Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 54 (1979)

Artikel: Abelian group extensions and the axiom of constructibility.

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DOI: https://doi.org/10.5169/seals-41589

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Abelian group extensions and the axiom of constructibility

by Paul C. Eklof and Martin Huber*

Introduction

Throughout this paper the word "group" will mean "abelian group". The results of Shelah's remarkable work on Whitehead's problem ($[Sh_1]$, $[Sh_2]$) suggested the investigation of the structure of $Ext(A, \mathbb{Z})$ for torsion-free A under the hypothesis of the Axiom of Constructibility, V = L. Applying Shelah's methods, H. Hiller, Shelah and the second-named author obtained a surprisingly simple description of the torsion-free part of $Ext(A, \mathbb{Z})$ in terms of A[H-H-S].

In this paper we study, in the same spirit, the group $\operatorname{Ext}(A,G)$ in the case where A is torsion-free and G is any group satisfying suitable cardinality conditions. We are interested in characterizing pairs (A,G) such that $\operatorname{Ext}(A,G)=0$ as well as in determining the structure of $\operatorname{Ext}(A,G)$. Herein we restrict our attention to its torsion-free part. Since in our case $\operatorname{Ext}(A,G)$ is always divisible, the structure of its torsion-free part is completely determined by its torsion-free rank. (Following $[F_1]$ we denote the torsion-free rank of a group B by $r_0(B)$.)

Our first task is to settle the case where A is countable. For this, of course, we do not need any additional axiom of set theory. We assume G to be a group of countable torsion-free rank, thus unifying the known cases $G = \mathbb{Z}[J, \S 2]$ and G = T, a torsion group ($[B_1]$, $[B_2]$). In Section 1 we consider the crucial case where A is of rank 1. For such A we give a group-theoretical characterization of pairs (A, G) such that $\operatorname{Ext}(A, G) = 0$ and show that $\operatorname{Ext}(A, G) \neq 0$ implies $r_0(\operatorname{Ext}(A, G)) \geq 2^{\aleph_0}$ (Theorem 1.2). In Section 2 we study $\operatorname{Ext}(A, G)$ in case A is any coutable torsion-free group. Applying Theorem 1.2 we obtain various conditions that are necessary and (or) sufficient for the vanishing of $\operatorname{Ext}(A, G)$ (Theorems 2.1 and 2.6, Corollaries 2.4 and 2.7). In particular we have the following analogue of Pontryagin's criterion: If $\operatorname{Ext}(B, G) = 0$ for every subgroup B of A of finite rank, then $\operatorname{Ext}(A, G) = 0$ (Corollary 2.7). Using Theorem 1.2 we conclude that also in this case, $\operatorname{Ext}(A, G) \neq 0$ implies $r_0(\operatorname{Ext}(A, G)) \geq 2^{\aleph_0}$ (Theorem 2.8).

^{*} Research of the first author partially supported by NSF grant MCS76-12014.

Section 3 is devoted to the vanishing of $\operatorname{Ext}(A,G)$ for uncountable A. From now on we have to assume V=L in order to be able to apply Shelah's methods. The main theorem of this section (Theorem 3.2) generalizes earlier results of the first-named author (see $[E_3]$). In particular it contains the following singular compactness theorem for $\operatorname{Ext}:(V=L)$. Let A be a group of singular cardinality κ and let G be a group of cardinality $<\kappa$. If $\operatorname{Ext}(B,G)=0$ for every subgroup B of A of cardinality $<\kappa$, then $\operatorname{Ext}(A,G)=0$. The proof of this is based on a new version $[\operatorname{Sh}_3]$ of the principal result of $[\operatorname{Sh}_2]$. Among other consequences of Theorem 3.2 we deduce a vanishing result for $\operatorname{Ext}(A,G)$ (Theorem 3.7) which corresponds to a theorem of Hill $[\operatorname{Hi}]$.

The final section deals with the structure of Ext (A, G) for uncountable A. We show that the main result of [H-H-S] generalizes to our situation; we proceed along the lines of that proof. Theorem 4.5 may be viewed as the principal result of Section 4: (V=L). Let A be torsion-free and G of countable torsion-free rank such that $\operatorname{Ext}(A, G) \neq 0$. Suppose that B is a pure subgroup of A and that $\operatorname{Ext}(A/B, G) = 0$. If B is of minimal cardinality, then $r_0(\operatorname{Ext}(A, G)) \geq 2^{|B|}$. (As usual |B| denotes the cardinality of B.) The case where A is of singular cardinality relies on a variant of Theorem 3.2 which is of interest in its own right (Theorem 4.3). Finally we deduce some corollaries concerning the torsion-free rank of $\operatorname{Ext}(A, G)$ which extend results of [H-H-S] and [Hu].

1. The rank one case

In this section we investigate the group Ext(A, G) in case A is torsion-free of rank 1 and G any group of (at most) countable torsion-free rank.

We first recall some definitions and known facts and state certain exact sequences which will be important tools in the proof of Theorem 1.2. Given a prime p, we denote the p-primary part of a group G by t_pG and the torsion subgroup of G by tG. The p-primary part of \mathbb{Q}/\mathbb{Z} is denoted by $\mathbb{Z}(p^{\infty})$ and its full preimage in \mathbb{Q} by $\mathbb{Q}^{(p)}$. A group G is called p-divisible if pG = G; G is divisible if it is p-divisible for every prime p. A group which does not contain any nontrivial divisible subgroup is called reduced. It is well known that every group is the direct sum of its maximal divisible subgroup and a reduced group.

Let A be a torsion-free group of rank 1. For a nonzero element $x \in A$ and any prime p let $h_p(x)$ be the largest integer k such that p^k divides x if it exists, or $h_p(x) = \infty$ otherwise; $h_p(x)$ is called the p-height of x. Suppose that for every p we are given k_p which is either a nonnegative integer or ∞ . Then there exists a nonzero $y \in A$ such that for every p, $h_p(y) = k_p$ if and only if the following two

conditions hold:

$$k_p \neq h_p(x)$$
 only for finitely many p's; (1.1a)

$$k_p = h_p(x)$$
 whenever $k_p = \infty$ or $h_p(x) = \infty$ (1.1b)

(see e.g. $(F_2, \S 85]$). By definition of the *p*-heights we can associate with every nonzero $x \in A$ a short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\mu} A \to \bigoplus_{p} \mathbf{Z}(p^{k_p}) \to 0, \tag{1.2}$$

where μ is given by $\mu(1) = x$ and $k_p = h_p(x)$ for every p. Here $k_p = 0$ means that the p-primary part does not occur. For any group G this sequence induces a long exact sequence

$$0 \to \prod_{p} \operatorname{Hom} \left(\mathbf{Z}(p^{k_{p}}), G \right) \to \operatorname{Hom} \left(A, G \right) \to G \to$$

$$\to \prod_{p} \operatorname{Ext} \left(\mathbf{Z}(p^{k_{p}}), G \right) \to \operatorname{Ext} \left(A, G \right) \to 0.$$
(1.3)

In particular we shall make use of the sequence

$$0 \to \operatorname{Hom}(\mathbf{Z}(p^{\infty}), G) \to \operatorname{Hom}(\mathbf{Q}^{(p)}, G) \to G \to$$

$$\to \operatorname{Ext}(\mathbf{Z}(p^{\infty}), G) \to \operatorname{Ext}(\mathbf{Q}^{(p)}, G) \to 0$$
(1.4)

which is induced by $0 \to \mathbb{Z} \to \mathbb{Q}^{(p)} \to \mathbb{Z}(p^{\infty}) \to 0$.

