'''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \frac{\Gamma(1-s)}{\Gamma(1+p+1-s)} \cdot (\lambda_k \xi)^{-s} ds = \sum_{p=0}^l \frac{a_p(\xi) \cdot I_p(\lambda_k \xi)}{(\lambda_k \xi)^{p+1}}, \tag{3.9}$$

provided that $c' + N > l + 1$, while

$$J'''(\xi) = \frac{1}{2\pi i} \int_{\mathcal{C}'''} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Phi_l(s, \xi) \cdot \lambda_k^{-s} ds.$$

If \mathcal{C}'''_0 is the path obtained by deforming \mathcal{C}''' , so that the two infinite half-lines in \mathcal{C}''' are moved to the right, then

$$J'''(\xi) = \frac{1}{2\pi i} \int_{\mathcal{C}'''_0} \frac{\Delta(s)}{\Delta(1-s)} \cdot \Phi_l(s, \xi) \cdot \lambda_k^{-s} ds,$$

and this integral converges absolutely for $n(\sigma - \frac{1}{2}) - l - 1 < -1$, or $\sigma < (l/n) + \frac{1}{2}$, where $\sigma = \text{Re} s$. Note that $\Phi_l(s)$ has no poles off the real axis. If we take $\sigma = l/n + \frac{1}{2} - \epsilon$, with $0 < \epsilon < \frac{1}{2}$, then

$$J'''(\xi) = O(\lambda_k^{-(l/n)-(1/2)+\epsilon}) = O(\lambda_k^{-l/n}). \tag{3.10}$$

Now (3.8)–(3.10) give the required result.

LEMMA 6. Let a be a real number, $a \neq 0$, $\alpha > 1/n$, $\mu = (h/2\pi\eta) \cdot \lambda_k^{1/n}$, $\eta > 0$, and h defined as in (2.4).

Let $\varphi(t) \equiv \varphi(t, \mu) = t^\alpha - \mu t^{1/n}$, $F_0(t) \equiv F_0(t, \mu) = t^\alpha / \varphi'(t)$, $F_{l+1}(t) \equiv F_{l+1}(t, \mu) = 1/\varphi'(t) \cdot (d/dt)F_l(t)$, for $l = 0, 1, 2, \dots$. Then

$$F_l(t) = \frac{t^{a-l}}{(\varphi'(t))^{l+1}} \cdot \sum_{p,q=0; q \leq p}^l c_{p,q,l} \frac{\left(\frac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^q}{\left(1 - \frac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^p},$$

where the $c_{p,q,l}$ are suitable constants.

Analogously, let $\psi(t) \equiv \psi(t, \mu) = t^\alpha + \mu t^{1/n}$, $G_0(t) \equiv G_0(t, \mu) = t^\alpha / \psi'(t)$, and $G_{l+1}(t) = G_{l+1}(t, \mu) = 1/\psi'(t) \cdot (d/dt)G_l(t)$, for $l = 0, 1, 2, \dots$. Then

$$G_l(t) = \frac{t^{a-l}}{(\psi'(t))^{l+1}} \cdot \sum_{p,q=0; q \leq p}^l d_{p,q,l} \frac{\left(\frac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^q}{\left(1 + \frac{\mu}{n\alpha} \cdot t^{1/n-\alpha}\right)^p}.$$

The proof follows by induction on l .

LEMMA 7. For a fixed ξ , such that $0 < \xi_0 \leq \xi \leq \xi_1 < \infty$, we have as $\mu \rightarrow \infty$, the following asymptotic expansion in decreasing powers of μ :

$$F_m(\xi, \mu) = \frac{1}{\mu^{m+1}} \left(\sum_{i=0}^L \frac{d_{i,m}}{\mu^i} + o(\mu^{-L}) \right).$$

For the proof we have only to use the expression for $F_m(\xi, \mu)$ given by Lemma 6 together with the Binomial Theorem.

In what follows we shall frequently use the notation $F_{l,\nu}(t) = F_l(t)$, and $G_{l,\nu}(t) = G_l(t)$, with $a = a(\nu) - \gamma = \omega_{-1} - (\nu/n) - \gamma = (1/2n) - (\frac{1}{2}) - (\nu/n) - \gamma$, for $l = 0, 1, 2, \dots$, $\nu = 0, 1, 2, \dots$, and $0 \leq \gamma < 1$.

LEMMA 8. Let $a(\nu) = \omega_{-1} - \nu/n = (1/2n) - (\frac{1}{2}) - (\nu/n)$, for $\nu = 0, 1, 2, \dots$, $0 \leq \gamma < 1$. Let $\delta > 0$, sufficiently small, and $0 < \eta_0 < \eta$, $\alpha > 1/n$. Let $\lambda_k \leq (c_0 - \delta)x^{n\alpha-1}$.

Then we have

$$\begin{aligned} & \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^{n(m+1)} \lambda_k^{a(\nu)} \sum_{0 \leq l \leq \{(m+1)/\alpha\}+1} \\ & \times \{ \exp(2\pi i \eta \varphi(x)) \cdot b_{l,\nu} F_{l,\nu} + \exp(2\pi i \eta \psi(x)) \cdot b'_{l,\nu} G_{l,\nu}(x) \} + O(x^{\omega-1-\nu-m}). \end{aligned}$$

Proof. The asymptotic expansion of $I_{-1}(\lambda_k t)$ leads us to consider integrals of the form

$$\int_x^\infty t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t)) dt, \quad \varphi(t) = t^\alpha - \mu t^{1/n}.$$

As in Lemma 6 of [4], we prove that

$$F_{l,\nu}(t) = O(t^{a(\nu)-\gamma+1-(l+1)\alpha}),$$

for $t \geq x$, $\lambda_k \leq (c_0 - \delta)x^{n\alpha-1}$, where the ‘ O ’ depends on $a(\nu)$, l , and δ , but *not* on t , x , or μ . Repeated integration by parts then gives

$$\int_x^\infty t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t)) dt = e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O(x^{a(\nu)-\gamma+1-(L+1)\alpha}),$$

and analogously

$$\begin{aligned} \int_x^\infty t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \psi(t)) dt &= e^{2\pi i \eta \psi(x)} \sum_{l=0}^L \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \\ &+ O(x^{a(\nu)-\gamma+1-(L+1)\alpha}). \end{aligned}$$

