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A universal space for normal bundles of *n*-manifolds

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§1. Introduction

In [3] the authors gave a simple criterion for deciding whether a polynomial in Stiefel-Whitney classes is zero on the normal bundles of all smooth *n*-manifolds. The ideal of relations among Stiefel-Whitney classes for all *n*-manifolds, $I_n \subset H^*(BO)$ was defined by

 $I_n = \{ w \in H^*(BO) \mid w(\nu_{\mathbf{M}^n}) = 0 \text{ for all } \mathbf{M}^n \}$

where M^n denotes a smooth *n*-manifold and ν_M is its stable normal bundle. Let $\Phi: H^*(BO) \simeq H^*(MO)$ be the Thom isomorphism and for $w \in H^*(BO)$, define wSq^i to be $\Phi^{-1}(\chi(Sq^i)\Phi(w))$. It was shown that I_n consists of all Z_2 -linear combinations of elements of the form wSq^i where 2i > n - |w| (|w| = dimension of w).

In this paper we give a stronger version of this result, namely:

THEOREM 1. There is a space BO/I_n and a map $\pi : BO/I_n \to BO$ such that (a) If M is a smooth, compact n-manifold and $h: M \to BO$ classifies ν_M , then there is a map $\overline{h}: M \to BO/I_n$ such that $\pi \overline{h} \simeq h$.

(b) The following sequence is exact.

$$0 \longrightarrow I_n \subset H^*(BO) \xrightarrow{\pi^*} H^*(BO/I_n) \longrightarrow 0.$$

Theorem 1 shows that BO/I_n is a universal space for normal bundles of *n*-manifolds in that stably, every such bundle is induced from the bundle over BO/I_n and BO/I_n is the space with the smallest cohomology having this property.

Our original result on I_n suggested the possibility of defining higher order characteristic classes, that is, one could form a space B over BO by killing the

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elements of I_n . Then an element of $H^*(B)$ might give a "new" characteristic class for *n*-manifolds. For example, with n = 4 or 5, the relation

$$(Sq^2 + w_1 \cup Sq^1 + w_2 U)(v_3) = v_3 Sq^2 = (1Sq^3)Sq^2 = 0$$

where v_3 is the Wu class, gives a class in $H^4(B)$ which is not a polynomial in Stiefel-Whitney classes. Theorem 1 shows that on an *n*-manifold this "new" class will be a polynomial in Stiefel-Whitney classes modulo indeterminacy.

The spaces BO/I_n are also related to the conjecture that any smooth *n*-manifold immerses in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the dyadic expansion of *n*. Since this conjecture is equivalent to the normal bundle map $h: \mathbb{M}^n \to BO$ lifting to $BO_{n-\alpha(n)}$ ([9]), the following is a stronger form of the conjecture:

CONJECTURE. $\pi: BO/I_n \to BO$ lifts to $BO_{n-\alpha(n)}$.

Using our proof of Theorem 1, our results in [4] can be restated in the following way which gives some plausibility to the above conjecture.

THEOREM 2. If ζ is the stable universal bundle over BO, MO is its Thom spectrum, MO/I_n is the Thom spectrum of $\pi^*\zeta$ and MO($n-\alpha(n)$) is the Thom spectrum of the universal bundle over BO_{$n-\alpha(n)$}, then MO/I_n lifts to MO($n-\alpha(n)$).

This paper is organized as follows: In §2 we give a detailed outline of the proof of Theorem 1 setting forth most of the notation and describing the various technical problems arising in the construction of BO/I_n . Then in Sections 3, 4, 5, and 6 we prove the various lemmas stated in §2. Throughout the remainder of this paper n is a fixed positive integer.

§2. Outline of the Proof of Theorem 1

All cohomology will be with Z_2 coefficients, A will be the mod two Steenrod algebra and $\chi: A \to A$ will be the canonical antiautomorphism. The semi-tensor product of A and $H^*(BO)$ ([6]) will be denoted by A(BO), that is, $A(BO) = A \otimes H^*(BO)$ with the algebra structure defined by

$$(a \otimes u)(b \otimes v) = \sum ab_i' \otimes (\chi(b_i'')u)v$$

where $b \to \sum b'_i \otimes b''_i$ under the diagonal of A. We denote $a \otimes u$ by $a \circ u$.

By a spectrum Y, we will mean a collection of spaces Y_q and maps $g_q: SY_q \rightarrow Y_{q+1}$. If X and Y are spectra, a map $f: X \rightarrow Y$ of degree p will be a collection of homotopy classes $f_q \in [X_q, Y_{q+p}]$ compatible with the maps g_q . If ξ is a real k-plane bundle, $T(\xi)$ will denote its Thom spectrum, i.e., $T(\xi)_q = S^{q-k}$ (Thom space of ξ). Thus the Thom class is in $H^0(T(\xi))$. If ξ is a vector bundle over B, $\Phi: H^*(B) \approx H^*(T(\xi))$ will be the Thom isomorphism. We make $H^*(T(\xi))$ into an A(BO) module as follows: Let $h: B \rightarrow BO$ classify ξ . If $u \in H^*(T(\xi))$, $w \in H^*(BO)$ and $a \in A$, $(a \circ w)u = a(h^*(w)u)$. One easily checks that $\Phi(I_n) \subset H^*(MO)$ is an A(BO) submodule.

We begin by constructing an A-free, acyclic resolution of $\Phi(I_n)$. In [3] the following was proved:

THEOREM 2.1. If $\{u_i\}$ is an A basis for $H^*(MO)$, then $\Phi(I_n)$ is the A module generated by

 $\{\chi(Sq^i)u_i | 2j > n - |u_i|\}.$

For a partition $\omega = \{j_1, j_2, \dots, j_l\}$ let $s_{\omega} \in H^*(BO)$ be the usual class ([17]) associated with the symmetric function $\sum t_1^{j_1} t_2^{j_2} \cdots t_1^{j_1}$. For each partition ω let ω_r be the partition consisting of odd integers j, one for each $j2^r \in \omega$. Let

$$u_{\omega} = \prod_{r} s_{\omega_{r}}^{2^{r}}$$

Since

$$u_{\omega} = s_{\omega} + \sum s_{\omega'}$$

where ω' has fewer entries than ω and $\{s_{\omega}\}$ is a basis for $H^*(BO)$, $\{u_{\omega}\}$ is also a basis for $H^*(BO)$. Also $\{\Phi(u_{\omega}) \mid 2^i - 1 \notin \omega\}$ is an A basis for $H^*(MO)$ since $\{\Phi(s_{\omega}) \mid 2^i - 1 \notin \omega\}$ is.

In [2] an A-free acyclic resolution of $A/A\{\chi(Sq^i) | i > h\}$ was constructed. Combining these resolutions with 2.1 and the $\Phi(u_{\omega})$ basis, we obtain the following resolution of $\Phi(I_n)$.

Let Λ be the graded free associative algebra over Z_2 with unit generated by λ_i , $i = 0, \pm 1, \pm 2, \ldots, |\lambda_i| = i$, modulo the relations: If 2i < j

$$\lambda_i \lambda_j = \sum {\binom{s-1}{2s-(j-2i)}} \lambda_{i+s} \lambda_{j-s}.$$

If $I = (i_1, i_2, \ldots, i_l)$, let $\lambda_I = \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_l}$, l(I) = l, $t(I) = i_l$, and $\lambda_{(.)} = 1$. We define I

to be admissible if $2i_i \ge i_{j+1}$. As we will see in §3, $\{\lambda_I \mid I \text{ admissible}\}$ is a Z_2 basis for Λ . Let $\{\lambda^I \mid I \text{ admissible}\}$ be the dual basis of $\Lambda^* = \text{Hom}(\Lambda, Z_2)$.

Let U_l be the vector space over Z_2 with basis the symbols $\lambda^I u_{\omega}$ where I is admissible, $2^i - 1 \notin \omega$, l(I) = l and $2(t(I) + 1) > n - |u_{\omega}|$. Grade U_l by $|\lambda^I u_{\omega}| = |\lambda^I| + |u_{\omega}|$. Let $d: A \otimes U_l \to A \otimes U_{l-1}$ be the A linear map defined by

$$d(1\otimes\lambda^{I}u_{\omega}) = \sum \lambda^{I}(\lambda_{j}\lambda_{J})\chi(Sq^{j})\otimes\lambda^{J}u_{\omega}$$

where the sum ranges over all j and admissible J. Note by 2.2, if $\lambda^{I}(\lambda_{i}\lambda_{J}) \neq 0$, $t(J) \geq t(I)$ and hence d is well defined. Let $\eta : A \otimes U_{0} \rightarrow H^{*}(MO)$ be given by $\eta(a \otimes \lambda^{(i)}u_{\omega}) = a\Phi(u_{\omega})$.

PROPOSITION 2.3. The following sequence is exact:

 $\longrightarrow A \otimes U_{l} \xrightarrow{d} A \otimes U_{l-1} \longrightarrow \cdots \longrightarrow A \otimes U_{0}$

and

 $\Phi(I_n) = \eta(\text{image} (d: A \otimes U_1 \to A \otimes U_0))$

We prove 2.3 in §3.

For a graded vector space V over Z_2 , let K(V) denote the Eilenberg-MacLane spectrum such that $\pi_*(K(V)) = V^*$ and $H^*(K(V)) = A \otimes V$.

PROPOSITION 2.4. There is a sequence of Ω -spectra X_l , l = 0, 1, 2, ... and maps $\alpha_l : X_{l-1} \rightarrow K(U_l)$ of degree +1 such that

(i) $X_0 = K(U_0)$

(ii) X_l is the fibration over X_{l-1} induced by α_l from the contractible fibration over $K(U_l)$.