The following facts will be applied several times; therefore we state them as a lemma.

LEMMA 1.1. (a) For any group G, $\operatorname{Ext}(\mathbf{Z}(p^n), G) \cong G/p^nG$. (b) $\operatorname{Ext}(\mathbf{Z}(p^\infty), G) = 0$ if and only if G is p-divisible.

Proof. Statement (a) is well-known (see e.g. $[F_1](D)$, p. 222) while (b) follows from Corollary 4.3 and Theorem 4.5 of $[N_1]$.

Finally we assign to every group G two sets of primes

$$D_1(G) = \{p \mid pG \neq G\}$$
 and
$$D_2(G) = \{p \mid p^{k+1}G \neq p^kG \text{ for all } k\}.$$

We are now ready to state the result of this section.

THEOREM 1.2. Let A be a torsion-free group of rank 1 and let G be any group of countable torsion-free rank. Then

- (a) Ext (A, G) = 0 if and only if for any nonzero $x \in A$ the following conditions hold:
 - (1) $\{p \in D_1(G) \mid h_p(x) \neq 0\}$ is finite;
 - (2) for all $p \in D_2(G)$, $h_p(x) < \infty$.
- (b) If Ext $(A, G) \neq 0$, then the torsion-free rank of Ext (A, G) is $\geq 2^{\aleph_0}$.

Remarks

- 1) Combining (a) with (1.1a, b) we obtain that $\operatorname{Ext}(A, G) = 0$ if and only if there is a nonzero $x \in A$ such that (1) and (2') are satisfied, where (2') means that $h_p(x) = 0$ for all $p \in D_2(G)$.
- 2) Let G be a group satisfying conditions (1) and (2) for any nonzero $x \in \mathbb{Q}$. It is not hard to see that such a G is the direct sum of a divisible group and a bounded torsion group. On the other hand, there are *cotorsion* groups G, i.e. groups satisfying $\operatorname{Ext}(\mathbb{Q}, G) = 0$, (of uncountable torsion-free rank) that are not of this form (cf. $[F_1]$, §55). We conclude that the countability hypothesis on G in statement (a) cannot be dropped.
- 3) In (b) the hypothesis on G cannot be omitted either. By [M-V, p. 119] there are groups A and G, A torsion-free of rank 1 and G torsion-free of rank 2^{\aleph} , such that $\operatorname{Ext}(A, G) \cong \bigoplus \mathbf{Q}$.

Proof of Theorem 1.2. We observe that it suffices to prove the following two statements:

- (a') If there is a nonzero $x \in A$ such that (1) and (2) are satisfied, then $\operatorname{Ext}(A, G) = 0$.
- (b') If there is a nonzero $x \in A$ such that (1) or (2) does not hold, then $r_0(\operatorname{Ext}(A, G)) \ge 2^{\aleph_0}$.

Proof of (a'): We consider the associated exact sequence (1.3). Since $\operatorname{Ext}(A,G)$ is divisible, the assertion follows if we can show that $\prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}),G)$ is divisible, the assertion follows if we can show that $\prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}),G)$ is a bounded torsion group. By conditin (1) and Lemma 1.1 $\operatorname{Ext}(\mathbf{Z}(p^{k_p},G))$ nonzero only for a finite set of primes, say I. Thus it remains to show that for all $p \in I$, $\operatorname{Ext}(\mathbf{Z}(p^{k_p}),G)$ is a bounded torsion group. If $k_p < \infty$ this is obvious. In case $k_p = \infty$ condition (2) implies that $p^{k+1}G = p^kG$ for some k. Therefore by Lemma 1.1(b) the first term of the exact sequence

$$\operatorname{Ext}(\mathbf{Z}(p^{\infty}), p^{k}G) \to \operatorname{Ext}(\mathbf{Z}(p^{\infty}), G) \to \operatorname{Ext}(\mathbf{Z}(p^{\infty}), G/p^{k}G) \to 0$$

is trivial. Consequently, $\operatorname{Ext}(\mathbf{Z}(p^{\infty}), G)$ is a bounded torsion group. This completes the proof of (a').

Proof of (b'): Suppose first that (1) does not hold. Again we consider the associated exact sequence (1.3). By hypothesis and Lemma 1.1 the product $\prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}), G)$ has infinitely many nontrivial factors. Therefore it has a quotient isomorphic to $\prod_{p \in J} \mathbf{Z}(p)$ for some infinite set of primes J. It follows that $r_0(\prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}), G)) \ge 2^{\aleph_0}$. As $r_0(G)$ is countable, we see from (1.3) that $r_0(\operatorname{Ext}(A, G)) \ge 2^{\aleph_0}$ as well.

Now suppose that x does not satisfy (2). So there is a prime p such that A contains a copy of $\mathbf{Q}^{(p)}$ and the chain

$$G \supseteq pG \supseteq \cdots \supseteq p^kG \supseteq \cdots$$

is properly descending. Since $\operatorname{Ext}(\mathbf{Q}^{(p)},G)$ is then an epimorphic image of $\operatorname{Ext}(A,G)$, it suffices to show that $r_0(\operatorname{Ext}(\mathbf{Q}^{(p)},G)) \ge 2^{\aleph_0}$. For this purpose we distinguish two cases. First assume that G/tG is not p-divisible. Then its p-adic completion $(G/tG)_p^{\wedge}$ is torsion-free of cardinality 2^{\aleph_0} . But by $[N_1, p. 233]$, $(G/tG)_p^{\wedge}$ is an epimorphic image of $\operatorname{Ext}(\mathbf{Z}(p^{\infty}), G/tG)$; hence $r_0(\operatorname{Ext}(\mathbf{Z}(p^{\infty}), G/tG)) \ge 2^{\aleph_0}$. Using the sequence (1.4) for G/tG, we conclude that $r_0(\operatorname{Ext}(\mathbf{Q}^{(p)}, G/tG)) \ge 2^{\aleph_0}$, and hence $r_0(\operatorname{Ext}(\mathbf{Q}^{(p)}, G)) \ge 2^{\aleph_0}$.