These expansions together with the asymptotic expansion of $I_{-1}(\lambda_k t)$ therefore

yield

$$\begin{aligned} & \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{\alpha(\nu)} \left(b_\nu \int_x^\infty t^{\alpha(\nu)-\gamma} e^{2\pi i \eta \varphi(t)} dt + b'_\nu \int_x^\infty t^{\alpha(\nu)-\gamma} e^{2\pi i \eta \psi(t)} dt \right) \\ & \quad + O_m(\lambda_k^{-1}(\lambda_k x)^{\alpha(m+1)-\gamma+1}) \\ &= \sum_{\nu=0}^m \lambda_k^{\alpha(\nu)} \left\{ b_\nu e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + b'_\nu e^{2\pi i \eta \psi(x)} \sum_{l=0}^L \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + O(x^{\alpha(\nu)-\gamma-(L+1)\alpha}) \right\} + O_m(x^{\alpha(m+1)-\gamma+1}). \end{aligned}$$

If $L = [(m + 1)/n\alpha]$, then $L + 1 > (m + 1)/n\alpha$, and we have

$$\begin{aligned} & \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{\alpha(\nu)} \left\{ b_\nu e^{2\pi i \eta \varphi(x)} \sum_{0 \leq l \leq (m+1)/n\alpha} \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + b'_\nu e^{2\pi i \eta \psi(x)} \sum_{0 \leq l \leq (m+1)/n\alpha} \frac{(-1)^{l+1} G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \right\} + O_m(x^{\alpha(m+1)-\gamma+1}) \end{aligned}$$

The lemma follows upon replacing m by $n(m + 1)$.

LEMMA 9. *Let $a(\nu) = \omega_{-1} - \nu/n = (1/2n) - \frac{1}{2} - (\nu/n)$, $\nu = 0, 1, 2, \dots, 0 \leq \gamma < 1$, $\delta > 0$, δ sufficiently small, $0 < 2\epsilon < \alpha$, $\alpha > 1/n$. Then we have for $(c_0 - \delta)x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1} - x^{n\alpha-1-\epsilon}$,*

$$\begin{aligned} & \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^{n(m+1)} \lambda_k^{\alpha(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x)} \sum_{0 \leq l \leq (m+1)/(n\alpha-2n\epsilon)} \frac{(-1)^{l+1} F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + b'_\nu e^{2\pi i \eta \psi(x)} \sum_{0 \leq l \leq (m+1)/(n\alpha-2n\epsilon)} \frac{(-1)^{l+1} G_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right) + O(x^{\omega_{-1}-\gamma-m}). \end{aligned}$$

Proof. The pattern of proof is similar to that of Lemmas 11 and 12 in [4]. We first prove that

$$\int_x^\infty t^{a(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t)) dt = e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + O_L(x^{a(\nu)-\gamma-\alpha L+(1-\alpha)+(L+1)2\varepsilon}).$$

An analogous expansion is valid with ψ in place of φ , and $G_{l,\nu}$ in place of $F_{l,\nu}$. As in Lemma 8, we then have

$$\begin{aligned} & \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x,\mu)} \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x,\mu)}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + b'_\nu e^{2\pi i \eta \psi(x,\mu)} \sum_{l=0}^L \frac{(-1)^{l+1} G_{l,\nu}(x,\mu)}{(2\pi i \eta)^{l+1}} \right) + O_m(x^{a(m+1)-\gamma+1}), \end{aligned}$$

provided that $L = [(m+1)/(n\alpha - 2n\varepsilon)]$, $0 < 2\varepsilon < \alpha$. The lemma follows upon replacing m by $n(m+1)$.

LEMMA 10. *If $\delta > 0$, and sufficiently small, and $0 < \eta_0 \leq \eta$, and $\alpha > 1/n$, then for $\lambda_k \geq (c_0 + \delta)x^{n\alpha-1}$, we have the asymptotic expansion*

$$\begin{aligned} & \int_0^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x,\mu)}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + b'_\nu e^{2\pi i \eta \psi(x)} \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x,\mu)}{(2\pi i \eta)^{l+1}} \right) + O(\mu^{-L-1}) + O(\lambda_k^{-m/n}). \end{aligned}$$

If $L = m$, the term $O(\mu^{-L-1})$ can be dropped.

Proof. Choose ξ such that $\lambda_1 < \xi < x$, and consider the given integral as the

sum of \int_0^ξ and \int_ξ^x . We then have

$$\int_\xi^x u(t) \cdot t^{a(\nu)-\gamma} \cdot e^{2\pi i \eta \varphi(t)} dt = e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} - e^{2\pi i \eta \varphi(\xi)} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} + O(\mu^{-L-1}),$$

where the ‘O’ does not depend on ξ . An analogous expansion is valid with ψ in place of φ , and $G_{l,\nu}$ in place of $F_{l,\nu}$ (see the proof of Lemma 10 in [4]). It follows that

$$\begin{aligned} &\int_\xi^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b''_\nu \exp(2\pi i \eta \varphi(x, \mu)) \cdot \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ &\quad \left. - b''_\nu \exp(2\pi i \eta \varphi(\xi, \mu)) \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} \right) \\ &\quad + \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b'''_\nu \exp(2\pi i \eta \psi(x, \mu)) \cdot \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right. \\ &\quad \left. - b'''_\nu \exp(2\pi i \eta \psi(\xi, \mu)) \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(\xi, \mu)}{(2\pi i \eta)^{l+1}} \right) + O(\mu^{-L-1}) + O(\lambda_k^{a(m+1)-\gamma}). \end{aligned} \tag{3.11}$$

On the other hand, if $F(t) = u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha)$, as in the proof of Lemma 1, we have

$$\begin{aligned} \int_0^\xi F(t) I_{-1}(\lambda_k t) dt &= \sum_{p=0}^L \frac{(-1)^p F^{(p)}(\xi) \cdot I_p(\lambda_k \xi)}{\lambda_k^{1+p}} \\ &\quad + (-1)^{L+1} \cdot \lambda_k^{-L-1} \int_0^\xi F^{(L+1)}(t) \cdot I_L(\lambda_k t) dt, \end{aligned} \tag{3.12}$$

and the last term is $O(\lambda_k^{-L/n})$. Since I_p has the asymptotic expansion (2.4), we can combine (3.12) and (3.11), and apply Lemma 9 of [4] to obtain the stated result.