(iii) If $i: K(U_l) \to X_l$ is the inclusion of the fibre of $X_l \to X_{l-1}$, $(\alpha_{l+1}i)^* = d: A \otimes U_{l+1} \to A \otimes U_l$.

(iv) If M is a smooth n-manifold, ν is its normal bundle, $g: MO \to K(U_0)$ realizes η and $h: T(\nu) \to MO$ comes from the classifying map of ν , then any lifting of $gh: T(\nu) \to X_0$ to X_{l-1} lifts to X_l .

Since the X_i 's are constructed from an acyclic complex,

 $\lim H^*(X_l) \approx \operatorname{Coker} (d: A \otimes U_1 \to A \otimes U_0) \approx H^*(MO)/\Phi(I_n).$

To construct BO/I_n we essentially construct a tower of spaces

 $\rightarrow B_{l} \rightarrow B_{l-1} \rightarrow \cdots \rightarrow B_{0} = BO$

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with fibres Eilenberg-MacLane spaces, such that if $T_l = T(\zeta_l)$ where $\zeta_l \to B_l$ is the pull back of the universal bundle over BO, then $T_l = X_l$ in dimensions $\leq n$. We can then, more or less, define $BO/I_n = \lim B_l$.

We recall how the cohomology of a Thom space of a vector bundle changes, in a stable range, when a cohomology class in the base is killed. Suppose $g: B \to BO$ is a map such that $g_*: \pi_q(B) \approx \pi_q(BO)$ for $2q \leq n$, V is a graded vector space with $V_q = 0$ for $2q \leq n$ and $p: B' \to B$ is the fibration induced by a map $\gamma: B \to K(V)_1$ ($K(V) = \{K(V)_q\}$). Let $T = T(g^*\zeta)$ and $T' = T(p^*g^*\zeta)$. Viewing $B' \subset B$ as the fibre of γ , γ factors as $B \xrightarrow{i} B/B' \xrightarrow{\gamma'} K(V)_1$. Let

$$\Psi: (A(BO) \otimes V)^q \to H^{q+1}(T/T')$$

be given by $\Psi(a \circ u \otimes v) = a(u\Phi((\gamma')^*(v_1)))$ where $v_1 \in H^*(K(V)_1)$ is the element corresponding to $v \in V$ and Φ is the relative Thom isomorphism. In §6 we show that Ψ is an isomorphism for $q \leq n$. (An equivalent form of this was proved in [1].) Combining this with the exact sequence of the pair (T, T') we obtain an exact sequence,

$$\rightarrow H^{q}(T) \rightarrow H^{q}(T') \rightarrow (A(BO) \otimes V)^{q} \rightarrow H^{q+1}(T) \rightarrow$$

for $q \leq n$.

The cohomology of X_l and X_{l-1} are related by the Serre exact sequence,

$$\rightarrow H^{q}(X_{l-1}) \rightarrow H^{q}(X_{l}) \rightarrow (A \otimes U_{l})^{q} \rightarrow H^{q+1}(X_{l-1}) \rightarrow .$$

Thus if we have constructed B_{l-1} such that $T_{l-1} = X_{l-1}$ in dimensions $\leq n$ and we wish to construct B_l , we should take $B = B_{l-1}$ in the above and choose V_l so that $A(BO) \otimes V_l = A \otimes U_l$ as A modules. Our main algebraic result asserts that this is possible. Let

 $V_l = \{\lambda^I u_\omega \in U_l \mid \omega_r = \{\} \text{ for } r \ge l\}$

PROPOSITION 2.5. There are A linear isomorphisms $\theta: A \otimes U_l \to A(BO) \otimes V_l$ and A(BO) linear maps $d: A(BO) \otimes V_l \to A(BO) \otimes V_{l-1}$, l > 1 and $d: A(BO) \otimes V_1 \to H^*(MO)$ such that the following diagram is commutative:



Furthermore, if $u \in V_l \subset U_l$, then $\theta(1 \otimes u) = 1 \otimes u$.

The construction of spaces B_l can now be made, modulo technical problems, using 2.5. Given B_{l-1} and f_{l-1} : $T_{l-1} \rightarrow X_{l-1}$, the k-invariant $\beta_l : B_{l-1} \rightarrow K(V_l)_1$ is defined by:

 $\Phi \beta_l^*(v_1) = f_{l-1}^* \alpha_l^*(v)$

where $\alpha_l: X_{l-1} \to K(U_l)$ is the k-invariant for X_l , $v \in V$ and $v_1 \in H^*(K(V)_1)$ corresponds to v. If M is an n-manifold and $h: M \to BO$ classifies its normal bundle, 2.4(iv) shows that any lifting of h to B_{l-1} lifts to B_l . The A(BO) linearity of d allows one (more or less) to construct $f_l: T_l \to X_l$. Actually, this straightforward procedure is marred by two technical details which we now describe.

Let s = [n/2]. To form B_1 from BO, one kills, among other things, the Wu class v_{s+1} , i.e. $d\lambda^s = \chi(Sq^{s+1})U = v_{s+1}U$, where the U is the Thom class. The map Ψ is zero on

$$\sum_{j>0} (Sq^j \circ v_{s+1-j}) \otimes \lambda^s \in (A(BO) \otimes V_1)^{2s+1}$$

As a result, there is a class $x \in H^{2s+1}(X_1)$ which goes to zero in $H^{2s+1}(T_1)$. The class x is killed in going from X_1 to X_2 . Hence if one were to follow the recipe given by 2.5, one would kill a class in B_1 which is already zero and thus produce a class in $H^{2s}(B_2)$ not coming from $H^{2s}(X_2)$. To avoid this, we omit a basis element from V_2 . This same phenomena occurs in dimension 2s + 2 so we omit some more elements from V_2 and V_3 . Namely, let $\overline{V_l} \subset V_l$ be spanned by $\lambda^I u_{\omega} \in V_l$ except $\lambda^{0,0} w_s^2$, $\lambda^{0,-1} w_{s+1}^2$, $\lambda^{-1,-2} w_{s+2}^2$ and for s odd, $\lambda^{-1,-2,-4} w_1^4 w_s^2$ ($w_s = u_{(1,1,\ldots,1)}$).

In §3 we define a certain A(BO) linear map

$$r: A(BO) \otimes V_l \to A(BO) \otimes \overline{V}_l \tag{2.6}$$

such that $r | A(BO) \otimes \overline{V}_l$ is the identity. We then use $r\theta$ in place of θ in our construction of B_l .

The second difficulty arises in the following fashion. Again suppose we have B_{l-1} and $f_{l-1}: T_{l-1} \to X_{l-1}$ and we construct B_l using \overline{V}_l instead of V_l as above. Let $g_l: T_{l-1}/T_l \to K(U_l)$ be the map such that $g_l^*(u) = \Psi r \theta(u)$ for $u \in U_l$. In order to construct $f_l: T_l \to X_l$ we need commutativity of the diagram

$$T_{l-1} \xrightarrow{J} T_{l-1}/T_l$$

$$\downarrow^{f_{l-1}} \qquad \downarrow^{g_l}$$

$$X_{l-1} \xrightarrow{\alpha_l} K(U_l).$$

We can only prove that this diagram commutes in dimensions $\leq 2s+1$. To correct for this we relabel B_l above, B_l' and we form B_l from B_l' by killing the obstructions to commutativity as follows:

Define $\Delta = \Delta(f_{l-1}) : U_l \to H^*(T_{l-1})$ by

$$\Delta(u) = f_{l-1}^* \alpha_l^* u - \sum x_i f_{l-1}^* \alpha_l^* u_i$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$, $u_i \in \overline{V}_l$. Then

$$j^* g_l^*(u) = j^* \Psi r \theta(u) = j^* \Psi \left(\sum x_i u_{i_1} \right) = \sum x_i j^* \Phi((\beta_l)^*(u_{i_1}))$$
$$= \sum x_i \Phi(\beta_l^*(u_i)) = \sum x_i f_{l-1}^* \alpha_l^*(u_i) = \Delta(u) + f_{l-1}^* \alpha_l^*(u)$$

Thus Δ is the deviation from commutativity of our diagram above. Let $W_l = U_l/\ker \Delta$. We kill $\Phi^{-1}(\Delta(W))$ in B'_l to form B_l .

To recapitulate, we inductively construct a sequence of spaces B_l , stable vector bundles ζ_l over B_l and maps $f_l: T_l = T(\zeta_l) \to X_l$ such that $\Delta(f_l) = 0$. We take $B_0 = BO$, $\zeta_0 = \zeta$ the universal bundle and f_0 the map such that $f_0^*(u_\omega) = \Phi(u_\omega)$ for $u_\omega \in U_0$. $(X_0 = K(U_0))$. Referring to 2.5, $f_0^* = \eta$, $\alpha_1^* = d$ and $\Delta(f_0) = \eta d - d\theta = 0$. Suppose B_{l-1} , ζ_{l-1} and f_{l-1} have been defined and $\Delta(f_{l-1}) = 0$. Let $p': B'_l \to B_{l-1}$ be the fibration induced by $\beta_l: B_{l-1} \to K(\overline{V}_l)_1$ where β_l is defined by

$$\Phi(\boldsymbol{\beta}_{l}^{*}(\boldsymbol{v}_{1})) = f_{l-1}^{*}\boldsymbol{\alpha}_{l}^{*}(\boldsymbol{v})$$

for $v \in \overline{V_l} \subset U_l$ and $v_1 \in H^*(K(\overline{V_l})_1)$ the element corresponding to v. Let $\zeta'_l = (p')^* \zeta_{l-1}$ and $T'_l = T(\zeta'_l)$.