In the second case assume that G/tG is p-divisible. Then the reduced part of t_pG must be unbounded. Therefore, by the main result of [Sz], there is an epimorphism

$$tG \longrightarrow \bigoplus_{k < \omega} \mathbf{Z}(p^k) = H.$$

We claim that $r_0(\operatorname{Ext}(\mathbf{Q}^{(p)}, H)) = 2^{\aleph_0}$. By Lemma 1 of [R] we have $|\operatorname{Ext}(\mathbf{Z}(p^{\infty}), H)| = 2^{\aleph_0}$. Thus, using exactness of (1.4) for G = H, we conclude that $|\operatorname{Ext}(\mathbf{Q}^{(p)}, H)| = 2^{\aleph_0}$. But $\operatorname{Ext}(\mathbf{Q}^{(p)}, H)$ is torsion-free, hence the claim is proved. Now $\operatorname{Ext}(\mathbf{Q}^{(p)}, H)$ is an epimorphic image of $\operatorname{Ext}(\mathbf{Q}^{(p)}, tG)$, and the latter fits into an exact sequence

$$\operatorname{Hom}(\mathbf{Q}^{(p)}, G/tG) \to \operatorname{Ext}(\mathbf{Q}^{(p)}, tG) \to \operatorname{Ext}(\mathbf{Q}^{(p)}, G).$$

As Hom $(\mathbf{Q}^{(p)}, G/tG)$ is countable, it follows that $r_0(\operatorname{Ext}(\mathbf{Q}^{(p)}, G)) \ge 2^{\aleph_0}$ also in this case. This completes the proof of Theorem 1.2.

2. The countable case

Throughout this section G denotes a given group of countable torsion-free rank. We now study the group $\operatorname{Ext}(A, G)$ in case A is any countable torsion-free group. We start with

THEOREM 2.1. For any countable torsion-free group A the following statements are equivalent:

- (a) Ext (A, G) = 0;
- (b) A is the union of an ascending chain of pure subgroups $\{A_n \mid n < \dot{\omega}\}$ such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \le 1$ and $\text{Ext}(A_{n+1}/A_n, G) = 0$.

For the proof of this theorem we need the following auxiliary result.

LEMMA 2.2. Suppose that G is reduced. If A is torsion-free of finite rank such that Ext(A, G) = 0, then the torsion-free rank of Hom(A, G) is countable.

Proof. We proceed by induction on the rank of A. Suppose first that A is of rank 1 and let $x \in A$, $x \ne 0$. Then we consider the associated exact sequences (1.2) and (1.3). Since G is reduced, we have $\operatorname{Hom}(\mathbf{Z}(p^{k_p}), G) = 0$ if $k_p = \infty$ or if pG = G. It remains to consider those primes for which $pG \ne G$ and $k_p < \infty$. But we know from Theorem 1.2(a) that $k_p \ne 0$ only for a finite number of them. Therefore $\prod_p \operatorname{Hom}(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group, and hence exactness of (1.3) implies that $r_0(\operatorname{Hom} A, G)$) is countable.

Now assume that the lemma holds for all torsion-free groups of rank $\leq n$. Let A be torsion-free of rank n+1 such that $\operatorname{Ext}(A, G) = 0$, and let B be a pure subgroup of A of rank n. Then there is an exact sequence

$$0 \rightarrow \operatorname{Hom}(A/B, G) \rightarrow \operatorname{Hom}(A, G) \rightarrow \operatorname{Hom}(B, G) \rightarrow \operatorname{Ext}(A/B, G) \rightarrow 0.$$

As by induction hypothesis $r_0(\text{Hom }(B,G))$ is countable, we conclude that $r_0(\text{Ext }(A/B,G))$ is countable too. But then we have Ext (A/B,G)=0 by Theorem 1.2(b). Therefore by the first part, $r_0(\text{Hom }(A/B,G))$ is countable; hence r (Hom (A,G)) is countable as well.

Proof of Theorem 2.1. The implication (b) \Rightarrow (a) is a special case of $[E_2, Theorem 1.2]$. Conversely, suppose that A is a countable torsion-free group such that Ext(A, G) = 0. Let A be represented as the union of an ascending chain of pure subgroups $\{A_n \mid n < \omega\}$ such that $A_0 = 0$ and $r_0(A_{n+1}/A_n) \le 1$. Clearly such a chain of subgroups exists. Then we have $Ext(A_n, G) = 0$ for all n, and therefore

there are exact sequences

$$\operatorname{Hom}(A_n,G) \to \operatorname{Ext}(A_{n+1}/A_n,G) \to 0.$$

Note that we may assume G to be reduced. Thus $r_0(\text{Hom }(A_n, G))$ is countable by Lemma 2.2 and hence $r_0(\text{Ext }(A_{n+1}/A_n, G))$ is countable too. But then by Theorem 1.2(b) Ext $(A_{n+1}/A_n, G) = 0$ for all n. This completes our proof.

Theorems 1.2 and 2.1 provide a number of interesting consequences. The first generalizes Stein's theorem (see e.g. $[E_1, Theorem 4.1]$).

COROLLARY 2.3. Suppose that G is countable torsion-free and for all primes p, G is not p-divisible. If A is any group of countable torsion-free rank, then $\operatorname{Ext}(A, G) = 0$ implies A free.

Proof. First it follows from $[N_1$, Theorem 4.5] that for any A, $\operatorname{Ext}(A, G) = 0$ implies A torsion-free. Therefore, if A is of countable rank, we may apply Theorem 2.1. Hence A is the union of an ascending chain $\{A_n \mid n < \omega\}$ of pure subgroups such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \le 1$ and $\operatorname{Ext}(A_{n+1}/A_n, G) = 0$. Now by the hypothesis on G we have $D_1(G) = D_2(G) = P$ (the set of all primes). Thus, by remark 1) from Theorem 1.2, there is for every n a nonzero $x \in A_{n+1}/A_n$ (except that $A_{n+1} = A_n$) such that $h_p(x) = 0$ for all p. But this means that for all n, A_{n+1}/A_n is free; hence by $[E_1$, Theorem 2.6] A is free.

COROLLARY 2.4. Let G' be a pure subgroup of G. If A is any countable torsion-free group, then $\operatorname{Ext}(A, G) = 0$ if and only if $\operatorname{Ext}(A, G') = 0$ and $\operatorname{Ext}(A, G/G') = 0$.

Proof. The "if" part holds trivially. Conversely, suppose that $\operatorname{Ext}(A, G) = 0$. Then clearly $\operatorname{Ext}(A, G/G') = 0$, and by Theorem 2.1, A is the union of an ascending chain of pure subgroups $\{A_n \mid n < \omega\}$ such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \le 1$ and $\operatorname{Ext}(A_{n+1}/A_n, G) = 0$. Now $D_i(G')$ is contained in $D_i(G)$ for i = 1, 2, since G' is pure in G. Therefore by Theorem 1.2(a) we have $\operatorname{Ext}(A_{n+1}/A_n, G') = 0$ for all n, and hence $\operatorname{Ext}(A, G') = 0$ by Theorem 2.1.

COROLLARY 2.5. There is a countable quotient H of G such that for any countable torsion-free group A, $\operatorname{Ext}(A, G) = 0$ if and only if $\operatorname{Ext}(A, H) = 0$.