LEMMA 11. *If $c_0 x^{n\alpha-1} + x^{n\alpha-1-\varepsilon} \leq \lambda_k \leq (c_0 + \delta) x^{n\alpha-1}$, $\delta > 0$, and sufficiently*

small, and $0 < 2\epsilon < \alpha$, $\alpha > 1/n$, then

$$\begin{aligned} & \int_0^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} + b'_\nu e^{2\pi i \eta \psi(x)} \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x)}{(2\pi i \eta)^{l+1}} \right) \\ &+ O(x^{\alpha(m+1)-\gamma+1}) + O(\lambda_k^{-m/n}), \end{aligned}$$

provided that $L = [(m + 1)/(\alpha n - 2n\epsilon)]$. As before $a(\nu) = \omega_{-1} - (\nu/n)$.

Proof. Choose λ such that $0 < \lambda < 1$, and consider the given integral as the sum of the integrals $\int_0^{x(1-\lambda)}$ and $\int_{x(1-\lambda)}^x$. Following the same pattern of proof as in Lemma 13 of [4], we obtain

$$\begin{aligned} & \int_{x(1-\lambda)}^x u(t) \cdot t^{\alpha(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t)) dt = \int_{x(1-\lambda)}^x t^{\alpha(\nu)-\gamma} \cdot \exp(2\pi i \eta \varphi(t)) dt \\ &= e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} - e^{2\pi i \eta \varphi(x(1-\lambda))} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x(1-\lambda), \mu)}{(2\pi i \eta)^{l+1}} \\ &+ O(x^{\alpha(\nu)-\gamma-\alpha L+(1-\alpha)+(L+1)2\epsilon}), \end{aligned}$$

as well as the analogue with ψ in place of φ , and $G_{l,\nu}$ in place of $F_{l,\nu}$. It follows that

$$\begin{aligned} & \int_{x(1-\lambda)}^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x)} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} + b'_\nu e^{2\pi i \eta \psi(x)} \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x, \mu)}{(2\pi i \eta)^{l+1}} \right) \\ &- \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x(1-\lambda))} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x(1-\lambda), \mu)}{(2\pi i \eta)^{l+1}} \right. \\ &\left. + b'_\nu e^{2\pi i \eta \psi(x(1-\lambda))} \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x(1-\lambda), \mu)}{(2\pi i \eta)^{l+1}} \right) + O(x^{\alpha(m+1)-\gamma}), \end{aligned} \tag{3.13}$$

provided that $L = [(m + 1)/(\alpha n - 2n\epsilon)]$.

On the other hand, by Lemma 10, (since $\lambda_k > (c_0 + \delta_1)\{x(1-\lambda)\}^{n\alpha-1}$), we have

$$\begin{aligned} & \int_0^{x(1-\lambda)} u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^m \lambda_k^{a(\nu)} \left(b_\nu e^{2\pi i \eta \varphi(x(1-\lambda))} \sum_{l=0}^L \frac{(-1)^l F_{l,\nu}(x(1-\lambda), \mu)}{(2\pi i \eta)^{l+1}} \right. \\ & \quad \left. + b'_\nu e^{2\pi i \eta \psi(x(1-\lambda), \mu)} \sum_{l=0}^L \frac{(-1)^l G_{l,\nu}(x(1-\lambda), \mu)}{(2\pi i \eta)^{l+1}} \right) + O(\lambda_k^{a(m+1)-\gamma}) + O(\lambda_k^{-m/n}). \end{aligned} \tag{3.14}$$

Combining (3.14) with (3.13), and applying Lemma 9 of [4], we obtain the stated result.

LEMMA 12. *If $c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon} < \lambda_k \leq c_0 x^{n\alpha-1}$, $\varepsilon > 0$, $\alpha > 1/n$, and, as hitherto, $a(\nu) = \omega_{-1} - \nu/n$, then*

$$\begin{aligned} & \int_x^\infty t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} u_{\nu,m} \cdot x^{a(\nu)-\gamma+1} \cdot \lambda_k^{a(\nu)} \cdot \frac{e^{2\pi i \eta \varphi(x)}}{(2\pi x^\alpha \eta)^{m+1}} \\ & \quad + \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} u'_{\nu,m} \cdot x^{a(\nu)-\gamma+1} \cdot \lambda_k^{a(\nu)} \cdot \frac{e^{2\pi i \eta x^\alpha P(v_0)}}{(2\pi x^\alpha \eta)^{m+(1/2)}} \\ & \quad + \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} u''_{\nu,m} x^{a(\nu)-\gamma+1} \cdot \lambda_k^{a(\nu)} \cdot \frac{e^{2\pi i \eta x^\alpha P(v_0)}}{(2\pi x^\alpha \eta)^m} \cdot \int_0^\tau s^{-1/2} \cdot e^{2\pi i x^\alpha \eta s} ds \\ & \quad + \sum_{\nu=0}^{nN} \sum_{m=0}^{N-1} \frac{(-1)^{m+1} G_{m,\nu}(x) \cdot e^{2\pi i \eta \psi(x)}}{(2\pi i \eta)^{m+1}} \cdot \lambda_k^{a(\nu)} \cdot + O(x^{a(\nu)-\gamma+1-\alpha(N+1)}), \end{aligned}$$

where $P(v) = v^{n\alpha} - \beta v$, $\beta = \mu x^{(1/n)-\alpha}$, $\tau = P(1) - v_0$, $v_0 = (\beta/n\alpha)^{1/(n\alpha-1)}$, and where the coefficients $u_{\nu,m}$, $u'_{\nu,m}$, $u''_{\nu,m}$ are continuous functions of β for fixed ν and m , while

$$\int_0^\tau s^{-1/2} \exp(2\pi i \eta x^\alpha s) ds = O(x^{-\alpha/2}),$$

uniformly in τ .

Proof. The pattern of proof here is the same as in Lemmas 16, 17 and 18 of

[4], with the difference that $P(v)$ is no longer a polynomial, but holomorphic in v for $\text{Re } v > 0$, and v_0 is a zero of $P'(v)$.

If $0 < \rho < 1$, then $\mu/\{x(1+\rho)\}^{\alpha-(1/n)} \leq n\alpha/(1+\rho)^{\alpha-(1/n)} \leq n\alpha - \delta_1$, say, so that, as in the proof of Lemma 8, we get

$$\int_{x(1+\rho)}^{\infty} t^{\alpha(v)-\gamma} \exp(2\pi i \eta \varphi(t)) dt = e^{2\pi i \eta \varphi(x(1+\rho))} \sum_{l=0}^L \frac{(-1)^{l+1} F_{l,\nu}(x(1+\rho))}{(2\pi i \eta)^{l+1}} + O(x^{\alpha(v)-\gamma+1-(L+1)\alpha}), \quad (3.15)$$

and a similar result holds with ψ in place of φ , and $G_{l,\nu}$ in place of $F_{l,\nu}$ for the entire integral \int_x^{∞} .