Viewing $B'_{l} \subset B_{l-1}$ as the fibre of β_{l} , β_{l} factors through β'_{l} . $B_{l-1}/B'_{l} \to K(\bar{V}_{l})_{1}$. Let $\Psi: A(BO) \otimes \bar{V}_{l} \to H^{*}(T_{l-1}/T'_{l})$ be the A(BO) linear map such that $\Psi(v) = \Phi((\beta'_{l})^{*}(v_{1}))$ for $v \in \bar{V}_{l}$. Let θ be as in 2.5, r as in 2.6, and let $g'_{l}: T_{l-1}/T'_{l} \to K(U_{l})$ be defined by $(g'_{l})^{*}(u) = \Psi r \theta(u)$. Since $\Delta(f_{l-1}) = 0$, there is a map f'_{l} making a commutative diagram

$$T_{l-1}/T'_{l} \longrightarrow T'_{l} \longrightarrow T_{l-1} \longrightarrow T_{l-1}/T'_{l}$$

$$\downarrow^{g'_{l}} \qquad \qquad \downarrow^{f'_{l}} \qquad \qquad \downarrow^{f'_{l-1}} \qquad \qquad \downarrow^{g'_{l}}$$

$$K(U_{l}) \xrightarrow{i} X_{l} \longrightarrow X_{l-1} \longrightarrow K(U_{l}).$$

Let $\Delta(f'_l): U_{l+1} \to H^*(T_l)$ be given by $\Delta(f'_l)(u) = (f'_l)^* \alpha^*_{l+1} u + \sum x_i (f')^* \alpha^*_{l+1} u_i$ where $r\theta u = \sum x_i u_i$. Let $W_{l+1} = U_{l+1}/\ker \Delta(f'_l)$ and let $p: B_l \to B'_l$ be the fibration induced

by $\gamma_l: B'_l \to K(W_{l+1})_1$ where $\Phi(\gamma_l^* u_1) = \Delta(f'_l)(u)$ for $u \in W_{l+1}$. Finally let $\zeta_l = p^* \zeta'_l$ and $f_l = f'_l T(p)$. Then $\Delta(f_l) = T(p)^* \Delta(f'_l) = 0$ and the inductive step is complete.

In §5 we prove:

LEMMA 2.7. If $l \ge 3$ and $q \le n$, $f_l^*: H^q(X_l) \approx H^q(T(\zeta_l))$. Furthermore, if M is a smooth n-manifold and $h: M \rightarrow B_0 = BO$ classifies its normal bundle, then any lifting of h to B_{l-1} lifts to B_l .

We next examine $H^*(B_l)$ for l large.

LEMMA 2.8. If $l \ge n$, $V_l^q = U_i^q = 0$ for q < n - 1, $W_l^q = 0$ for $q \le n$ and $V_l^{n-1} = U_l^{n-1} = \{\lambda^{(0,0,...,0)} u_\omega \mid u_\omega \in U_0^{n-1}\}$. Furthermore, $\Phi(\beta_l^*(\lambda^{(0,...,0)} u_\omega)) = \delta_l \tilde{u}_\omega$

 $\tilde{u}_{\omega} \in H^*(T_{l-1}; Z_{2l}), u_{\omega}U \in H^*(T_{l-1})$ is the mod two reduction of \tilde{u}_{ω} and δ_l is the Bockstein associated with $Z_2 \rightarrow Z_{2l+1} \rightarrow Z_{2l}$. Thus for $l \ge n$,

$$H^{q}(B_{l}) \approx H^{q}(BO)/I_{n}^{q} \qquad q < n$$

$$H^n(B_l)/\Phi^{-1}{\{\delta_{l+1}\tilde{u}_{\omega}\}}\approx H^n(BO)/I_n^n$$

We form B_{∞} from B_{l} , $l \ge n$, by killing classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_{\omega}) \in H^{n+1}(B_{l}; Z_{\tau})$ where Z_{τ} denotes twisted integer coefficients, twisted by w_{1} , $\Phi: H^{*}(B_{l}; Z_{\tau}) \approx H^{*}(T(\zeta_{l}); Z)$ is the Thom isomorphism and δ^{l} is the Bockstein associated with $Z \rightarrow Z \rightarrow Z_{2l}$. Let \tilde{B}_{l} be the two sheeted cover of \tilde{B}_{l} defined by w_{1} . The classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_{\omega})$ may be represented by $Z_{2^{-}}$ equivariant maps $x_{\omega}: \tilde{B}_{l} \rightarrow K(Z, n)$ where K(Z, n) has the action defined by the nontrivial action of Z_{2} on Z. Let \tilde{B}_{∞} be the fibration over \tilde{B}_{l} induced by

$$x = \prod x_{\omega} : \tilde{B}_{l} \to \prod K(Z, n)$$

Since x is Z_2 -equivariant, Z_2 acts freely on \tilde{B}_{∞} . Let $B_{\infty} = \tilde{B}_{\infty}/Z_2$. The map $B_{\infty} = \tilde{B}_{\infty}/Z_2 \rightarrow \tilde{B}_l/Z_2 = B_l$ has fibre $\Pi K(Z, n)$. With Z_2 coefficients, $\pi_1(B_l)$ acts trivially on the cohomology of the fibre. The Serre spectral sequences, with Z_2 coefficients has its usual, nonlocal coefficient form and the usual argument shows

that in dimensions $\leq n$,

$$H^{\ast}(BO_{\infty}) = H^{\ast}(B_{l})/\{\Phi^{-1}(\delta^{l+1}\tilde{u}_{\omega})\}$$

Thus for $q \leq n$

$$0 \rightarrow I_n^q \rightarrow H^q(BO) \rightarrow H^q(B_\infty) \rightarrow 0$$

is exact. Also if M is an *n*-manifold and $h: M \to B$ is covered by a bundle map $g: \nu \to \zeta_l$, $T(g)^*(\delta^{l+1}\tilde{u}_{\omega}) = \delta^{l+1}T(g^*)(\tilde{u}_l) = 0$ since the top homology class of $T(\nu)$ is spherical. Therefore, h lifts to B_{∞} .

Finally, assume B_{∞} is a CW complex and let

 $BO/I_n = B_\infty^n \cup e_1^{n+1} \cup e_2^{n+1} \cdots e_m^{n+1}$

where e_i^{n+1} is attached by $f_i | S^n$, $f_i : (D^{n+1}, S^n) \to (B^{n+1}, B^n)$ and $[f_i] \in \pi_{n+1}(B_{\infty}^{n+1}, B_{\infty}^n)$ give a \mathbb{Z}_2 -basis for the image of

 $\pi_{n+1}(B^{n+1}_{\infty}, B^n_{\infty}) \xrightarrow{\rho} H_{n+1}(B^{n+1}_{\infty}, B^n_{\infty}) \xrightarrow{\partial^*} H_n(B^n_{\infty}, B^{n-1}_{\infty})$

The maps f_i give an extension of $B_{\infty}^n \subset B_{\infty}$, $f: BO/I_n \to B_{\infty}$ and

$$f^*: H^q(B_{\infty}) \approx H^q(BO/I_n) \quad \text{for} \quad q \leq n$$
$$H^q(BO/I_n) = H^q(BO)/I_n = 0 \quad \text{for} \quad q > n$$

Also any map of an *n*-manifold into B_{∞} is homotopic to a map factoring through *f*. The proof of Theorem 1 is thus complete, modulo the lemmas and propositions of this section.

§3. Proofs of 2.3, 2.5, and 2.6

Let Λ_l^k be the Z_2 -subspace of Λ^* generated by λ^I with l(I) = l, $t(I) \ge k$, and I admissible. Let

 $d: A \otimes \Lambda_{l}^{k} \to A \otimes A_{l-1}^{k}$

be defined by

$$d(1 \otimes \lambda^{I}) = \sum \lambda^{I}(\lambda_{j}\lambda_{J})\chi(Sq^{j+1}) \otimes \lambda^{J}$$
(3.1)

where the sum is over all j and admissible J. Proposition 2.3 follows from 2.1 and 3.2(ii) below:

PROPOSITION 3.2.

- (i) $\{\lambda_I \mid I \text{ admissible}\}$ is a \mathbb{Z}_2 -basis for Λ .
- (ii) The following is exact:

$$\longrightarrow A \otimes \Lambda_{l}^{k} \xrightarrow{d} A \otimes \Lambda_{l-1}^{k} \longrightarrow \cdots \longrightarrow A \otimes A_{0}^{k} \xrightarrow{\epsilon} A/A\{\chi(Sq^{i}) \mid i > k\}$$

where $\epsilon(a \otimes \lambda^{()}) = \{a\}.$

(iii) If I and J are admissible, l(I) = l, l(J) = l-1, and $I_l = (1, 2, 4, ..., 2^{l-1})$, then $\lambda^{I+rI_l}(\lambda_{j+r}\lambda_{J+2rI_{l-1}}) = \lambda^I(\lambda_j\lambda_J)$.

Proof. For any sequence $T = (t_1, t_2, ..., t_l)$ and integer r, let $h^r(\lambda_T) = \lambda_{T+rI_l}$. Extending linearly, h^r gives a well defined map $h^r : \Lambda \to \Lambda$ since for any element of Λ of the form $\alpha = \lambda_{I_1} \beta \lambda_{I_2}$ where β is a relation for Λ as in 2.2, $h^r(\alpha)$ also has this form. Since $h^r h^{-r}$ is the identity, h^r is an isomorphism for all r. Furthermore, $h^r(\lambda_I)$ is admissible if and only if λ_I is admissible.