Proof. By the main result of [Sz] and the proof of [E₃, Theorem 2.2] there is a countable torsion group T and an epimorphism $\varepsilon: tG \longrightarrow T$ such that for any countable torsion-free A, Ext (A, tG) = 0 if and only if Ext (A, T) = 0. We denote

the induced homomorphism $\operatorname{Ext}(G/tG, tG) \to \operatorname{Ext}(G/tG, T)$ by ε_* and let H be a representative of the class $\varepsilon_*[G]$. Thus there is a commutative diagram

$$0 \to tG \to G \to G/tG \to 0$$

$$\downarrow^{\varepsilon} \qquad \downarrow^{\eta} \qquad \parallel$$

$$0 \to T \to H \to G/tG \to 0$$

with exact rows. We claim that H has the required properties. First it is clear that H is countable. Furthermore we see from the diagram that η is an epimorphism. Hence for any group A, $\operatorname{Ext}(A,G)=0$ implies $\operatorname{Ext}(A,H)=0$. Conversely, suppose that A is countable torsion-free such that $\operatorname{Ext}(A,H)=0$. Then we have $\operatorname{Ext}(A,G/tG)=0$ and $\operatorname{Ext}(A,T)=0$ by Corollary 2.4. Thus $\operatorname{Ext}(A,tG)=0$ as well and hence $\operatorname{Ext}(A,G)=0$. This completes our proof.

THEOREM 2.6. If A is a countable torsion-free group such that Ext(A, G) = 0, then Ext(A/B, G) = 0 for every pure subgroup B of A of finite rank.

Proof. Let B be any pure subgroup of A of finite rank. We can choose an ascending chain of pure subgroups $\{B_n \mid n < \omega\}$ of A of finite rank with union A such that $B_0 = B$ and for all n, $r_0(B_{n+1}/B_n) \le 1$. Then the same argument as in the proof of Theorem 2.1 shows that $\operatorname{Ext}(B_{n+1}/B_n, G) = 0$ for all n. Now let $A_n = B_n/B$, so we have $A/B = \bigcup_{n < \omega} A_n$ where $A_0 = 0$ and for all n, A_n is a pure subgroup of A/B of finite rank and $A_{n+1}/A_n \cong B_{n+1}/B_n$. Therefore we have $\operatorname{Ext}(A/B, G) = 0$ by Theorem 2.1.

Remark. Combining Theorems 1.2, 2.1 and 2.6 we obtain another proof of Baer's criterion which characterizes pairs of groups (A, T), A countable torsion-free and T a torsion group, such that every extension of T by A splits $[B_1]$.

The following result is an analogue of Pontryagin's criterion (see e.g. [F₁], Theorem 19.1).

COROLLARY 2.7. If A is a countable torsion-free group such that $\operatorname{Ext}(B,G)=0$ for every subgroup B of A of finite rank, then $\operatorname{Ext}(A,G)=0$.

Proof. Let A be represented as the union of an ascending chain of pure subgroups $\{A_n \mid n < \omega\}$ such that $A_0 = 0$ and for all n, $r_0(A_{n+1}/A_n) \le 1$. Then for all n, Ext $(A_n, G) = 0$ by hypothesis. Using Theorem 2.6, we conclude that for all n, Ext $(A_{n+1}/A_n, G) = 0$. Hence we have Ext (A, G) = 0 by Theorem 2.1.

The next result contains Proposition 5 of [Hu] and, in particular, the well-known fact that for every countable torsion-free nonfree group A, $r_0(\text{Ext}(A, \mathbb{Z}))$ is 2^{\aleph_0} (see e.g. [J], Théorème 2.7).

THEOREM 2.8. Let A be a countable torsion-free group such that $\operatorname{Ext}(A, G) \neq 0$. Then the torsion-free rank of $\operatorname{Ext}(A, G)$ is $\geq 2^{\aleph_0}$.

Proof. Clearly we may assume that G is reduced. If $\operatorname{Ext}(A, G) \neq 0$, then by Corollary 2.7 there exists a subgroup B of A of finite rank such that $\operatorname{Ext}(B, G) \neq 0$. Suppose that B is of minimal rank, and let B' be a pure subgroup of B such that $r_0(B/B') = 1$. Then we consider the exact sequence

$$\operatorname{Hom}(B',G) \to \operatorname{Ext}(B/B',G) \to \operatorname{Ext}(B,G) \to \operatorname{Ext}(B',G)$$
.

The minimality of $r_0(B)$ implies that $\operatorname{Ext}(B',G)=0$. Thus the exact sequence yields that $\operatorname{Ext}(B/B',G)\neq 0$, and hence by Theorem 1.2(b), $r_0(\operatorname{Ext}(B/B',G))\geq 2^{\aleph_0}$. On the other hand, by Lemma 2.2, $r_0(\operatorname{Hom}(B',G))$ is countable. It follows that $r_0(\operatorname{Ext}(B,G))\geq 2^{\aleph_0}$, hence $r_0(\operatorname{Ext}(A,G))\geq 2^{\aleph_0}$.

Remark. For A a countable torsion-free group and T an arbitrary torsion group, Baer $[B_2]$ has shown that Ext(A, T) is torsion-free. If in addition T is countable, we conclude that

Ext
$$(A, T) \cong \prod_{\aleph_0} \mathbf{Q}$$
 (cf. $[B_2]$, pp. 229–230).

It is well-known that this group admits a *compact topology*. Furthermore we know from [J, Corollaire 2.8] that the same holds for groups of the form $Ext(A, \mathbf{Z})$, A being countable torsion-free. These facts led us to ask the following

Question. Does Ext(A, G) admit a compact topology whenever A is countable torsion-free and G countable of finite torsion-free rank?

3. The uncountable case: vanishing of Ext (A, G)

In order to extend the results of the previous sections to groups of uncountable cardinality, we shall need to assume the Axiom of Constructibility, V = L. Before stating the main result of this section, let us recall a definition from $[E_3]$. For any set U and infinite cardinal λ , let $K_{\lambda^+}(U)$ denote the filter on $\mathcal{P}(U)$, the

power set of U, generated by all $X \in \mathcal{P}(\mathcal{P}(U))$ satisfying

- (i) X is closed under unions of chains; and
- (ii) for all $S \subseteq U$, there exists $H \in X$ such that $S \subseteq H$ and $|H| \le |S| + \lambda$.

(Such an X will be called a generating element of $K_{\lambda^+}(U)$.) We shall say that a property P of subsets of U holds for almost all subsets (w.r.t. $K_{\lambda^+}(U)$) if $\{S \subseteq U \mid S \text{ satisfies } P\}$ belongs to $K_{\lambda^+}(U)$.

LEMMA 3.1. (1) If $\lambda \leq \mu$, then $K_{\lambda^+}(U) \subseteq K_{\mu^+}(U)$.