The integral $\int_x^{x(1+\rho)}$ is then dealt with as in Lemma 17 of [4], and gives

$$\begin{aligned} & \int_x^{x(1+\rho)} t^{\alpha(v)-\gamma} \exp(2\pi i \eta \varphi(t)) dt \\ &= \frac{1}{2} n x^{\alpha(v)-\gamma+1} \left\{ \sum_{p=0}^N \frac{A_p(1+\rho) \exp(2\pi i \eta \varphi(x(1+\rho))) - B_p(1) \exp(2\pi i \eta \varphi(x))}{(2\pi x^\alpha \eta)^{p+1}} \right. \\ & \quad \left. + \sum_{m=0}^N \frac{\alpha'_m \exp(2\pi_c x^\alpha \eta P(\nu_0))}{(2\pi x^\alpha \eta)^{m+(1/2)}} - \sum_{m=0}^N \frac{\alpha''_m}{(2\pi x^\alpha \eta)^m} \int_0^\tau s^{-1/2} e^{2\pi i \eta x^\alpha s} ds \right\} \\ & \quad + O(x^{\alpha(v)-\gamma+1-\alpha(N+1)}), \end{aligned} \quad (3.16)$$

where $A_p, B_p, \alpha'_m, \alpha''_m$ depend on β , and, for fixed p, m , are continuous functions of β in a neighbourhood of the point $\beta = n\alpha$.

We now combine (3.15) and (3.16), and apply Lemma 9 of [4] to get rid of the ρ from the expansions.

LEMMA 13. *If $c_0 x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1} + x^{n\alpha-1-\epsilon}$, $\epsilon > 0$, then*

$$\int_0^x u(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot L_{-1}(\lambda_k t) dt$$

has the same expansion as in Lemma 12.

Proof. We consider the integrals $\int_0^{x(1-\rho)}$ and $\int_{x(1-\rho)}^x$ separately, where $0 < \rho < 1$.

The first integral is dealt with as in Lemma 10, while the second is handled as in Lemma 12, and Lemma 9 of [4] is then called into play.

LEMMA 14. For $\alpha > 1/n$, and $0 \leq \gamma < 1$, the integral

$$\int_0^{\infty} u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt$$

has the asymptotic expansion

$$c_1(\alpha, \gamma, \eta, K) \sum_{\nu=0}^m m_k^{\theta - [\nu/(\alpha-1)]} \{ C_\nu \cos(qm_k^{1/(\alpha-1)} + k_\nu \pi) - i D_\nu \sin(qm_k^{1/(\alpha-1)} + k_\nu \pi) \} + O(m_k^{\theta - (m+1)/(\alpha-1)}),$$

where c_1 is a constant determined by Lemma 4, and $\theta = [2(1-\gamma) - \alpha n] / [2\alpha(\alpha-1)]$.

Proof. We have

$$\begin{aligned} & \int_0^{\infty} u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= \int_0^{\infty} t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &+ \int_0^{\lambda_1} (u_1(t) - 1) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \end{aligned}$$

The first integral on the right has an asymptotic expansion given by Lemma 4.

If we choose a ρ , such that $\rho < c$ (defined along with the function u at the beginning), then

$$\begin{aligned} & \int_0^{\lambda_1} (u_1(t) - 1) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= - \int_0^{\rho} t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &+ \int_{\rho}^{\lambda_1} (u_1(t) - 1) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \end{aligned}$$

Lemma 5 gives an asymptotic expansion of the first integral on the right-hand

side:

$$\sum_{p=0}^l \frac{a_p(\rho) \cdot I_p(\lambda_k \rho)}{(\lambda_k \rho)^{p+1}} + \sum_{p=0}^l \frac{b_p}{\lambda_k^{1-\gamma+\alpha p}} + O(\lambda_k^{-l/n}),$$

where $b_p = (2\pi i \eta)^p / p! \cdot \Delta(1 - \gamma + \alpha p) / \Delta(\gamma - \alpha p)$. For the second integral, we apply repeated integration by parts, together with the fact that $u_1 - 1$ (and all its derivatives) vanish at $t = \lambda_1$, to get an asymptotic expansion as in (3.12). By proper choice of l , the term $\sum_{p=0}^l b_p \cdot \lambda_k^{-1+\gamma-\alpha p}$ cancels out with the residual term given by Lemma 4, and the application of Lemma 9 of [4] leads to the stated result.

§4. Proofs of the theorems

Proof of Theorem 1. From Lemma 1 we have

$$\begin{aligned} \sum_{\lambda_k \leq x} a_k F(\lambda_k) &= \sum_{\lambda_k \leq x} a_k \cdot \lambda_k^{-\gamma} \cdot u(\lambda_k) \cdot \exp(2\pi i \eta \lambda_k^\alpha) \\ &= \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &\quad - \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k H_1(x, \lambda_k) + \sum_{c_0 x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1+\epsilon_0}} a_k H_1(x, \lambda_k) \\ &\quad + O\{x^{1-\alpha_1-\gamma} + x^\epsilon\}, \end{aligned} \tag{4.1}$$

for $\alpha > 1/n$, $\frac{1}{2} + (1/2n) - \alpha < \gamma < 1$, $\gamma \geq 0$, $c_0 = (2\pi \eta n \alpha / h)^n$, $h = n \cdot 2^{r_1/n}$. We shall estimate the first three terms on the right-hand side of (4.1). By Lemma 14, we have

$$\begin{aligned} &\int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= c_1(\alpha, \gamma, \eta, K) \cdot \lambda_k^{\alpha\theta} \exp\{-iqm_k^{1/(n\alpha-1)}\} + O(m_k^{\theta-1/(n\alpha-1)}), \end{aligned}$$

where $\theta = \{2(1 - \gamma) - \alpha n\} / \{2\alpha(n\alpha - 1)\}$. If $\gamma < \frac{1}{2}n\alpha$, then $\alpha\theta > -1$, and we obtain

$$\begin{aligned} &\sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \int_0^\infty u_1(t) \cdot t^{-\gamma} \cdot \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= c_1(\alpha, \gamma, \eta, K) \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{\alpha\theta} \cdot \exp\{-iqm_k^{1/(n\alpha-1)}\} + O(x^{[(n-1)/2]\alpha-\gamma}). \end{aligned} \tag{4.2}$$