Let $\overline{\Lambda} \subset \Lambda$ be the subalgebra generated by $\lambda_0, \lambda_1, \lambda_2, \ldots$ In [8] it is proved that $\{\lambda_I \mid I \text{ admissible}\}$ is a basis for $\overline{\Lambda}$. For any λ_I , $h^r(\lambda_I) \in \overline{\Lambda}$ for r sufficiently large. Thus $\{\lambda_I \mid I \text{ admissible}\}$ is a basis for Λ .

In [2], 3.2(ii) was proved for $k \ge 0$. From 2.2 one sees that $\lambda_{-1}\lambda_{-1} = 0$ and if $t(J) \ge 0$, $\lambda_{-1}\lambda_J$ is a sum involving $\lambda_{J'}$'s with t(J') > 0 and $\lambda_J\lambda_{-1}$. Suppose $J_1 = (j_1, \ldots, j_m)$, $J_2 = (j_{m+1}, \ldots, j_l)$ and $J = (j_1, \ldots, j_l)$ are admissible with J_1 or J_2 possibly the empty sequence (). Define $\lambda^{J_1}\lambda^{J_2} = \lambda^J$. Suppose $j_m \ge 0$ and $j_{m+1} < -1$. Then 3.1 yields

$$d(\lambda^{J_1}\lambda^{-1}\lambda^{J_2}) = (d\lambda^{J_1})\lambda^{-1}\lambda^{L_2} + \lambda^{J_1}\lambda^{J_2}$$
$$d(\lambda^{J_1}\lambda^{J_2}) = (d\lambda^{J_1})\lambda^{J_2}.$$

Let

$$D(\lambda^{J_1}\lambda^{J_2}) = \lambda^{J_1}\lambda^{-1}\lambda^{J_2}, D(\lambda^{J_1}\lambda^{-1}\lambda^{J_2}) = 0.$$

Then for k < 0, $D: A \otimes \Lambda_{l}^{k} \to A \otimes \Lambda_{l+1}^{k}$ satisfies dD + Dd = identity. Therefore 3.2(ii) holds for k < 0.

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Finally we prove 3.2(iii). Note that if I is admissible, $I + rI_l$ is admissible and if $(h^r)^* : \Lambda^* \to \Lambda^*$ is the dual of h^r , h^r , $(h^r)^* \lambda^I = \lambda^{I-rI_l}$. Therefore

$$\lambda^{I}(\lambda_{j}\lambda_{J}) = (h^{r})^{*}(\lambda^{I+rI_{l}})(\lambda_{j}\lambda_{J})$$
$$= \lambda^{I+rI_{l}}(h^{r}(\lambda_{j}\lambda_{J})) = \lambda^{I+rI_{l}}(\lambda_{j+r}\lambda_{J+2rI_{l-1}})$$

Proof of 2.5. Let $C_l = A \otimes U_l$, $D_l = A(BO) \otimes V_l$, l > 0, and $D_0 = H^*(MO)$. Denote $a \otimes u \in C_l$ by au and $a \circ v \otimes w \in D_l$, l > 0, by $(a \circ v)w$. We filter C_l and D_l as follows: $F_q(C_l)$ is spanned by $a\lambda^I u_l$ with $|u_{\omega}| \leq q$ and $F_q(D_l)$, l > 0, is spanned by all $a \circ v\lambda^I u_l$ with $|u_{\omega}| + 2^l |v| \leq q$. $F_q(D_0)$ is spanned by all au_{ω} where $a \in A$, $u_{\omega} \in U_0 = \{u_{\omega} \mid 2^i - 1 \notin \omega\}$ and $|u_{\omega}| \leq q$.

The chain complex (C_{l}, d) is a direct sum of chain complexes of the form described in 3.2, indexed by the $u_{\omega} \in U_{0}$. Hence d is filtration preserving and:

(3.3) The following is exact.

$$\longrightarrow F_q(C_l) \xrightarrow{d} F_q(C_{l-1}) \longrightarrow \cdots \longrightarrow F_q(C_0)$$

Using induction on l we define A linear maps $\theta: C_l \to D_l$ and A(BO) linear maps $d: D_l \to D_{l-1}$ such that

(i) θ is an isomorphism and $\theta: C_0 \to D_0$ is given by $\theta(a \otimes u_{\omega}) = a \Phi(u_{\omega}) \in H^*(MO), u_{\omega} \in U_0$.

- (iii) If $u \in V_l \subset U_l$, $\theta(u) = u$
- (iv) $\theta(F_q(C_l)) = F_q(D_l)$
- (v) Suppose $\lambda^{I} u_{\omega} \in U_{l}$. Let α and β be the partitions

$$\alpha = \bigcup_{r < l} 2^r \omega_r, \qquad \beta = \bigcup_{r \ge l} 2^{r-l} \omega_r$$

Note $u_{\omega} = u_{\alpha} u_{\beta}^{2^{1}}$. Then θ satisfies

$$\theta(\lambda^{T} u_{\omega}) = u_{\beta} \lambda^{T} u_{\alpha} \mod F_{|u_{\omega}|-1}(D_{l})$$

where $I' = I + |u_{\beta}| I_{l}$.

Note that Proposition 2.5 consists of statements (i), (ii), and (iii) above.

For l=0, θ is defined by (i) and d=0 on D_0 .

Suppose θ and d have been defined on C_k and D_k k < l, and satisfy (i)-(v). Define $d = d_D : D_l \to D_{l-1}$ to be the A(BO) linear map such that for $u \in V_l$,

⁽ii) $d\theta = \theta d$

 $d_{D}(u) = \theta(d_{C}u)$. We next define $\theta: C_{l} \to D_{l}$. Suppose $\lambda^{I}u_{\omega} \in U_{l}$ and $u_{\omega} = u_{\alpha}u_{\beta}^{2^{1}}$ as in (v). If $u_{\beta} = 1$, $\lambda^{I}u_{\omega} \in V_{l}$ and we define $\theta(\lambda^{I}u_{\omega}) = \lambda^{I}u_{\omega}$. In this case (i) – (v) are satisfied. Suppose $u_{\beta} \neq 1$. Let

$$X = \theta(d(\lambda^{I} u_{\omega})) + u_{\beta} \theta(d\lambda^{I'} u_{\alpha})$$

where $I' = I + |u_{\beta}| I_{l}$. By induction, $\theta d = d\theta$ on C_{l-1} and hence $\partial X = 0$. We show that $X \in F_{p-1}(D_{l})$ where $p = |u_{\omega}|$. Decompose u_{α} into $u_{\alpha_{1}}u_{\alpha_{2}}^{2^{l-1}}$ as in (v).

$$\theta(d\lambda^{I}u_{\omega}) = \sum \lambda^{I}(\lambda_{j}\lambda_{K})\chi(Sq^{j+1})\theta(\lambda^{K}u_{\omega})$$

= $\sum \lambda^{I}(\lambda_{j}\lambda_{K})(\chi(Sq^{j+1}) \circ u_{\alpha_{2}}u_{\beta}^{2})\lambda^{K'}u_{\alpha_{1}} \mod F_{p-1}$

where $K' = K + |u_{\alpha_2}u_{\beta}^2| I_{l-1}$. On the other hand,

$$u_{\beta}\theta(d\lambda^{I'}u_{\alpha}) = \sum \lambda^{I'}(\lambda_{j}\lambda_{J})u_{\beta}\chi(Sq^{j+1})\theta(\lambda^{J}u_{\alpha})$$

In A(BO),

$$u_{\beta}\chi(Sq^{j+1}) = \chi(Sq^{j-q+1}) \circ u_{\beta}^2 + \sum_{k < q} \chi(Sq^{j-k+1}) \circ Sq^k u_{\beta}$$

where $q = |u_{\beta}|$.

$$\theta(\lambda^J u_{\alpha}) = u_{\alpha_2} \lambda^{J'} u_{\alpha_1} \mod F_{|u_{\alpha}|-1}$$

where $J' = J + |u_{\alpha_2}| I_{l-1}$. If $u\lambda^I v$ has filtration less than $|u_{\alpha}| - 1$ and k < q, $Sq^k u_{\beta} u\lambda^I v$ has filtration less than $p = |u_{\omega}|$.

Hence

$$u_{\beta}\theta(d\lambda^{I'}u_{\alpha}) = \sum_{j,J} \lambda^{I'}(\lambda_{j}\lambda_{J})(\chi(Sq^{j-q+1}) \circ u_{\alpha}u_{\beta}^{2})\lambda^{J'}u_{\alpha_{1}} \mod F_{p-1}$$

In the above sum, replace j by j+q and J by $K+2qI_{l-1}$. Then

$$u_{\beta}\theta(d\lambda^{I'}u_{\alpha}) = \sum_{j,K} \lambda^{I'}(\lambda_{j+q}\lambda_{K+2qI_{l-1}})\chi(Sq^{j+1}) \circ u_{\alpha_2}u_{\beta}^2\lambda^{K'}u_{\alpha_1} \mod F_{p-1}$$

where $K' = K + |u_{\alpha_2}u_{\beta}^2| I_{l-1}$. But $I' = I + qI_l$ and hence by 3.2(iii),

$$\lambda^{I'}(\lambda_{j+q}\lambda_{K+2qI_{l-1}}) = \lambda^{I}(\lambda_{j}\lambda_{K})$$

Hence $X \in F_{p-1}(D_l)$.

