

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 54 (1979)

Artikel: A universal space for normal bundles of n -manifolds.
Autor: Brown, E.H., Jr. / Peterson, F.P.
DOI: <https://doi.org/10.5169/seals-41587>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A universal space for normal bundles of n -manifolds

E. H. BROWN, JR, and F. P. PETERSON¹

§1. Introduction

In [3] the authors gave a simple criterion for deciding whether a polynomial in Stiefel–Whitney classes is zero on the normal bundles of all smooth n -manifolds. The ideal of relations among Stiefel–Whitney classes for all n -manifolds, $I_n \subset H^*(BO)$ was defined by

$$I_n = \{w \in H^*(BO) \mid w(\nu_{M^n}) = 0 \text{ for all } M^n\}$$

where M^n denotes a smooth n -manifold and ν_M is its stable normal bundle. Let $\Phi: H^*(BO) \simeq H^*(MO)$ be the Thom isomorphism and for $w \in H^*(BO)$, define wSq^i to be $\Phi^{-1}(\chi(Sq^i)\Phi(w))$. It was shown that I_n consists of all \mathbb{Z}_2 -linear combinations of elements of the form wSq^i where $2i > n - |w|$ ($|w|$ = dimension of w).

In this paper we give a stronger version of this result, namely:

THEOREM 1. *There is a space BO/I_n and a map $\pi: BO/I_n \rightarrow BO$ such that*

(a) *If M is a smooth, compact n -manifold and $h: M \rightarrow BO$ classifies ν_M , then there is a map $\bar{h}: M \rightarrow BO/I_n$ such that $\pi\bar{h} \simeq h$.*

(b) *The following sequence is exact.*

$$0 \longrightarrow I_n \subset H^*(BO) \xrightarrow{\pi^*} H^*(BO/I_n) \longrightarrow 0.$$

Theorem 1 shows that BO/I_n is a universal space for normal bundles of n -manifolds in that stably, every such bundle is induced from the bundle over BO/I_n and BO/I_n is the space with the smallest cohomology having this property.

Our original result on I_n suggested the possibility of defining higher order characteristic classes, that is, one could form a space B over BO by killing the

¹ During the work on this paper the authors were supported by NSF grant MCS76-08804 A01 and MCS 76-06323.

elements of I_n . Then an element of $H^*(B)$ might give a “new” characteristic class for n -manifolds. For example, with $n = 4$ or 5 , the relation

$$(Sq^2 + w_1 \cup Sq^1 + w_2 U)(v_3) = v_3 Sq^2 = (1Sq^3)Sq^2 = 0$$

where v_3 is the Wu class, gives a class in $H^4(B)$ which is not a polynomial in Stiefel–Whitney classes. Theorem 1 shows that on an n -manifold this “new” class will be a polynomial in Stiefel-Whitney classes modulo indeterminacy.

The spaces BO/I_n are also related to the conjecture that any smooth n -manifold immerses in $R^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the dyadic expansion of n . Since this conjecture is equivalent to the normal bundle map $h : M^n \rightarrow BO$ lifting to $BO_{n-\alpha(n)}$ ([9]), the following is a stronger form of the conjecture:

CONJECTURE. $\pi : BO/I_n \rightarrow BO$ lifts to $BO_{n-\alpha(n)}$.

Using our proof of Theorem 1, our results in [4] can be restated in the following way which gives some plausibility to the above conjecture.

THEOREM 2. *If ζ is the stable universal bundle over BO , MO is its Thom spectrum, MO/I_n is the Thom spectrum of $\pi^*\zeta$ and $MO(n-\alpha(n))$ is the Thom spectrum of the universal bundle over $BO_{n-\alpha(n)}$, then MO/I_n lifts to $MO(n-\alpha(n))$.*

This paper is organized as follows: In §2 we give a detailed outline of the proof of Theorem 1 setting forth most of the notation and describing the various technical problems arising in the construction of BO/I_n . Then in Sections 3, 4, 5, and 6 we prove the various lemmas stated in §2. Throughout the remainder of this paper n is a fixed positive integer.

§2. Outline of the Proof of Theorem 1

All cohomology will be with Z_2 coefficients, A will be the mod two Steenrod algebra and $\chi : A \rightarrow A$ will be the canonical antiautomorphism. The semi-tensor product of A and $H^*(BO)$ ([6]) will be denoted by $A(BO)$, that is, $A(BO) = A \otimes H^*(BO)$ with the algebra structure defined by

$$(a \otimes u)(b \otimes v) = \sum ab'_i \otimes (\chi(b''_i)u)v$$

where $b \rightarrow \sum b'_i \otimes b''_i$ under the diagonal of A . We denote $a \otimes u$ by $a \circ u$.