For a $\delta > 0$, chosen sufficiently small, we then write

$$\begin{aligned} & \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k H_1(x, \lambda_k) \\ &= \sum_{\lambda_k \leq (c_0 - \delta)x^{n\alpha-1}} + \sum_{(c_0 - \delta)x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon}} + \sum_{c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon} < \lambda_k \leq c_0 x^{n\alpha-1}} \\ &= V_1 + V_2 + V_3, \quad \text{say.} \end{aligned}$$

In V_1 we use Lemma 8, and note that $\lambda_k \leq (c_0 - \delta)t^{n\alpha-1}$ for $0 < x \leq t \leq x_1$, so that

$$H(t, \lambda_k) \ll \lambda_k^{\omega-1} \cdot |F_{0,0}(t, \lambda_k)| \ll \lambda_k^{\omega-1} \cdot t^{\omega-1-\gamma+1-\alpha},$$

hence

$$\begin{aligned} V_1 &= \sum_{\lambda_k \leq (c_0 - \delta)x^{n\alpha-1}} a_k H_1(x, \lambda_k) \ll x^{\omega-1-\gamma+1-\alpha} \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \lambda_k^{\omega-1} \\ &\ll x^{\omega-1-\gamma+1-\alpha} \cdot x^{(n\alpha-1)(\omega-1)} \\ &\ll x^{[(n-1)/2]\alpha-\gamma}. \end{aligned} \tag{4.3}$$

(cf. the estimate of $W_{5,1}$ in [4]).

In V_2 we observe that $(c_0 - 2\delta)t^{n\alpha-1} < \lambda_k \leq c_0 t^{n\alpha-1} - t^{n\alpha-1-\varepsilon}$, and use Lemma 8, so that

$$H(t, \lambda_k) \ll \frac{\lambda_k^{\omega-1} \cdot t^{\omega-1-\gamma}}{|\varphi'(t)|}, \quad \text{for } 0 < x \leq t \leq x_1.$$

We have $\varphi'(t) = \alpha t^{\alpha-1} (1 - c_0^{-1/n} \lambda_k^{1/n} t^{(1/n)-\alpha})$, and in the given range

$$|1 - c_0^{-1/n} \lambda_k^{1/n} x^{(1/n)-\alpha}| \gg x^{-\varepsilon}.$$

However,

$$1 - c_0^{-1/n} \lambda_k^{1/n} t^{(1/n)-\alpha} = 1 - (c_0^{-1/n} \lambda_k^{1/n}) x^{(1/n)-\alpha} + O(x^{-\alpha}),$$

so that

$$\frac{1 - c_0^{-1/n} \cdot \lambda_k^{1/n} t^{(1/n)-\alpha}}{1 - c_0^{-1/n} \cdot \lambda_k^{1/n} x^{(1/n)-\alpha}} = 1 + O(x^{\varepsilon-\alpha}), \quad (\varepsilon < \frac{1}{2}\alpha).$$

Hence

$$H_1(x, \lambda_k) \ll \frac{\lambda_k^{\omega_{-1}} \cdot x^{\omega_{-1}-\gamma+1-\alpha}}{|1 - c_0^{-1/n} \lambda_k^{1/n} x^{(1/n)-\alpha}|},$$

so that

$$V_2 \ll x^{\omega_{-1}-\gamma+1-\alpha} \sum_{(c_0-\delta)x^{n\alpha-1} < \lambda_k \leq c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon}} \frac{a_k \cdot \lambda_k^{\omega_{-1}}}{|1 - c_0^{-1/n} \lambda_k^{1/n} \cdot x^{(1/n)-\alpha}|}.$$

The last sum is estimated in the same way as $W_{5,2}$ in [4], so as to yield

$$V_2 \ll x^{[(n-1)/2]\alpha-\gamma} \cdot \log(1+x). \tag{4.4}$$

In V_3 we observe that $c_0 t^{n\alpha-1} - 2t^{n\alpha-1-\varepsilon} < \lambda_k \leq c_0 t^{n\alpha-1}$, for $0 < x \leq t \leq x_1$, so that Lemma 12 applies, and we get

$$H_1(x, \lambda_k) \ll \lambda_k^{\omega_{-1}} \cdot x^{\omega_{-1}-\gamma+1-(\alpha/2)},$$

so that

$$\begin{aligned} V_3 &\ll x^{\omega_{-1}+1-\gamma-\alpha/2} \sum_{c_0 x^{n\alpha-1} - x^{n\alpha-1-\varepsilon} < \lambda_k \leq c_0 x^{n\alpha-1}} a_k \lambda_k^{\omega_{-1}}, \quad (\omega_{-1} < 0) \\ &\ll x^{\omega_{-1}+1-\gamma-\alpha/2} \cdot x^{\omega_{-1}(n\alpha-1)} \sum a_k, \quad a_k = O(k^{\varepsilon'}), \text{ say.} \\ &\ll x^{\varepsilon'+(n\alpha-1-\varepsilon)+\omega_{-1}+1-\gamma-(\alpha/2)+\omega_{-1}(n\alpha-1)} \\ &\ll x^{(\omega_{-1}+1)n\alpha-\gamma-(\alpha/2)-\varepsilon+\varepsilon'} \\ &\ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1}, \end{aligned} \tag{4.5}$$

where $\varepsilon_1 = \varepsilon - \varepsilon'$, for any positive ε , such that $\varepsilon < \frac{1}{2}\alpha$, $\varepsilon < n\alpha - 1$, and all $\varepsilon' > 0$, so that (4.5) holds for any $\varepsilon_1 < \alpha/2$, $\varepsilon_1 < n\alpha - 1$. From (4.3), (4.4), and (4.5) we obtain

$$\sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k H_1(x, \lambda_k) \ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1} + x^{[(n-1)/2]\alpha-\gamma} \log(1+x). \tag{4.6}$$

Next we consider

$$\begin{aligned} & \sum_{c_0x^{n\alpha-1} < \lambda_k \leq c_0x^{n\alpha-1-\varepsilon_0}} a_k H_1(x, \lambda_k) \\ &= \sum_{c_0x^{n\alpha-1} < \lambda_k \leq c_0x^{n\alpha-1} + x^{n\alpha-1-\varepsilon}} + \sum_{c_0x^{n\alpha-1} + x^{n\alpha-1-\varepsilon} < \lambda_k \leq (c_0+\delta)x^{n\alpha-1}} \\ &+ \sum_{(c_0+\delta)x^{n\alpha-1} < \lambda_k \leq x^{n\alpha-1+\varepsilon_0}} \\ &= V_4 + V_5 + V_6, \quad \text{say.} \end{aligned}$$