- (2) $K_{\lambda^+}(U)$ is λ^+ -complete, i.e. if X_{ν} , $\gamma < \lambda$ are elements of $K_{\lambda^+}(U)$, then $\cap \{X_{\gamma} \mid \nu < \lambda\}$ belongs to $K_{\lambda^+}(U)$.
- (3) If $V \subseteq U$ and $X \in K_{\lambda^+}(V)$, then $\{H \subseteq U \mid H \cap V \in X\}$ belongs to $K_{\lambda^+}(U)$.
- (4) If A is a group, B a subgroup of A and $X \in K_{\lambda^+}(A/B)$, then $\{H \subseteq A \mid (H+B)/B \in X\}$ belongs to $K_{\lambda^+}(A)$.
- (5) If A is a group, then almost all subsets of A (w.r.t. $K_{\omega_1}(A)$) are pure subgroups of A.
- (6) If B is a pure subgroup of A, then for almost all subsets H of A (w.r.t. $K_{\omega}(A)$), B+H is a pure subgroup of A.

Proof. (1)–(4) are easy consequences of the definition. Part (5) follows from the facts that (i) the union of a chain of pure subgroups is a pure subgroup, and (ii) every infinite subset of A is contained in a pure subgroup of A of the same cardinality (cf. $[F_1]$, Proposition 26.2). By (5) applied to the group A/B we obtain an element X of $K_{\omega_1}(A/B)$ consisting of pure subgroups of A/B. Then by part (4), $Y = \{H \subseteq A \mid (H+B)/B \in X\}$ belongs to $K_{\omega_1}(A)$; it is readily verified that for all $H \in Y$, B + H is a pure subgroup of A. This proves (6).

For any infinite cardinal κ , let cf (κ) denote the confinality of κ . By definition, κ is singular if cf (κ) < κ and otherwise κ is regular. Recall that a group is said to be κ -generated if it has a set of generators of cardinality < κ .

THEOREM 3.2. (V = L). Let A be a group of uncountable cardinality κ and let G be a group of cardinality $\lambda < \kappa$. Then

- (1) If κ is singular and $\operatorname{Ext}(B,G)=0$ for every κ -generated subgroup B of A, then $\operatorname{Ext}(A,G)=0$.
- (2) Ext (A, G) = 0 if and only if A is the union of a continuous ascending chain $\{A_{\nu} \mid \nu < \text{cf}(\kappa)\}\$ of κ -generated pure subgroups of A such that Ext $(A_0, G) = 0$ and for all $\nu < \text{cf}(\kappa)$, Ext $(A_{\nu+1}/A_{\nu}, G) = 0$.
- (3) If Ext (A, G) = 0, then for almost all subgroups H of A (w.r.t. $K_{\lambda^+}(A)$), Ext (A/H, G) = 0.

The above result is proved in $[E_3]$ for the case of a torsion-free group A and G a torsion group, but the same proof works for arbitrary A and G any countable

group. That proof was based on $[Sh_2]$; here we shall give a proof based on a new simplified version $[Sh_3]$ of the principal result of $[Sh_2]$. For the convenience of the reader we shall state a version (for abelian groups) of the main theorem of $[Sh_3]$.

THEOREM 3.3 (Shelah). Let κ be a singular cardinal and let $\{\kappa_i \mid i < \text{cf }(\kappa)\}$ be an increasing and continuous sequence of cardinals satisfying: $\kappa_0 = 0$, $\text{cf }(\kappa) \le \kappa_1$ and $\sup \{\kappa_i \mid i < \text{cf }(\kappa)\} = \kappa$. Let A be a group of cardinality κ ; let $S_i = \text{the set of all}$ subgroups of A of cardinality κ_i and let $S_i' = \{0\} \cup S_i$. Suppose \mathcal{F} is a class of pairs (C_2, C_1) of subgroups of A such that $C_1 \subseteq C_2$. Suppose \mathcal{F} satisfies the following two properties:

- (HI) For every $i < cf(\kappa)$, there is a function $g_i : S'_i \times S_i \to S_i$ such that whenever $A_1 \subsetneq A_2$ are in S'_i and $A_1 \in \{0\} \cup (range \ g_i)$, then $A_2 \subseteq g_i(A_1, A_2)$ and $(g_i(A_1, A_2), A_1) \in \mathcal{F}$;
- (HII) For every $i < cf(\kappa)$ and every $A_1 \not\subseteq A_2$ in S'_{i+1} , if $(A_2, A_1) \in \mathcal{F}$ then player II has a winning strategy in the following game: in the n^{th} move $(n < \omega)$, player I chooses $B_n \in S'_i$ such that $C_{n-1} \subseteq B_n$ (where $C_{-1} = 0$) and then player II chooses $C_n \in S'_i$ such that $B_n \subseteq C_n$. Player II wins if

$$(A_2 + \bigcup_{n < \omega} C_n, A_1 + \bigcup_{n < \omega} C_n) \in \mathscr{F}.$$

Then A is the union of a continuous ascending chain $\{A_{\nu} \mid \nu < \omega \text{ cf }(\kappa)\}$ of κ -generated subgroups of A such that $A_0 = 0$ and $(A_{\nu+1}, A_{\nu}) \in \mathcal{F}$ for all $\nu < \omega \text{ cf }(\kappa)$.

Proof of Theorem 3.2. Let A and G be as in he hypotheses of 3.2. We shall prove (1), (2) and (3) simultaneously by induction on κ . Part (3) is proved as in the proof of Theorem 3.4 of $[E_3]$. (In place of Lemma 3.5(3) we require the straightforward generalization in which K_{ω_1} is replaced by K_{λ^+}). The sufficiency of the condition in part (2) is just Theorem 1.2 of $[E_2]$, and when κ is regular, necessity is Theorem 1.5 of $[E_2]$. (Note that we can assume the chain $\{A_{\nu} \mid \nu < \kappa\}$ consists of *pure* subgroups by Lemma 3.1(5).) Thus it remains only to prove (1) and necessity in (2) when κ is singular.

Let \mathcal{F} be the class of pairs (C_2, C_1) of subgroups of A such that C_1 is a pure subgroup of C_2 and $\operatorname{Ext}(C_2/C_1, G) = 0$. Choose an increasing sequence $\{\kappa_i \mid i < \operatorname{cf}(\kappa)\}$, whose limit is κ such that $\kappa_0 = 0$ and for all $i \ge 0$, $\kappa_{i+1} \ge \max\{\operatorname{cf}(\kappa), \lambda\}$. We shall show that \mathcal{F} satisfies (HI) and (HII) of Theorem 3.3. First we prove a lemma.

LEMMA 3.4. (V = L). Let κ be a limit cardinal, let A be a group of cardinality κ and let G be a group of cardinality $\lambda < \kappa$. Suppose that Ext (B, G) = 0 for every

 κ -generated subgroup B of A. Let C be a subgroup of A of infinite cardinality μ , where $\lambda \leq \mu^+ < \kappa$. Then there is a pure subgroup C^* of A of cardinality μ such that C^* contains C and

(*) Ext $(C'/C^*, G) = 0$ for all subgroups C' of A of cardinality μ that contain C^* .