By (iv) there is a $Y \in F_{p-1}(C_{l-1})$ such that $\theta(Y) = X$ and by (i) and (ii), dY = 0. Hence for l > 1, by 3.3, there is a $Z \in F_{p-1}(C_l)$ such that dZ = Y. We verify that there is such a Z for l = 1 by showing that when l = 1, $X \in \Phi(I_n)$. In this case

$$X = \chi(Sq^{i+1})\Phi(u_{\alpha}u_{\beta}^{2}) + u_{\beta}\chi(Sq^{i+q+1})\Phi(u_{\alpha})$$
$$= \sum_{j < q} \chi(Sq^{i+q+1-j})\Phi((Sq^{j}u_{\beta})u_{\alpha})$$

where 2(i+1) > n-q, $q = |u_{\beta}|$. But then, $2(i+q-j+1) > n-|(Sq^{i}u_{\beta})u_{\alpha}|$ and hence $X \in \Phi(I_{n})$.

We now define $\theta(\lambda^{I}u_{\omega})$ by induction on $|u_{\omega}| =$ filtration degree of $\lambda^{I}u_{\omega}$. For $|u_{\omega}| = 0$, $\theta(\lambda^{I}1) = \lambda^{I}1$. If θ is defined on $F_{|u_{\omega}|-1}(C_{l})$, let

 $\theta(\lambda^{I} u_{\omega}) = u_{\beta} \lambda^{I'} u_{\alpha} + \theta(Z)$

where Z, α , β , and I' are as above. Then $d\theta(Z) = \theta(dZ) = \theta(Y) = X$ and

$$d\theta(\lambda^{I}u_{\omega}) = d(u_{\beta}\lambda^{I'}u_{\alpha}) + d\theta(Z)$$
$$= u_{\beta}\theta(d(\lambda^{I'}u_{\alpha})) + X = \theta(d(\lambda^{I}u_{\omega}))$$

Note that elements of the form $u_{\beta}\lambda^{I'}u_{\alpha}$, as above, together with $F_{p-1}(D_l)$, span $F_p(D_l)$ over A. Thus $\theta: C_l \to D_l$ is an epimorphism. (It is at this point that we use λ^I where I has negative entries. For each $u_{\beta}\lambda^{I'}u_{\alpha} \in H^*(BO)V_l$ we need $\lambda^I u_{\alpha}u_{\beta_l}^{2'} \in U_l$ such that $I' = I + |u_l| I_l$.) Elements of the form $\lambda^I u_{\alpha}u_{\beta}^{2'}$ are an A basis for C_l and elements of the form $u_{\beta}\lambda^{I'}u_{\alpha}$ are an A basis for D_l . Hence $\theta: C_l \to D_l$ is an isomorphism and the proof of 2.5 is complete.

Proof of 2.6. Let $v_i \in H^*(BO)$ be the Wu classes, that is, $\Phi(v_i) = \chi(Sq^i)\Phi(1)$ where $\Phi: H^*(BO) \to H^*(MO)$ is the Thom isomorphism.

LEMMA 3.4.

$$v_i = \sum s_{\omega}$$

where the sum ranges over all ω with entries only of the form $2^{i} - 1$ and $|s_{\omega}| = i$.

Proof. We view $H^*(BO) \subset Z_2[t_1, t_2, ...], |t_i| = 1$, and $t_1 t_2 ...$ as the Thom class. Let $Sq = Sq^0 + Sq^1 + \cdots$ and $v = v_0 + v_1 + \cdots$. Then

 $\chi(Sq)t_i = \sum t_i^{2^j}$

and

$$v(t_1, t_2, \ldots)(t_1 t_2 \cdots) = \chi(Sq)(t_1 t_2 \cdots)$$
$$= \prod_i \left(\sum_j t_i^{2j-1}\right)(t_1 t_2 \cdots) = \left(\sum_{\omega} s_{\omega}\right)(t_1 t_2 \cdots)$$

where the sum ranges over ω with entries only of the form $2^{j} - 1$.

Let x_1 and $x_2 \in A(BO)$ be given by

$$x_1 = \sum_{j>0} Sq^j \circ v_{s+1-j}, \qquad x_2 = \sum Sq^j \circ v_{s+2-j}$$

Recall s = [n/2] and n is the dimension of the manifolds we are considering. Let $y_i^1 \in D_1$ be defined by

$$y_1^1 = x_1 \lambda^s$$
, $y_2^1 = x_2 \lambda^s$, $y_3^1 = v_{s+1} \lambda^{s+1} + v_{s+2} \lambda^s + x_2 \lambda^s$

LEMMA 3.5. There are elements $y_i^2 \in D_2$ such that $dy_i^2 = y_i^1$ and

$$y_1^2 = \lambda^{0,0} v_s^2 \mod F_{2s-1}$$

$$y_2^2 = \lambda^{0,-1} v_{s+1}^2 \mod F_{2s+1}$$

$$y_3^2 = \lambda^{-1,-2} v_{s+2} \mod F_{2s+3}$$

If s is odd, there is an element y_2^3 such that $y_2^3 = (Sq^1 + w_1)y_2^2$ and

$$y_2^3 = \lambda^{-1,-2,-4} w_1^4 v_{s+2}^2 \mod F_{2s+7}$$

Proof. We first show that $dy_i^1 = 0$, $d: D_1 \rightarrow D_0 = H^*(MO)$. Let $U \in H^0(MO)$ be the Thom class.

$$dy_{1}^{1} = x_{1}d\lambda^{s} = \sum Sq^{i}(v_{s+1-j}\chi(Sq^{s+1})U) + v_{s+1}\chi(Sq^{s+1})U$$

$$= (Sq^{s+1}v_{s+1})U + v_{s+1}^{2}U = 0$$

$$dy_{2}^{1} = \sum Sq^{i}(v_{s+2-j}\chi(Sq^{s+1})U)$$

$$= \sum Sq^{i}(v_{s+1}\chi(Sq^{s+2-j})U) = (Sq^{s+2}v_{s+1})U = 0$$

$$dy_{3}^{1} = v_{s+1}\chi(Sq^{s+2})U + v_{s+2}\chi(Sq^{s+1})U + dy_{2}^{1} = 0$$

We next show that y_1^2 exists. In $A \otimes \Lambda^*$ one may easily calculate $d\lambda^{0,0} = Sq^1\lambda^0$.

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Hence, by the arguments in the proof of 2.5,

$$d\lambda^{0,0}v_s^2 = \theta(d\lambda^{0,0}v_s^2) = \theta(Sq^1\lambda^0v_s^2)$$
$$= Sq^1 \circ v_s\lambda^s \mod F_{2s-1}$$
$$= \sum_{j>0} Sq^j \circ v_{s+1-j}\lambda^s \mod F_{2s-1} = y_1^1 \mod F_{2s-1}$$

Thus $u = d\lambda^{0,0}v_s^2 + y_1^1 \in F_{2s-1}$ and du = 0. Therefore there is a $z \in F_{2s-1}(D_2)$ such that dz = u. Let $y_1^2 = \lambda^{0,0}v_s^2 + z$. The existence of y_2^2 , y_3^2 , and y_3^3 are proven in an analogous fashion.

We now define $r: A(BO) \otimes V_l \to A(BO) \otimes \overline{V}_l$. For $l \neq 2$ and $l \neq 3$, s odd, $\overline{V}_l = V_l$ and r is the identity; $\overline{V}_l \subset V_l$ and $r \mid A(BO) \otimes \overline{V}_l$ is the identity. \overline{V}_2 is formed from V_2 by omitting the basis elements $\lambda^{0,0} w_s^2$, $\lambda^{0,-1} w_{s+1}^2$ and $\lambda^{-1,-2} w_{s+2}^2$. By 3.4, v_i involves $w_i = s_{(1,1,\dots,1)}$ when v_i is expressed in the u_{ω} basis. Let

$$r(\lambda^{0,0}w_s^2) = y_1^2 - \lambda^{0,0}w_s^2$$

$$r(\lambda^{0,-1}w_{s+1}^2) = y_2^2 - \lambda^{0,-1}w_{s+1}^2$$

$$r(\lambda^{-1,-2}w_{s+2}^2) = y_2^3 - \lambda^{-1,-2}w_{s+2}^2$$

We define r on $A(BO) \otimes V_3$ analogously. Then $r(y_i^2) = r(y_2^3) = 0$.

We conclude this section with an algebraic lemma about the y_i^{i} 's. Let $L_l \subset A(BO) \otimes V_l$ be defined as follows: $L_l = 0$ for l = 0, l = 3 and s even, and l > 3.

$$L_1 = A(BO)(\{y_i^1\} + S_1)$$

where $S_1 = \{v_3 Sq^2 \lambda^2\}$ when s = 2 and $S_1 = 0$ for $s \neq 2$.

$$L_2 = A(BO)(\{y_i^2\} + S)$$

where $S_2 = \{v_3 \lambda^{1,2}\}$ when s = 2 and $S_2 = 0$, $s \neq 2$.

$$L_3 = A(BO)\{y_3^2\}$$

 $(d(v_3\lambda^{1,2})=v_3\mathbf{S}q^2\lambda^2).$

LEMMA 3.6. $d(L_l) \subset L_{l-1}$, $r(L_l) = 0$ for l > 1 and the sequence

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \longrightarrow \cdots \longrightarrow L_0$$

is exact at L_l^q for all l and $q \leq 2s+2$.