In V_6 we note that $t^{n\alpha-1+\varepsilon_0} \geq \lambda_k \geq (c_0 + \delta_1)t^{n\alpha-1}$, for $1 < x \leq t \leq x_1$, with $\delta_1 = \frac{1}{2}\delta$, say, so that $H(t, \lambda_k)$ can be estimated with the help of Lemma 10, leading to an estimate of $H_1(x, \lambda_k)$, and thence

$$V_6 \ll x^{[(n\alpha)/2] - \gamma - \varepsilon_1}, \quad \varepsilon_1 < \frac{1}{2}\alpha, \quad \varepsilon_1 < n\alpha - 1. \tag{4.7}$$

In V_5 we note that $c_0t^{n\alpha-1} = c_0x^{n\alpha-1} + O(x^{n\alpha-1-\alpha})$, for $1 < x \leq t \leq x_1$, so that $c_0t^{n\alpha-1} + \frac{1}{2}t^{n\alpha-1-\varepsilon} < \lambda_k \leq (c_0 + \delta)t^{n\alpha-1}$, and Lemma 11 can be used. We further note that, as in the case of V_2 , we can replace $\varphi'(t)$ by $\varphi'(x)$, and obtain in the same way

$$V_5 \ll x^{[(n-1)/2]\alpha - \gamma} \log(1+x). \tag{4.8}$$

In V_4 we consider the integral for $H(t)$ as the sum $\int_0^{t(1-\rho)} + \int_{t(1-\rho)}^t$, for a sufficiently small ρ , such that $0 < \rho < 1$. In the first integral, we have, for $x \leq t \leq x_1$,

$$\begin{aligned} c_0t^{n\alpha-1}(1-\rho)^{n\alpha-1} &\leq c_0(x_1(1-\rho))^{n\alpha-1} = c_0(x^{n\alpha-1}(1+O(x^{-\alpha}))) \cdot (1-\rho)^{n\alpha-1} \\ &\ll c_0x^{n\alpha-1} < \lambda_k, \end{aligned}$$

so that Lemma 10 can be applied to yield

$$H(t, \lambda_k) \ll \lambda_k^{\omega-1} \cdot t^{\omega-1+1-\gamma-\alpha},$$

while in the second integral, we have

$$\frac{\lambda_k}{c_0t^{n\alpha-1}} \geq \frac{\lambda_k}{c_0x^{n\alpha-1}(1+O(x^{-\alpha}))}, \quad \frac{\lambda_k}{c_0t^{n\alpha-1}} \leq \frac{\lambda_k}{c_0x^{n\alpha-1}} \leq 1 + O(x^{-\varepsilon}).$$

Here Lemma 12 can be applied to obtain (as in V_3)

$$H(t, \lambda_k) \ll \lambda_k^{\omega-1} t^{\omega-1+1-\gamma-(\alpha/2)}.$$

Altogether, we therefore have

$$H_1(x, \lambda_k) \ll \lambda_k^{\omega-1} \cdot x^{\omega-1+1-\gamma-(\alpha/2)},$$

and

$$V_4 \ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we obtain

$$\sum_{c_0 x^{n\alpha-1} < \lambda_k < c_0 x^{n\alpha-1+\varepsilon_0}} a_k H_1(x, \lambda_k) \ll x^{[(n\alpha)/2]-\gamma-\varepsilon_1} + x^{[(n-1)/2]\alpha-\gamma} \cdot \log(1+x). \tag{4.10}$$

If we use (4.10), (4.6), and (4.2) in (4.1), we obtain:

$$\begin{aligned} & \sum_{\lambda_k \leq x} a_k \cdot \lambda_k^{-\gamma} \cdot \exp(2\pi i \eta \lambda_k^\alpha) \\ &= c_1(\alpha, \gamma, \eta, K) \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{\alpha\theta} \cdot \exp\{-iqm_k^{1/(n\alpha-1)}\} + O(x^{[(n\alpha)/2]-\gamma-\alpha}) \\ & \quad + O(x^{[(n-1)/2]\alpha-\gamma} \log(1+x)) + O(x^{[(n\alpha)/2]-\gamma-\varepsilon_1}) + O(x^{1-\alpha_1-\gamma}) + O(x^\varepsilon). \end{aligned} \tag{4.11}$$

Since $\varepsilon_1 < \frac{1}{2}\alpha$, the exponent $\frac{1}{2}(n\alpha) - \gamma - \varepsilon_1 > 0$, if $\frac{1}{2}(n\alpha) - \gamma - \frac{1}{2}\alpha > 0$, that is, if $\gamma < \frac{1}{2}(n-1)\alpha$, in which case the term x^ε can be dropped. Such a choice of γ can indeed be made. If $\alpha > \frac{1}{2} + (1/2n)$, we choose $\gamma = 0$. Otherwise $\frac{1}{2} + (1/2n) - \alpha < \frac{1}{2}(n-1)\alpha$ for $\alpha > 1/n$, so that if γ is greater than $\frac{1}{2} + (1/2n) - \alpha$ and close enough to it, we have $\frac{1}{2}(n-1)\alpha > \gamma$. Clearly the only O -terms that remain in (4.11) are:

$$O(x^{[(n\alpha)/2]\alpha-\gamma} \log(1+x)) + O(x^{[(n\alpha)/2]-\gamma-\varepsilon_1}) + O(x^{1-\alpha_1-\gamma}). \tag{4.12}$$

Thus (4.11) can be rewritten as

$$\begin{aligned} & \sum_{\lambda_k \leq x} a_k \cdot \lambda_k^{-\gamma} \cdot \exp(2\pi i \eta \lambda_k^\alpha) \\ &= c_1 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{-[\gamma/(n\alpha-1)] + [(2-\alpha n)/2(n\alpha-1)]} \exp\{-iqm_k^{1/(n\alpha-1)}\} \\ & \quad + O(x^{[(n\alpha)/2] - \gamma - \varepsilon_1}) + O(x^{[(n\alpha)/2]\alpha - \gamma} \log(1+x)) + O(x^{1-\alpha_1-\gamma}). \end{aligned} \tag{4.13}$$