Proof. If no such group C^* exists, then we can construct by induction a continuous ascending chain $\{C_{\nu} \mid \nu < \mu^+\}$ of subgroups of A of cardinality μ such that $C_0 = C$ and for all ν , $1 \le \nu < \mu^+$, C_{ν} is pure in A and Ext $(C_{\nu+1}/C_{\nu}, G) \ne 0$. Let $\tilde{C} = \bigcup_{\nu < \mu^+} C_{\nu}$. Then $|\tilde{C}| = \mu^+ < \kappa$, but Ext $(\tilde{C}, G) \ne 0$ by Lemma 1.4 of $[E_2]$, which contradicts the hypothesis. This completes the proof of the lemma.

Now we can verify (HI) by defining g_i so that for all $(A_1, A_2) \in S_i' \times S_i$, $A_2 \subseteq g_i(A_1, A_2)$ and $C^* = g_i(A_1, A_2)$ satisfies (*) of Lemma 3.4. (Note that if $A_1 = 0$, then $(g_i(A_1, A_2), 0) \in \mathcal{F}$ by the hypothesis of Theorem 3.2(1).)

It remains to verify (HII). Given $A_1 \subsetneq A_2$ in $S'_{i+1}(i \ge 1)$ such that $(A_2, A_1) \in \mathcal{F}$, there exists by Lemma 3.1(6) a generating element X of $K_{\lambda^+}(A_2)$ such that for all $H \in X$, $A_1 + H$ is a pure subgroup of A_2 . Moreover since by inductive hypothesis A_2/A_1 satisfies Theorem 3.2(3) we may assume – using Lemma 3.1(4) – that every element H of X satisfies $\operatorname{Ext}(A_2/(A_1+H),G)=0$. Hence for all $H \in X$, $(A_2,A_1+H)\in \mathcal{F}$. Now the winning strategy of player II is as follows. Suppose $C_{n-1} \in S'_i$ has been chosen so that $C_{n-1} \cap A_2 \in X$. If player I chooses $B_n \in S'_i$ with $C_{n-1} \subseteq B_n$, then player II chooses $C_n \in S'_i$ such that $B_n \subseteq C_n$ and $C_n \cap A_2 \in X$; this is possible because $\kappa_i \ge \lambda$. Now

$$(A_2 + \bigcup_{n < \omega} C_n)/(A_1 + \bigcup_{n < \omega} C_n) \cong A_2/(A_1 + \bigcup_{n < \omega} (C_n \cap A_2)),$$

and $(A_2, A_1 + \bigcup_{n < \omega} (C_n \cap A_2)) \in \mathcal{F}$ since X is closed under unions of chains. It follows easily that $(A_2 + \bigcup_{n < \omega} C_n, A_1 + \bigcup_{n < \omega} C_n) \in \mathcal{F}$.

Therefore by Theorem 3.3 we have a continuous ascending chain $\{A_{\nu} \mid \nu < \omega \text{ cf }(\kappa)\}$ of κ -generated subgroups with union A such that $(A_{\nu+1}, A_{\nu}) \in \mathcal{F}$ for all $\nu < \omega \text{ cf }(\kappa)$. Note that the continuity of the chain implies that for all ν , A_{ν} is pure in A. By choosing a continuous subchain of length cf (κ) we obtain the chain required for Theorem 3.2(2). This completes the proof of Theorem 3.2.

COROLLARY 3.5 (V = L). Let G be a group of countable torsion-free rank. There is a countable quotient H of G such that for any torsion-free group A, $\operatorname{Ext}(A, G) = 0$ if and only if $\operatorname{Ext}(A, H) = 0$.

Proof. Let H be the countable quotient of G given by Corollary 2.5. For any group A, $\operatorname{Ext}(A, G) = 0$ implies $\operatorname{Ext}(A, H) = 0$. We shall prove by induction on |A| that the converse is also true if A is torsion-free. For countable A this is

Corollary 2.5. If A is uncountable and Ext (A, H) = 0, then by Theorem 3.2(2), A is the union of a continuous chain $\{A_{\nu} \mid \nu < \operatorname{cf}(\kappa)\}$ of κ -generated subgroups such that Ext $(A_0, H) = 0$ and for all $\nu < \operatorname{cf}(\kappa)$, $A_{\nu+1}/A_{\nu}$ is torsion-free and Ext $(A_{\nu+1}/A_{\nu}, H) = 0$. By induction, Ext $(A_0, G) = 0$ and for all $\nu < \operatorname{cf}(\kappa)$, Ext $(A_{\nu+1}/A_{\nu}, G) = 0$; hence (by Theorem 1.2 of $[E_2]$) Ext (A, G) = 0.

Remark. As an immediate consequence of Corollary 3.5 we obtain Theorem 3.2 for A torsion-free and G any group of countable torsion-free rank (and arbitrary cardinality) with $\lambda = \omega$ in 3.2(3). This generalizes Theorem 3.4 of $[E_3]$.

We can also generalize Shelah's solution of Whitehead's problem in L using Corollary 2.3.

COROLLARY 3.6 (V = L). Suppose that G is a countable torsion-free group such that for all primes p, G is not p-divisible. For any group A, if Ext(A, G) = 0 then A is free.

Proof. We proceed by induction on the cardinality of A, using Corollary 2.3 and Theorem 3.2(2).

Remark. By $[Sh_1]$, Corollary 3.6 is independent of ZFC. Moreover the same holds for Corollary 3.5 and Theorem 3.2(2) and (3) (see $[E_3]$). We do not know, however, if Theorem 3.2(1) is independent of ZFC.

The following result is related to Corollary 2.5 as Hill's theorem ([Hi]) is related to Pontryagin's criterion.

THEOREM 3.7 (V=L). Let A be a torsion-free group and G a group of countable torsion-free rank. Suppose $A = \bigcup_{n < \omega} A_n$, where $\{A_n \mid n < \omega\}$ is a chain of pure subgroups of A such that $\operatorname{Ext}(A_n, G) = 0$ for all $n < \omega$. Then $\operatorname{Ext}(A, G) = 0$.