Proof. The first part of 3.6 is clear from the definition of L_l . One easily checks that if $x \in A(BO)$, $|x| \leq 1$ and $d(xy_2^3) = 0$, then x = 0 and therefore $d: L_3^q \rightarrow L_2^{q+1}$ is an injection for $q \leq 2s+2$. $d: L_2 \rightarrow L_1$ is clearly onto. To check exactness at L_2^q , $q \leq 2s+2$ one must verify that if $y = x_1y_1^1 + x_2y_2^1 + x_3y_3^1 + x_4v_3Sq^2\lambda^2 = 0$, $x_i \in A(BO)$ and $|y| \leq 2s+3$, then $x_1 = x_3 = x_4 = 0$ and $x_2 = 0$ or s is odd and $x_2 = Sq^1 + w_1$. This is a tedious but straightforward calculation, made somewhat simpler by the following observation. Let

$$F: A(BO) \otimes \{\lambda^s\} \rightarrow H^*(MO \wedge K(Z_2, N))$$

be given by

$$F(a \circ u\lambda^{s}) = a(u\chi(Sq^{s+1})U \otimes \iota_{N})$$

Then

 $F(y_1^1) = v_{s+1}U \otimes \iota_N + U \otimes Sq^{s+1}\iota_N$ $F(y_2^1) = U \otimes Sq^{s+2}\iota_N$ $F(v_3Sq^2\lambda^2) = v_3^2U \otimes Sq^2\iota_N$

We leave the details to the reader.

§4. Proofs of 2.4 and 2.8

Let $\{A \otimes \Lambda_{l}^{k}, d\}$ be the chain complex described in Proposition 3.2.

PROPOSITION 4.1. For each integer k, there are Ω -spectra $Y_l = Y_l(k)$ and maps $\rho_l = \rho_l(k): Y_{l-1} \rightarrow K(\Lambda_l^k)$ of degree one, $l = 0, 1, 2, \ldots$ such that

(i) $Y_0 = K(\Lambda_0^k)$. Y_l is a fibration over Y_{l-1} induced by ρ_l from the contractible fibration over $K(\Lambda_l^k)$.

(ii) If $i: K(\Lambda_{l-1}^k) \to Y_{l-1}$ is the inclusion of the fibre,

 $(\rho_l i)^* = d : A \otimes \Lambda_l^k \to A \otimes \Lambda_{l-1}^k$

where d is as in 3.2.

(iii) If M is a smooth, compact n-manifold and v is its normal bundle, then

 $[T(\nu), Y_l]_p \rightarrow [T(\nu), Y_{l-1}]_p$

is an epimorphism for p < 2k + 2.

(iv) Suppose k = 0. Let I(l, 0) = (0, ..., 0) have length *l*.

$$\rho_l^* \lambda^{I(l,0)} = \delta_l \tilde{\iota}$$

where $\iota \in H^0(Y_{l-1}; Z_{2l})$, $\tilde{\iota}$ reduced modulo two is the generator $\iota \in H^0(Y_{l-1}) \approx Z_2$ and δ_l is the Bockstein associated to $Z_2 \rightarrow Z_{2l+1} \rightarrow Z_{2l}$.

Proof. For $k \ge 0, 4.1(i), (ii)$, and (iii) were proved in [5]. For $k < 0, \{A \otimes \Lambda_l^k, d\}$ is a free acyclic resolution of the zero A module so that the existence of Y_l and ρ_l easily follow by induction on l. If M is as in (iii), $v: T(v) \to Y_{l-1}$ has degree p, p < 2k+2 and k < 0, then $|(\rho_l v)^*(\lambda^I)| > n$ and (iii) follows.

Finally we prove (iv). The formula for d in 3.1 shows that $d\lambda^{I(l,0)} = Sq^1\lambda^{I(l-1,0)}$ The complex,

$$\longrightarrow A \otimes \{\lambda^{I(\mathfrak{l},0)}\} \stackrel{d}{\longrightarrow} A \otimes \{\lambda^{I(\mathfrak{l}-1,0)}\} \longrightarrow \cdots A \otimes \{\lambda^{I(0,0)}\}$$

is realized by the tower

$$\rightarrow K(Z_{2}) \rightarrow K(Z_{2l-1}) \rightarrow \cdots \rightarrow K(Z_{2})$$

with k-invariants, $\delta_l : K(Z_{2l}) \to K(Z_2)$. Except for $\lambda^{I(l,0)}$, the generators of Λ_l^0 have dimension >0 and hence kill classes of dimension >1. Thus $Y_l = K(Z_{2l+1})$ in dimensions ≤ 1 . Therefore (iv) holds.

Proof of 2.4: We wish to realize the complex $\{A \otimes U_l, d\}$ by a tower of spectra, X_l . Let $Y_l(k)$ and $\rho_l(k)$ be as in 4.1. For a spectrum Z, let SZ denote the shift suspension, i.e., $(SZ)_q = Z_{q+1}$. Define X_l and $\alpha_l : X_{l-1} \to K(Y_l)$ by

$$X_{l} = \prod_{u_{\omega} \in U_{0}} S^{|u_{\omega}|} Y_{l}([(n - |u_{\omega}|)/2])$$
$$\alpha_{l} = \prod S^{|u_{\omega}|} \rho_{l}([(n - |u_{\omega}|)/2])$$

The map α_l takes X_{l-1} into $K(U_l)$ since

$$\prod S^k K(\Lambda_l^k) = K(U_l)$$

where k ranges over $[(n-|u_{\omega}|/2], |u_{\omega}| \in U_0$. Proposition 2.4 now follows directly from 4.1.

Proof of 2.8: Using induction on l, one easily proves that if I is admissible and l = l(I),

$$|\lambda^{I}| \geq 2t(I) \left(1 - \frac{1}{2^{l}}\right)$$

Suppose $l \ge n$ and $\lambda^{I} u_{\omega} \in U_{l}$. Then $2(t(I)+1) > n - |u_{\omega}|$. Therefore

$$|\lambda^{I}u_{\omega}| \ge 2t(I)\left(1-\frac{1}{2^{l}}\right)+|u_{\omega}| \ge n-1-\frac{n-|u_{\omega}|-1}{2^{l}}>n-2$$

Also if $|u_{\omega}| > n-1$, $|\lambda^{I}u_{\omega}| > n-1$. If $|u_{\omega}| < n-1$, $t(I) \ge 1$ and hence $|\lambda^{I}| \ge l \ge n$. Therefore $U_{l}^{q} = 0$ for q < n-1 and $U_{l}^{n-1} = \{\lambda^{I(l,0)}u_{\omega} \mid u_{\omega} \in U_{0}^{n-1}\}$ since $\lambda^{I(l,0)}$ is the only λ^{I} with $t(I) \ge 0$ and $|\lambda^{I}| = 0$. If r > l and $\omega_{r} \ne \{\}$, $|u_{\omega}| \ge |u_{\omega_{r}}^{2r}| \ge 2^{r} > n$. Hence $V_{l}^{q} = U_{l}^{q}$ for $q \le n-1$.

By the definition of $\beta_l : B_{l-1} \to K(V_l)$,

$$\Phi(\beta_{l}^{*}(\lambda^{I(l,0)}u_{\omega})) = f_{l-1}^{*}\alpha_{l}^{*}(\lambda^{I(l,0)}u_{\omega})$$

By 4.1(iv) $\alpha_l^*(\lambda^{I(l,0)}u_{\omega}) = \delta_l \tilde{\iota}$ where $\tilde{\iota} \in H^*(X_{l-1}; Z_{2^l})$ comes from the factor of X_{l-1} , $Y([n-|u_{\omega}|/2])$. Since the diagram



commutes, $\tilde{u} = f_{0-1}^* \tilde{\iota}$ reduced modulo two is $p_1^* f_0^* u_\omega = p_1^* u_\omega U_0 = u_\omega U_{l-1}$, where U_l is the Thom class of T_l and the proof of 2.8 is complete.

§5. Proof of 2.7

If G_1 and G_2 are graded groups and $h: G_1 \to G_2$ is a homomorphism of degree *i*, we will say that *h* is *k* connected if $h: G_1^q \to G_2^{q+i}$ is an epimorphism for q < k and a monomorphism if $q \leq k$. We will say that a sequence of graded groups and homorphisms,

$$\cdots \to G_{\mathfrak{l}} \to G_{\mathfrak{l}-1} \to \cdots$$

is k-exact if

$$G_{l+1}^{q-i} \rightarrow G_l^q \rightarrow G_{l-1}^{q+j}$$

is exact for all l and $q \leq k$.

In §3 we constructed isomorphisms $\theta: A \otimes U_l \to A(BO) \otimes V_l$ and a subcomplex $\{L_l, d\} \subset \{A(BO) \otimes V_l, d\}$ such that

 $\longrightarrow L_{l} \xrightarrow{d} L_{l-1} \xrightarrow{d} \cdots \longrightarrow L_{0} = 0$

is 2s+2 exact, s = [n/2]. In §4 we constructed a tower of fibrations $\rightarrow X_l \rightarrow X_{l-1} \rightarrow$ with k-invariants $\alpha_l : X_{l-1} \rightarrow K(U_l)$ associated to the complex $\{A \otimes U_l, d\}$. Let

 $\bar{H}^{*}(K(U_{l})) = H^{*}(K(U_{l}))/\theta^{-1}(L_{l})$ $\bar{H}^{*}(X_{l}) = H^{*}(X_{l})/\alpha_{l-1}^{*}\theta^{-1}(L_{l-1})$

LEMMA 5.1: The maps

 $K(U_l) \xrightarrow{i} X_l \xrightarrow{p} X_{l-1} \xrightarrow{\alpha_l} K(U_l)$

induce a 2s+2-exact sequence

 $\rightarrow \overline{H}^*(K(U)) \rightarrow \overline{H}^*(X_{l-1}) \rightarrow \overline{H}^*(X_l) \rightarrow$

Proof: Let E_l be the kernel of

$$H^*(X_l) \to \lim_{k \to \infty} H^*(X_k)$$

Then $H^*(X_l) \approx H^*(MO)/\Phi(I_n) \oplus E_l$ and E_l and $A \otimes U_l$ are related by the diagram



where the $\bar{\alpha}_l$ and $\bar{\iota}_l$ are defined by α_l^* and i_l^* and each pair of composable arrows is exact. Dividing $A \otimes U_l$ and E_{l-1} by $\theta^{-1}(L_l)$ and $\bar{\alpha}_l \theta^{-1}(L_{l-1})$, respectively, produces the same type of diagram with exactness replaced by 2s + 2-exactness. The desired result then follows.