Since $\gamma < \frac{1}{2}(n\alpha)$, the exponent of λ_k is greater than -1 , and we can apply partial summation, and obtain

$$\begin{aligned} & \sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^\alpha) \\ &= c_2 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{(2-\alpha n)/2(n\alpha-1)} \cdot \exp\{-iqm_k^{1/(n\alpha-1)}\} \\ & \quad + O(x^{[(n\alpha)/2] - \varepsilon_1}) + O(x^{[(n-1)/2]\alpha + \varepsilon}) + O(x^{1-\alpha_1}), \end{aligned} \tag{4.14}$$

where $c_2 = c_1 \cdot c_0^{-\gamma/(n\alpha-1)}$ is independent of γ , for all $\varepsilon_1 > 0$, such that $\varepsilon_1 < \frac{1}{2}\alpha$, $\varepsilon_1 < n\alpha - 1$. These conditions on ε_1 imply that

$$x^{[(n\alpha)/2] - \varepsilon_1} \ll x^{[(n-1)/2]\alpha + \varepsilon} + x^{1-(n\alpha/2) + \varepsilon}$$

Hence (4.14) leads to

$$\begin{aligned} & \sum_{\lambda_k \leq x} a_k \cdot \exp(2\pi i \eta \lambda_k^\alpha) \\ &= c_2 \sum_{\lambda_k \leq c_0 x^{n\alpha-1}} a_k \cdot \lambda_k^{(2-\alpha n)/2(n\alpha-1)} \exp\{-iqm_k^{1/(n\alpha-1)}\} \\ & \quad + O(x^{[(n-1)/2]\alpha + \varepsilon}) + O(x^{1-(n\alpha/2) + \varepsilon}) + O(x^{1-\alpha_1}). \end{aligned} \tag{4.15}$$

If $n \geq 3$, then $\frac{1}{2}(n\alpha) > \alpha \geq \alpha_1$, so that the term $O(x^{1-(n\alpha/2) + \varepsilon})$ may be dropped. If $n = 2$, the only O -terms in (4.15) are $O(x^{(\alpha/2) + \varepsilon}) + O(x^{1-\alpha + \varepsilon})$, and that completes the proof of Theorem 1.

Proof of Theorem 2. If $\alpha = 2/n$, (4.15) gives the approximate reciprocity

formula:

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^{2/n}) = c_3 \sum_{\lambda_k \leq c_0 x} a_k \exp\left(\frac{-iq \lambda_k^{2/n}}{2\pi \eta}\right) + O(x^{1-(1/n)+\epsilon}), \tag{4.16}$$

for every $\epsilon > 0$.

In this special case, we have

$$c_0 = \left(\frac{4\pi\eta}{h}\right)^n, \quad h = n2^{r_1/n}, \quad c_3 = \left(\frac{h}{4\pi\eta}\right)^{n/2} e^{i\pi[(1/2)-(r_1/4)]}, \quad q = \frac{h^2}{4},$$

while, in general, $\lambda_k = B \cdot k$, $B = 2^{r_2} \pi^{n/2} |D|^{-1/2} = (2\pi n/h)^{n/2} \cdot |D|^{-1/2}$.

If we choose $4\pi\eta = h$, then $c_0 = 1$, $c_3 = e^{i\pi[(1/2)-(r_1/4)]}$. Setting $Y = x \cdot B^{-1}$, we get:

$$\sum_{k \leq Y} a_k \cdot e^{i\pi n |D|^{-1/n} k^{2/n}} = e^{i\pi[(1/2)-(r_1/4)]} \sum_{k \leq Y} a_k \cdot e^{-i\pi n |D|^{-1/n} k^{2/n}} + O(Y^{1-(1/n)+\epsilon}),$$

which is (1.4).

Similarly if we choose $4\pi\eta = h |D|^{-1/n} \cdot m$, where m is an integer, $m \neq 0$, then we get a formula which, in the case $n = 2$, gives the Corollary to Theorem 1 in [3] with Y^ϵ in place of $\log Y$.

If we take $n = 2$ in (4.16), so that $\alpha = 1$, and $r_1 = 2$ (this is the case when K is a real quadratic field), then $c_0 = (\pi\eta)^2$, $c_3 = (\pi\eta)^{-1}$, $q = 4$; if $n = 2$, $\alpha = 1$, $r_1 = 0$ (the case of an imaginary quadratic field), then $c_0 = (2\pi\eta)^2$, $c_3 = i(2\pi\eta)^{-1}$, $q = 1$. Formula (4.16) then reduces to the one which we proved sometime ago [3] by a different method, but with $\log x$ taking the place of x^ϵ in the error-term.

Proof of Theorem 3. The sum on the right-hand side of (4.15) is $O(x^{(n\alpha)/2})$, for $\alpha < 2/n$. We can therefore conclude that if $1/n < \alpha < 2/n$, and $n \geq 3$, then

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^\alpha) = O(x^{(n\alpha)/2}) + O(x^{1-\alpha_1}), \tag{4.17}$$

with $\alpha_1 = \alpha$ if $\alpha \leq 2/(n+1)$, and $\alpha_1 = \alpha - \epsilon < \alpha$, if $\alpha > 2/(n+1)$. If $n = 2$, and $\frac{1}{2} < \alpha < 1$, the sum is $O(x^\alpha)$, since $1 - \alpha < \alpha$. If $n \geq 3$, and we take $\alpha = 1/(n-1)$ in

(4.17), we get

$$\begin{aligned} \sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^{1/(n-1)}) &= O(x^{n/2(n-1)}) + O(x^{1-1/(n-1)}) \\ &= \begin{cases} O(x^{3/4}), & \text{if } n=3; \\ O(x^{1-1/(n-1)}), & \text{if } n \geq 4. \end{cases} \end{aligned} \quad (4.18)$$

We note that if $n \geq 3$, (4.18) gives a stronger result than the one we obtained before [4].