Proof. By Corollary 3.5 we may assume that G is countable, the proof of the theorem will be by induction on |A|. If A is countable the result follows easily from Corollary 2.7. Suppose now that $|A| = \kappa > \aleph_0$. Theorem 3.2(3) and Lemma 3.1(2) and (3) imply that there is a generating element X of $K_{\omega_1}(A)$ consisting of subgroups H such that for all $n < \omega$, $\operatorname{Ext}(A_n/(H \cap A_n), G) = 0$. Moreover by Lemma 3.1(6) we may assume that for all $H \in X$ and all $n < \omega$, $A_n + H$ is pure in A. Now using the properties of a generating element we can define by transfinite induction a continuous ascending chain $\{H_{\nu} \mid \nu < \kappa\}$ of elements of X such that $H_0 = 0$, $A = \bigcup_{\nu < \kappa} H_{\nu}$, and for all $\nu < \kappa$, $|H_{\nu}| < \kappa$. For all $\nu < \kappa$

$$\begin{split} H_{\nu+1}/H_{\nu} &= \bigcup_{n < \omega} ((H_{\nu+1} \cap A_n) + H_{\nu})/H_{\nu} \quad \text{and} \quad ((H_{\nu+1} \cap A_n) + H_{\nu})/H_{\nu} \\ &\cong (H_{\nu+1} \cap A_n)/(H_{\nu} \cap A_n). \end{split}$$

Hence Ext $(((H_{\nu+1} \cap A_n) + H_{\nu})/H_{\nu}, G) = 0$ since by choice of X, Ext $(A_n/(H_{\nu} \cap A_n), G) = 0$. Moreover $((H_{\nu+1} \cap A_n) + H_{\nu})/H_{\nu}$ is pure in $H_{\nu+1}/H_{\nu}$ since by choice of X, $A_n + H_{\nu}$ is pure in A. Therefore by inductive hypothesis, Ext $(H_{\nu+1}/H_{\nu}, G) = 0$. Since this is true for all $\nu < \kappa$, it follows that Ext (A, G) = 0. This completes the proof of the theorem.

4. The uncountable case: structure of Ext (A, G)

The aim of this final section is to determine the torsion-free rank of $\operatorname{Ext}(A, G)$ in the case where A is uncountable torsion-free and G satisfies suitable cardinality conditions. The results of this section extend those of [H-H-S], when Shelah's solution of Whitehead's problem in L is taken into account. We do not know, however, whether our results remain valid without additional axioms of set theory. We start with

THEOREM 4.1 (V = L). Let A be a torsion-free group of uncountable cardinality κ and let G be any group of cardinality $<\kappa$. Suppose that for every κ -generated pure subgroup B of A, $\operatorname{Ext}(A/B, G) \neq 0$. Then the torsion-free rank of $\operatorname{Ext}(A, G)$ is 2^{κ} .

To prove this we follow the pattern of the proof of the corresponding result (Theorem 1) of [H-H-S]. The regular case is an easy consequence of the subsequent proposition. Recall that Ext(A, G) can be defined as the quotient group Fact(A, G)/Trans(A, G), where Fact(A, G) is the abelian group of all factor sets on A to G and Trans(A, G) is the subgroup of transformation sets (see e.g. $[F_1]$, pp. 209-211).

PROPOSITION 4.2 (V = L). Let A be a torsion-free group of regular uncountable cardinality κ and let G be any group of cardinality κ . Suppose that for every κ -generated pure subgroup B of A, $\operatorname{Ext}(A/B, G) \neq 0$. If A_0 is any κ -generated pure subgroup of A, then for every $f_0 \in \operatorname{Fact}(A_0, G)$ there exists a subset $\{f^{\alpha} \mid \alpha < 2^{\kappa}\}$ of $\operatorname{Fact}(A, G)$ such that

- (i) for all $\alpha < 2^{\kappa}$, f^{α} extends f_0 ;
- (ii) for each pair $\alpha \neq \beta$, $f^{\alpha} f^{\beta}$ represents an element of infinite order of Ext (A, G).

The proof of this proposition is almost identical with that of Proposition 1 in [H-H-S]. We only have to replace the statement "A is free" by "Ext (A, G) = 0". Instead of $[E_1, Theorem 2.6]$ we make use of Theorem 1.2 of $[E_2]$. Note that the

cardinality hypothesis on G is needed in order that Lemma 3 of [H-H-S] can be applied.

The proof of Theorem 4.1 in case κ is singular relies on the above proposition and

THEOREM 4.3 (V = L). Let A be any group of singular cardinality κ . Let G be a group of cardinality $\lambda < \kappa$ and let γ be any infinite cardinal $< \kappa$. Suppose that every κ -generated subgroup B of A contains a γ^+ -generated subgroup C such that C is pure in B and Ext (B/C, G) = 0. Then A contains a γ^+ -generated pure subgroup C such that Ext (A/C, G) = 0.

Proof. Let \mathscr{F}_1 be the class of pairs (B,0) where B is a subgroup of A that contains a γ^+ -generated subgroup C such that C is pure in B and $\operatorname{Ext}(B/C,G)=0$. Let \mathscr{F}_2 be the class of pairs (B,C) of subgroups of A where C is a non-trivial pure subgroup of B such that $\operatorname{Ext}(B/C,G)=0$. We show that the class $\mathscr{F}=\mathscr{F}_1\cup\mathscr{F}_2$ satisfies (HI) and (HII) of Theorem 3.3, assuming that the sequence $\{\kappa_i\mid i<\operatorname{cf}(\kappa)\}$ is chosen such that $\kappa_1\geq \max\{\operatorname{cf}(\kappa),\lambda,\gamma\}$. Condition (HI) is easily verified by means of the following analogue of Lemma 3.4.

LEMMA 4.4 (V = L). Suppose that A and G satisfy the hypotheses of Theorem 4.3. Let B be a subgroup of A of cardinality μ , where $\max \{\lambda, \gamma\} \leq \mu < \kappa$. Then there is a pure subgroup B^* of A of cardinality μ such that B^* contains B and $\operatorname{Ext}(B'/B^*, G) = 0$ for all subgroups B' of A of cardinality μ that contain B^* .

Proof. We proceed as in the proof of Lemma 3.4. Supposing that no such B^* exists, we obtain a subgroup \tilde{B} of A of cardinality μ^+ which is the union of a continuous ascending chain $\{B_{\nu} \mid \nu < \mu^+\}$ of subgroups of cardinality μ such that $B_0 = B$ and for all ν , $1 \le \nu < \mu^+$, B_{ν} is pure in A and $\operatorname{Ext}(B_{\nu+1}/B_{\nu}, G) \ne 0$. By hypothesis \tilde{B} contains a γ^+ -generated pure subgroup C such that $\operatorname{Ext}(\tilde{B}/C, G) = 0$. As $|\tilde{B}|$ is regular, we may assume that C is contained in B_0 ; so we have $\tilde{B}/C = \bigcup_{\nu < \mu^+} B_{\nu}/C$. But then Lemma 1.4 of $[E_2]$ yields a contradiction, and our lemma is proved.

It remains to check (HII). Given $A_1 \subseteq A_2$ in S'_{i+1} such that $(A_2, A_1) \in \mathcal{F}$, we distinguish the following two cases. First if $A_1 \neq 0$, we proceed exactly as in the proof of Theorem 3.2. In the second case, suppose that $A_1 = 0$. So $(A_2, A_1) \in \mathcal{F}$ means that A_2 contains a γ^+ -generated pure subgroup C such that $\operatorname{Ext}(A_2/C, G) = 0$. Then by Theorem 3.2(3) and Lemma 3.1 there exists a generating element X of $K_{\lambda^+}(A_2)$ consisting of subgroups H of A_2 such that $(A_2, C+H) \in \mathcal{F}$. Now the winning strategy of player II is to choose $C_n \in S'_i$ such that C_0 contains C and $C_n \cap A_2 \in X$. This is possible by the assumption on κ_i .