In §2 we defined maps

 $g'_l: K(U_l) \rightarrow T_{l-1}/T'_l$

In §6 we prove:

LEMMA 5.2. The map g'_i induces a 2s+2-connected map

$$F_l: \overline{H}^*(K(U_l)) \to H^*(T_{l-1}/T'_l)$$

for $l \geq 1$.

Proof of 2.7: We first prove 2.7(ii). Suppose M is a smooth *n*-manifold, $h: M \to B_0 = BO$ classifies ν , the normal bundle of M and $\tilde{h}: M \to B_{l-1}$ is a lifting of h. Let $T(\tilde{h}): T(\nu) \to T_{l-1}$ denote the associated Thom space map. Then $f_{l-1}T(\tilde{h}): T(\nu) \to X_{l-1}$ is a lifting of $f_0T(h): T(\nu) \to X_0$ and hence by 2.4(iv), $f_{l-1}T(\tilde{h})$ lifts to X_l and therefore $\alpha_l f_{l-1}T(\tilde{h}) = 0$. Thus for $v \in V_l$

$$\Phi h^* \beta_l^*(v_1) = T(\tilde{h})^* \Phi(\beta_l^*(v_1)) = T(\tilde{h})^* f_{l-1}^* \alpha_l^*(v) = 0$$

Thus $\beta_l \tilde{h} = 0$ and \tilde{h} lifts to $h': M \to B'_l$

If $u \in U_{l+1}$, $\bar{u} = \{u\} \in W_{l+1} = U_{l+1}/\ker \Delta$ and $\nu \theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$, then

$$\Phi((h')^*\gamma_l^*(\bar{u}_1)) = T(h')^*\Phi(\gamma_l^*\bar{u}_1) = T(h')\Delta(u).$$

Recall,

$$\Delta(u) = (f'_{l})^{*} \alpha^{*}_{l+1} u - \sum x_{i} (f'_{l+1})^{*} \alpha^{*} u_{i}$$

But $T(h')^*$ is A(BO) linear and $\alpha_{l+1}f'T(h')=0$ as above. Thus $T(h')^*\Delta(u)=0$ and hence $\gamma_l h'=0$. Therefore h' lifts to B_l and the proof of 2.7(ii) is complete. We note for further reference:

LEMMA 5.3: $T(h')^* \Delta(u) = 0$ for $u \in U_{l+1}$.

LEMMA 5.4. If $\delta^*: H^*(T'_l) \to H^*(T_{l-1}/T'_l), \ \delta^*\Delta(u) = 0$ for $u \in U_{l+1}$.

Proof. Consider the commutative diagram:

$$H^{*}(X_{l}) \xrightarrow{i^{*}} H^{*}(K(U_{l}))$$

$$\downarrow^{f_{l}} \qquad \qquad \downarrow^{(g_{l})}$$

$$H^{*}(T_{l}') \xrightarrow{\alpha^{*}} H^{*}(T_{l-1}/T_{l}')$$

Recall, g'_0 realizes $\Psi r\theta$, $i^*\alpha^*_{l+1} = d$ and Ψ , r, and $d:A(BO) \otimes V_{l-1} \rightarrow A(BO) \otimes V_{l-1}$ are A(BO) linear. Hence,

$$\delta^* \Delta(u) = \delta^*((f'_l)^* \alpha^*_{\alpha+1} u + \sum x_i (f')^* \alpha^*_{l+1} u_i)$$

= $(g'_l)^* i^* \alpha^*_{l+1} u + \sum x_i (g'_l)^* i^* \alpha^*_{l+1} u_i$
= $\Psi r \theta du + \sum x_i \Psi r \theta du_i = \Psi r d \theta u + \sum \Psi r dx_i \theta(u_i)$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$. But for $v \in V_{l+1}$, $\theta(v) = v$. Thus

$$\sum x_i \theta(u_i) = \sum x_i u_i = r \theta u = \theta u + z$$

where $z \in L_{l+1}$. Furthermore $dz \in L_l$. Hence $\delta^* \Delta(u) = \Psi r dz = \Psi r \theta \theta^{-1} dz = (g')^* \theta^{-1} dz$.

But by 5.2, $\theta^{-1}(L_l)$ is the kernel of $(g'_l)^*$.

We now prove that f_l induces a 2s+2-connected map $\overline{f}_l: \overline{H}(X_l) \to H^*(T_l)$ by induction on $l \ge 0$. We first show that \overline{f}_l is well defined.

$$\bar{H}^{*}(X_{l}) = H^{*}(X_{l})/\alpha_{l+1}^{*}(\theta^{-1}(L_{l+1}))$$

From the commutative diagram:

$$T_{l} \xrightarrow{j} T_{l}/T'_{l+1}$$

$$\downarrow^{f_{l}} \qquad \qquad \downarrow^{g'_{l+1}}$$

$$X_{l} \xrightarrow{\alpha_{l+1}} K(U_{l+1})$$

we see that

$$f_{l}^{*}\alpha_{l+1}^{*}(\theta^{-1}(L_{l+1})) = j^{*}(g'_{+1})^{*}(\theta^{-1}(L_{l+1}))$$

By 5.2, $\theta^{-1}(L_{l+1})$ is in the kernel of $(g'_{l+1})^*$.

Since f_0^* is an isomorphism, $\overline{f}_0 = f_0^*$ and \overline{f}_0 is an isomorphism.

Suppose \overline{f}_{l-1} is 2s+2 connected. If $u \in U_{l+1}$, $\Delta(u) \in H^q(T'_l)$ pulls back to $H^q(T_{l-1})$ since, by 5.4, $\delta^* \Delta(u) = 0$ and it pulls back to $H^q(X_{l-1})$ if q < 2s+2, that is, if |u| < 2s+1, $\Delta(u) = (f'_l)^* p^* x$ where $p: X_l \to X_{l-1}$. But since the X_l 's are constructed from an acyclic complex, image $p^* = \text{image } (H^*(X_0) \to H^*(X_l))$. Therefore image $(f'_l)^* p^* = \text{image } (H^*(T_0) \to H^*(T'_l)) = H^*(MO)/\Phi(I_n)$. But by 5.3, $\Delta(u)$ is zero on all *n*-manifolds. Hence $\Delta(u) = 0$ and we have shown that $W^q_{l+1} = (U_{l+1}/\ker \Delta)^q = 0$ for q < 2s+1. Therefore $H^q(B'_l) \to H^q(B_l)$ is an isomorphism for $q \leq 2s+2$ since B_l is a fibration over B'_{l-1} induced by $\gamma_l: B'_l \to K(W_{l+1})_1$. Then $H^q(T'_l/T_l) = H^q(B'_l, B_l) = 0$ for q < 2s+2 and hence

 $H^{\ast}(T_{l-1}/T'_{l}) \rightarrow H^{\ast}(T_{l-1}/T_{l})$

is (2s+2)-connected. Let g_l be the composition

$$T_{l-1}/T_l \longrightarrow T_{l-1}/T'_l \xrightarrow{g_l} K(U_l)$$

and let $\bar{g}_l: \bar{H}^*(K(U_l)) \to H^*(T_{l-1}/T_l)$ be induced by g_l . Then \bar{g}_l is (2s+2)-connected by 5.2. Consider the commutative diagram:

$$\longrightarrow \bar{H}^{*}(K(U_{l})) \longrightarrow \bar{H}^{*}(X_{l-1})) \longrightarrow \bar{H}^{*}(X_{l}) \longrightarrow \bar{H}^{*}(K(U)) \longrightarrow$$

$$\downarrow_{\tilde{s}_{l}} \qquad \qquad \qquad \downarrow_{\tilde{f}_{l-1}} \qquad \qquad \downarrow_{\tilde{f}^{-}} \qquad \qquad \qquad \downarrow$$

$$\longrightarrow H^{*}(T_{l-1}/T_{l}) \longrightarrow H^{*}(T_{l-1}) \longrightarrow H^{*}(T_{l}) \longrightarrow H^{*}(T_{l-1}/T_{l}) \longrightarrow$$

A five lemma argument and the fact that \overline{f}_{l-1} and \overline{g}_l are (2s+2)-connected shows that \overline{f}_l is 2s+2-connected.

Since $L_l = 0$ for l > 3, $\overline{H}^*(X_l) = H^*(X_l)$ for $l \ge 3$ and therefore $f_l^*: H^q(X_l) \to H^q(T_l)$ is an isomorphism for $q \le n < 2s + 2$. This completes the proof of 2.7.

§6. Proof of 5.2

LEMMA 6.1.