Proof of Theorems 4 and 5. If $0 < \alpha < 1/n$, the proof is much simpler than if $\alpha > 1/n$. Proceeding as in Lemma 1, we consider here the infinite series

$$\begin{aligned} (-1)^{r+1} \sum_{k=1}^{\infty} a_k \cdot \lambda_k^{-1-r} \int_c^{x_1} F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt, \quad \gamma=0, \quad F(t) = u(t) \cdot e^{2\pi i \eta t^\alpha}. \\ \ll \sum_{k=1}^{\infty} a_k \cdot \lambda_k^{-(1/2)-(1/2n)-(r/n)} = O(1), \end{aligned}$$

if r is chosen sufficiently large. On the other hand,

$$\int_0^\infty F^{(r+1)}(t) \cdot Q_r(t) dt = O(x^{1-\alpha}), \quad (4.18)'$$

as before, while $\sum_{x < \lambda_k \leq x_1} a_k = O(x^{1-\alpha})$. Hence

$$\sum_{\lambda_k \leq x} a_k \exp(2\pi i \eta \lambda_k^\alpha) = O(x^{1-\alpha}), \quad \text{for } 0 < \alpha < \frac{1}{n}.$$

If $\alpha = 1/n$, and $\varepsilon > 0$, we first obtain the estimate

$$\begin{aligned} (-1)^{r+1} \sum_{\lambda_k > x^\varepsilon} a_k \cdot \lambda_k^{-1-r} \int_c^{x_1} F^{(r+1)}(t) \cdot I_r(\lambda_k t) dt \\ = O(x^{\varepsilon[(1/2)-(1/2n)-(r/n)]+(1/2)+(1/2n)}) = O(x^{-\varepsilon}) \end{aligned} \quad (4.19)$$

for any $q > 0$, provided that r is chosen sufficiently large. We next consider

$$\begin{aligned} & \sum_{\lambda_k \leq x^e} a_k \int_c^{x_1} F(t) \cdot I_{-1}(\lambda_k t) dt, \quad F(t) = u(t) \cdot \exp(2\pi i \eta t^\alpha) \\ &= \sum_{\lambda_k \leq x^e} a_k \int_c^{x_1} u(t) \cdot dH(t, \lambda_k), \quad H(t, \lambda_k) = \int_c^t \exp(2\pi i \eta t^\alpha) \cdot I_{-1}(\lambda_k t) dt \\ &= - \sum_{\lambda_k \leq x^e} a_k \int_c^{x_1} H(t, \lambda_k) \cdot u'(t) dt, \end{aligned}$$

since $H(t, \lambda_k)$ vanishes for $t = c$, while $u(t)$ vanishes for $t = x_1$. Now

$$\int_c^{x_1} H(t, \lambda_k) u'(t) dt = \left(\int_c^{\lambda_1} + \int_{\lambda_1}^{x_1} \right) H(t, \lambda_k) u'(t) dt,$$

and the first integral on the right-hand side is $\ll \lambda_k^{\omega-1} = O(1)$, since $I_{-1}(\lambda_k t) \ll \lambda_k^{\omega-1}$, for $c \leq t \leq \lambda_1$, while

$$\int_x^{x_1} H(t, \lambda_k) u'(t) dt \ll \sup_{x \leq t \leq x_1} |H(t, \lambda_k)|.$$

We shall estimate the order of magnitude of $H(t, \lambda_k)$ by using the full asymptotic expansion of $I_{-1}(\lambda_k t)$, (cf. Lemma 3). We then have to consider integrals of the form

$$\int_c^y t^{-(1/2)+(1/2n)-(\nu/n)} \exp\{2\pi i t^{1/n}(\eta - \lambda_k^{1/n} \cdot h)\} dt, \quad \nu = 0, 1, 2, \dots,$$

together with an O -term of the order $O(\lambda_k^{-(1/2)+(1/2n)-[(\nu+1)/n]})$, if $-(1/2)+(1/2n)-[(\nu+1)/n] < -1$. It is sufficient to consider the case $\nu = 0$. If $\eta = \lambda_{k_0}^{1/n} \cdot h$ for some $k_0 \geq 1$, we have

$$\begin{aligned} H(y, \lambda_{k_0}) &= \lambda_{k_0}^{\omega-1} \left\{ c_{k_0} \int_c^y t^{-(1/2)+(1/2n)} dt + O\left(\int_c^y t^{-[(1/2)+(1/2n)]} dt \right) \right\} \\ &= c'_{k_0} y^{(n+1)/2n} + O(y^{(n-1)/2n}), \end{aligned}$$

while

$$H(y, \lambda_k) \ll \lambda_k^{\omega-1} y^{(n-1)/2n}, \quad \text{for } k \neq k_0,$$

(which follows by replacing $t^{1/n}$ by t and integrating by parts). Hence, if $n \geq 3$,

$$\begin{aligned}
 - \sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} H(t, \lambda_k) \cdot u'(t) dt &\ll x^{(n+1)/2n} + \sum_{\lambda_k \leq x^\varepsilon, k \neq k_0} a_k \cdot \lambda_k^{\omega-1} x^{(n-1)/2n} \\
 &\ll x^{(n+1)/2n} \ll x^{1-(1/n)}.
 \end{aligned}
 \tag{4.20}$$

If $n = 2$, and $\eta = h \cdot \lambda_{k_0}^{1/2}$, we have

$$\begin{aligned}
 \int_c^{x_1} H(t, \lambda_{k_0}) u'(t) dt &= c_{k_0} \lambda_{k_0}^{\omega-1} \int_c^{x_1} t^{3/4} u'(t) dt + O(x^{1/4}) \\
 &= c_{k_0} \lambda_{k_0}^{\omega-1} \int_x^{x_1} t^{3/4} u'(t) dt + O(x^{1/4}) \\
 &= -c_{k_0} \lambda_{k_0}^{\omega-1} \left\{ x^{3/4} + \frac{3}{4} \int_x^{x_1} u(t) \cdot t^{-1/4} dt \right\} + O(x^{1/4}) \\
 &= -c_{k_0} \lambda_{k_0}^{\omega-1} x^{3/4} + O(x^{1/4}).
 \end{aligned}$$

Hence, in this case,

$$- \sum_{\lambda_k \leq x^\varepsilon} a_k \int_c^{x_1} H(t, \lambda_k) \cdot u'(t) dt = c_{k_0} \cdot a_{k_0} \cdot \lambda_{k_0}^{\omega-1} \cdot x^{3/4} + O(x^{(1/4)+\varepsilon}),
 \tag{4.21}$$

while, if $\eta \neq h \lambda_{k_0}^{1/2}$, for all $k \geq 1$, the sum is $O(x^{1/2})$. The result follows from this and (4.18)'.

Remark. The method of proof adopted here makes it possible to prove corresponding results for the coefficient-sums of Dirichlet series satisfying a functional equation of the type studied in [1, 2]. Particular cases are the zeta-function of an ideal class and Hecke's zeta-function with Grössencharacters. Since no new ideas are required, we do not go into the details.

Our estimates prove that the sequence $\{\eta(N\mathfrak{A})^\alpha\}$, for $0 < \alpha < 2/n$, η real, $\eta \neq 0$, and or an integral ideal in K with norm $N\mathfrak{A}$, when arranged in order of increasing norms, is uniformly distributed modulo 1. It is likely that this is true for $0 < \alpha < 1$.

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