Then

$$(A_2 + \bigcup_{n < \omega} C_n)/(A_1 + \bigcup_{n < \omega} C_n) \cong A_2/\bigcup_{n < \omega} (C_n \cap A_2)$$

$$\cong A_2/(C + \bigcup_{n < \omega} (C_n \cap A_2)).$$

But $(A_2, C + \bigcup_{n < \omega} (C_n \cap A_2))$ is in \mathcal{F} since X is closed under unions of chains; so indeed $(A_2 + \bigcup_{n < \omega} C_n, A_1 + \bigcup_{n < \omega} C_n) \in \mathcal{F}$.

Therefore by Theorem 3.3 we obtain a continuous chain $\{A_{\nu} \mid \nu < \omega \text{ cf }(\kappa)\}$ of κ -generated pure subgroups of A such that $A = \bigcup_{\nu < \omega \text{ cf}(\kappa)} A_{\nu}$, A_1 contains a γ^+ -generated pure subgroup C with $\text{Ext}(A_1/C, G) = 0$ and for all ν , $1 \le \nu < \omega \text{ cf }(\kappa)$, $\text{Ext}(A_{\nu+1}/A_{\nu}, G) = 0$. Hence we have Ext(A/C, G) = 0 by Theorem 3.2(2). This completes the proof of Theorem 4.3.

Proof of Theorem 4.1. Clearly 2^{κ} is an upper bound for $r_0(\operatorname{Ext}(A, G))$. If κ is regular, Proposition 4.2 implies that the quotient group of $\operatorname{Ext}(A, G)$ modulo torsion is of cardinality 2^{κ} . Hence in this case 2^{κ} is also a lower bound for $r_0(\operatorname{Ext}(A, G))$. Note that for κ regular the theorem still holds if the cardinality of G is κ .

Now assume that κ is singular. In this case we define by induction a chain of pure subgroups $\{A_{\nu} \mid \nu < cf(\kappa)\}\$ of A such that

- (i) $A = \bigcup_{\nu < \mathrm{cf}(\kappa)} A_{\nu};$
- (ii) $|A_{\nu}|$ is a regular cardinal $> \max\{|G|, |\bigcup_{\mu < \nu} A_{\mu}|\};$
- (iii) if C is a $|A_{\nu}|$ -generated pure subgroup of A_{ν} , then Ext $(A_{\nu}/C, G) \neq 0$. The definition of the chain is similar to the one in the proof of Theorem 1 of [H–H–S]. Instead of Theorem 2 of [H–H–S] we apply our Theorem 4.3, while condition (iii) is checked by making use of Theorem 3.2(3).

Let $\tilde{A}_{\nu} = \bigcup_{\mu < \nu} A_{\mu}$. As in [H-H-S], we deduce from Proposition 4.2 that to each sequence η of ordinals of length ν with $\eta(\mu) \in 2^{|A_{\nu}|}$, $\mu < \nu$, a factor set $f^{\eta} \in \text{Fact}(\tilde{A}_{\nu}, G)$ can be assigned such that

- (iv) if ξ is an initial segment of η , then f^{η} extends f^{ξ} ;
- (v) if $\xi \neq \eta$ are of the same length ν , then $f^{\xi} f^{\eta}$ represents an infinite order element of Ext (\tilde{A}_{ν}, G) .

We conclude that there are $\prod_{\nu < cf(\kappa)} 2^{|A_{\nu}|} = 2^{\kappa}$ factor sets on A to G which represent pairwise different elements of $\operatorname{Ext}(A, G)$ modulo torsion. This completes the proof of Theorem 4.1.

THEOREM 4.5 (V = L). Let A be a torsion-free group and let G be any group of countable torsion-free rank such that $\text{Ext}(A, G) \neq 0$. Suppose that B is a pure

subgroup of A such that $\operatorname{Ext}(A/B, G) = 0$. If B is of minimal cardinality, then $r_0(\operatorname{Ext}(A, G)) \ge 2^{|B|}$.

Proof. By hypothesis we have $\operatorname{Ext}(A, G) \cong \operatorname{Ext}(B, G)$. The case where B is countable is therefore settled by Theorem 2.8. For uncountable B the result follows from Corollary 3.5 and Theorem 4.1.

Remark. The following special case of the above theorem is implicit in $[N_2]$: If A is a torsion-free group and T a torsion group such that $\operatorname{Ext}(A, T) \neq 0$, then $r_0(\operatorname{Ext}(A, T)) \geq 2^{\aleph_0}$. Note that this does not require V = L.

The next two results are immediate consequences of Theorem 4.5.

COROLLARY 4.6. (V=L). Let A be torsion-free and let G be countable such that Ext $(A, G) \neq 0$. Then

- (a) $r_0(\text{Ext}(A, G)) = 2^{\mu}$ for some infinite cardinal μ ;
- (b) $r_0(\text{Ext}(A, G)) = |\text{Ext}(A, G)|$.

COROLLARY 4.7. (V = L). Let A be κ -free for some infinite cardinal κ and let G be countable. If Ext $(A, G) \neq 0$, then $r_0(\text{Ext}(A, G)) = 2^{\mu}$ for some $\mu \geq \kappa$.

Recall that a group A is called κ -free if every κ -generated subgroup of A is free. We already mentioned that the results of this section extend those of [H-H-S]. Corollaries 4.7 and 4.8 generalize, moreover, Théorème 1 and Corollaire 2 of [Hu], respectively.

COROLLARY 4.8 (V = L). Let A be any group and let G be countable. If Ext (A, G) is nonzero and divisible, then $r_0(\text{Ext}(A, G)) = 2^{\mu}$ for some infinite μ .

Proof. Clearly we may assume that G is reduced. We consider the exact sequence

$$\operatorname{Hom}(tA, G) \to \operatorname{Ext}(A/tA, G) \xrightarrow{\varphi} \operatorname{Ext}(A, G) \to \operatorname{Ext}(tA, G) \to 0.$$

From Lemma 55.3 of $[F_1]$ we know that $\operatorname{Ext}(tA, G)$ is reduced. On the other hand, the hypothesis implies that $\operatorname{Ext}(tA, G)$ is divisible; hence $\operatorname{Ext}(tA, G) = 0$. Using Lemma 1.1 we conclude that G is p-divisible for every prime p for which $t_pA \neq 0$. Therefore we have $t_pG = 0$ whenever $t_pA \neq 0$. It follows that $\operatorname{Hom}(TA, G) \cong \prod_p \operatorname{Hom}(t_pA, t_pG) = 0$. Hence by exactness of the above sequence φ is an isomorphism. Thus it sufficies to consider the case where A is torsion-free. But this case has already been settled by Corollary 4.6(a).

Acknowledgement. The second-named author would like to thank Professor R. Baer for his stimulating interest and helpful suggestions.

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Received July 8, 1978