 $H^q(B_{l-1}) \rightarrow H^q(B'_l)$

is an isomorphism for l > 1 and $q \le s + 1$. For l = 1 it is an epimorphism for $q \le s + 1$ and v_{s+1} , w_1v_{s+1} , Sq^1v_{s+1} and v_{s+2} generate the kernel for $q \le s + 2$. Proof. As we saw in the proof of 2.8, if $\lambda^{I}u_{\omega} \in V_{l}$, $|\lambda^{I}u_{\omega}| \ge (n-1)-(n-|u_{\omega}|-1)/2^{l}$. Hence the lowest dimensional element in V_{l} is of the form λ^{I} with t(I) = s. For such an I, $|\lambda^{I}| \ge s+2$ except for l=1 or l=2 and s=1 and 2. The space B'_{l} is a fibration over B_{l-1} induced by $\beta_{l} : B_{l-1} \to K(V_{l})_{1}$ and for l > 1, $K(V_{l})_{1}$ is s+2 connected except when l=2 and s=1 or 2. For s=1 or 2, the lowest dimensional elements in V_{2} are $\lambda^{1,1}$ and $\lambda^{1,2}$ respectively; $d\lambda^{1,1} \neq 0$ and $d\lambda^{1,2} \neq 0$ so these elements kill nonzero classes in B_{1} . Thus for l > 1, $H^{q}(B_{l-1}) \approx H^{q}(B'_{l})$ for $q \le s+1$.

Suppose l = 1. From 3.1 one sees that $d\lambda^i = \chi(Sq^{i+1})U = \Phi(v_{i+1})$ where U is the Thom class and v_{i+1} is the Wu class. Hence $\beta_1: B_0 \to K(V_1)_1$ takes λ^i into v_{i+1} . One easily checks that $V_1^q = 0$ for q < s, $V_1^s = \{\lambda^s\}$ and $V_1^{s+1} = \{\lambda^{s+1}\}$. The remainder of 6.1 now follows by a simple Serre spectral sequence argument.

Let $K_l = K(V_l)_1$. Viewing $\beta_l : B_{l-1} \to K_l$ as a fibre map with fibre B'_l , consider the pair of fibrations p_1 and p_2 :



where p_1 is defined by β_i , p_2 is projection on the second factor and $c = id \times p$. Note c is a fibre preserving map so we may use it to compare the Serre spectral sequences of p_1 and p_2 .

LEMMA 6.2. For l>1, $c^*: H^q(B_{l-1} \times K_l, B_{l-1} \times \{^*\}) \rightarrow H^q(B_{l-1}, B'_l)$ is an isomorphism for $q \leq 2s+3$. For l=1, c^* is an epimorphism for $q \leq 2s+2$ and for $q \leq 2s+3$ the kernel is generated by

$$v_{s+1} \otimes \lambda_{1}^{s} + 1 \otimes (\lambda_{1}^{s})^{2}$$

$$v_{s+1} \otimes Sq^{1}\lambda_{1}^{s} + 1 \otimes \lambda_{1}^{s}Sq^{1}\lambda_{1}^{s}$$

$$v_{s+1} \otimes \lambda_{1}^{s+1} + 1 \otimes \lambda_{1}^{s}\lambda_{1}^{s+1}$$

$$w_{1}v_{s+1} \otimes \lambda_{1}^{s} + w_{1} \otimes (\lambda_{1}^{s})^{2}$$

$$Sq^{1}v_{s+1} \otimes \lambda_{1}^{s} + 1 \otimes \lambda_{1}^{s}Sq^{1}\lambda_{1}^{s}$$

$$v_{s+2} \otimes \lambda_{1}^{s} + 1 \otimes \lambda_{1}^{s}\lambda_{1}^{s+1}$$

Proof. Let $E_r^{p,q}$ and $\overline{E}_r^{p,q}$ denote the Serre spectral sequences for p_1 and p_2 respectively.

$$E_2^{p,q} = H^p(K_l, *) \otimes H^q(B_{l-1})$$

$$\bar{E}_2^{p,q} = H^p(K_l, *) \otimes H^q(B')$$

As we saw above, for l > 1, K_l is s+2 connected and $H^q(B_{l-1}) \approx H^q(B'_l)$ for $q \leq s+1$. Therefore c induces an isomorphism at the E_2 level for $p+q \leq 2s+3$ and the differentials are trivial for p_2 because it is a product fibration. This proves 6.2 for l > 1.

For l = 1, 6.2 is true at the E_2 level with the first summands in the above list of elements as a basis for the kernel; the second summands are of lower filtration. The same is true at the E_{∞} level, so to complete the proof, we must show that these elements are in the kernel of c^* .

Under the map $H^*(B_0, B'_1) \to H^*(B_0)$, $c^*(1 \otimes \lambda_1^s)$ goes to v_{s+1} . Hence

$$c^{*}(v_{s+1} \otimes \lambda_{1}^{s} + 1 \otimes (\lambda_{1}^{s})^{2}) = v_{s+1}c^{*}(1 \otimes \lambda_{1}^{s}) + c^{*}(1 \otimes \lambda_{1}^{s})^{2} = 0$$

(If $j: X \subset (X, A)$ and $x \in H^*(X, A)$, $x^2 = (j^*x)x$.) The same argument applies to the other five elements.

Let

$$\phi: (A(BO) \otimes V_l)^q \to H^{q+1}(T_{l-1} \wedge K_l)$$

be defined by

 $\phi((a \otimes w)u) = a(wU \otimes u_1)$

where U is the Thom class, $a \in A$, $w \in H^*(BO)$ and $u \in \overline{V}_1$.

LEMMA 6.3. For $q \le 2s+1$, ϕ is an epimorphism. For $q \le 2s+2$ the kernel of ϕ is zero for l > 1 and $(l, s) \ne (2, 2)$, is $\{v_3\lambda^{1,2}\}$ for (l, s) = (2, 2) and is $\{(\sum Sq^i \circ v_{s+2-i})\lambda^s\}$ for l = 1.

Proof. Let μ , $\mu': A(BO) \rightarrow A(BO)$ be defined by

$$\mu(a \circ w) = \sum a'_i \circ w\varsigma(a''_i)$$
$$\mu'(a \circ w) = \sum a'_i \circ w\chi(a''_i)$$

(Recall, wa is defined by $(wa/U = \chi(a)(wU))$.) Where $a \to \sum a'_i \otimes a''_i$ in the diagonal in A. Then $\mu \mu' = \mu' \mu = \text{identity}$ and thus μ is a Z_2 -isomorphism. Let $\phi' = \phi(\mu \otimes id)$. Then

$$\phi'((a \circ w)u) = \sum a'_i(\chi(a''_i)(wU) \otimes u_1) = wU \otimes au_1$$

Let λ^{I} be the lowest dimensional element in \overline{V}_{l} ; $|\lambda^{I}| > s$ for l = 1. The lowest

dimensional element in $H^*(T_{l-1} \wedge K_l)$ not in the image of ϕ' is $U \otimes (\lambda_1^I \cup Sq^i \lambda_1^I)$, an element of dimension $\geq 2s+3$. Hence ϕ is an epimorphism for q < 2s+2. The lowest dimensional elements in the kernel of ϕ' are $1 \circ v_{s+1}\lambda_1^I$ or $(Sq^m \circ 1)\lambda_1^I$ where $m = |\lambda_1^I| + 1$. For l > 2, $(l, s) \neq (2, 2)$, $\lambda_1^I > s+1$ and hence these elements occur in dimensions > 2s+3. For (l, s) = (2, 2), $\phi(v_3\lambda^{1,2}) = \phi'(v_3\lambda^{1,2}) = 0$. For l = 1

$$0 = \phi'((Sq^{s+2} \circ 1)\lambda^s) = \phi\left(\left(\sum Sq^i \circ v_{s+2-i}\right)\lambda^s\right)$$

This proves the last part of 6.3.

Proof of 5.2: We must show that

$$(g'_l)^* = \Psi r \theta : (A \otimes U_l)^q \rightarrow H^{q+1}(T_{l-1}/T'_l)$$

is an epimorphism for $q \leq 2s+2$ and $(L_l)^q$ is the kernel for $q \leq 2s+2$. By 2.5, θ is an isomorphism. Let ϕ be the map in 6.3 and c the map in 6.2. Lifting c to the Thom space level we obtain a map

$$T(c): T_{l-1}/T'_l \to T_{l-1} \wedge K_l$$

Furthermore $\Psi = T(c)^*$. Thus by 6.2 and 6.3, Ψ is an epimorphism for $q \le 2s+1$ and since r is an epimorphism, $(g'_l)^*$ is an epimorphism for $q \le 2s+1$. For l > 1and $(l, s) \ne (2, 2)$, $T(c)^*$ and ϕ are monomorphisms for $q \le 2s+2$ and L^q_l is the kernel of r. When $(l, s) = (2, 2) r(L_l) = \{v_3 \lambda^{1,2}\}$. This completes the proof of 5.2 for l > 1.

Suppose l = 1. Then r =identity. We wish to show that $L_1 = \phi^{-1}(\ker T(c)^*)$. In 6.2 a basis for ker c^* was given for $q \le 2s + 2$. Since image $\phi =$ image ϕ' cannot involve cup products (except squares) in $H^*(K_l)$, the above basis shows that the following is a basis for image $\phi \cap \ker T(c)^*$:

$$v_{s+1}U \otimes \lambda_1^s + U \otimes Sq^{s+1}\lambda_1^s$$
$$w_1v_{s+1}U \otimes \lambda_1^s + w_1U \otimes Sq^{s+1}\lambda_1^s$$
$$v_{s+1}U \otimes Sq^1\lambda_1^s + (Sq^1v_{s+1})U \otimes \lambda_1^s$$
$$v_{s+1}U \otimes \lambda_1^{s+2} + v_{s+2}U \otimes \lambda_1^s$$

Thus a basis for $\phi^{-1}(\ker c^*)$ is ϕ^{-1} of these elements and $(\sum Sq^i \circ v_{s+2-i})\lambda^s$ from the kernel of ϕ . A simple calculation shows that these elements form a basis for L_1^q , $q \leq 2s+2$, completing the proof of 5.2.

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