By a spectrum Y , we will mean a collection of spaces Y_q and maps $g_q: SY_q \rightarrow Y_{q+1}$. If X and Y are spectra, a map $f: X \rightarrow Y$ of degree p will be a collection of homotopy classes $f_q \in [X_q, Y_{q+p}]$ compatible with the maps g_q . If ξ is a real k -plane bundle, $T(\xi)$ will denote its Thom spectrum, i.e., $T(\xi)_q = S^{q-k}$ (Thom space of ξ). Thus the Thom class is in $H^0(T(\xi))$. If ξ is a vector bundle over B , $\Phi: H^*(B) \approx H^*(T(\xi))$ will be the Thom isomorphism. We make $H^*(T(\xi))$ into an $A(BO)$ module as follows: Let $h: B \rightarrow BO$ classify ξ . If $u \in H^*(T(\xi))$, $w \in H^*(BO)$ and $a \in A$, $(a \circ w)u = a(h^*(w)u)$. One easily checks that $\Phi(I_n) \subset H^*(MO)$ is an $A(BO)$ submodule.

We begin by constructing an A -free, acyclic resolution of $\Phi(I_n)$. In [3] the following was proved:

THEOREM 2.1. *If $\{u_i\}$ is an A basis for $H^*(MO)$, then $\Phi(I_n)$ is the A module generated by*

$$\{\chi(Sq^j)u_i \mid 2j > n - |u_i|\}.$$

For a partition $\omega = \{j_1, j_2, \dots, j_l\}$ let $s_\omega \in H^*(BO)$ be the usual class ([17]) associated with the symmetric function $\sum t_1^{j_1} t_2^{j_2} \cdots t_l^{j_l}$. For each partition ω let ω' be the partition consisting of odd integers j , one for each $j2^r \in \omega$. Let

$$u_\omega = \prod_r s_{\omega_r}^{2^r}$$

Since

$$u_\omega = s_\omega + \sum s_{\omega'}$$

where ω' has fewer entries than ω and $\{s_\omega\}$ is a basis for $H^*(BO)$, $\{u_\omega\}$ is also a basis for $H^*(BO)$. Also $\{\Phi(u_\omega) \mid 2^i - 1 \notin \omega\}$ is an A basis for $H^*(MO)$ since $\{\Phi(s_\omega) \mid 2^i - 1 \notin \omega\}$ is.

In [2] an A -free acyclic resolution of $A/A\{\chi(Sq^i) \mid i > h\}$ was constructed. Combining these resolutions with 2.1 and the $\Phi(u_\omega)$ basis, we obtain the following resolution of $\Phi(I_n)$.

Let Λ be the graded free associative algebra over Z_2 with unit generated by λ_i , $i = 0, \pm 1, \pm 2, \dots$, $|\lambda_i| = i$, modulo the relations: If $2i < j$

$$\lambda_i \lambda_j = \sum \binom{s-1}{2s-(j-2i)} \lambda_{i+s} \lambda_{j-s}.$$

If $I = (i_1, i_2, \dots, i_l)$, let $\lambda_I = \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_l}$, $l(I) = l$, $t(I) = i_l$, and $\lambda_{(\)} = 1$. We define I

to be admissible if $2i_i \geq i_{i+1}$. As we will see in §3, $\{\lambda_I \mid I \text{ admissible}\}$ is a Z_2 basis for Λ . Let $\{\lambda^I \mid I \text{ admissible}\}$ be the dual basis of $\Lambda^* = \text{Hom}(\Lambda, Z_2)$.

Let U_l be the vector space over Z_2 with basis the symbols $\lambda^I u_\omega$ where I is admissible, $2^i - 1 \notin \omega$, $l(I) = l$ and $2(t(I) + 1) > n - |u_\omega|$. Grade U_l by $|\lambda^I u_\omega| = |\lambda^I| + |u_\omega|$. Let $d : A \otimes U_l \rightarrow A \otimes U_{l-1}$ be the A linear map defined by

$$d(1 \otimes \lambda^I u_\omega) = \sum \lambda^I (\lambda_j \lambda_J) \chi(Sq^j) \otimes \lambda^J u_\omega$$

where the sum ranges over all j and admissible J . Note by 2.2, if $\lambda^I (\lambda_j \lambda_J) \neq 0$, $t(J) \geq t(I)$ and hence d is well defined. Let $\eta : A \otimes U_0 \rightarrow H^*(MO)$ be given by $\eta(a \otimes \lambda^{(\cdot)} u_\omega) = a\Phi(u_\omega)$.

PROPOSITION 2.3. *The following sequence is exact:*

$$\longrightarrow A \otimes U_l \xrightarrow{d} A \otimes U_{l-1} \longrightarrow \cdots \longrightarrow A \otimes U_0$$

and

$$\Phi(I_n) = \eta(\text{image}(d : A \otimes U_1 \rightarrow A \otimes U_0))$$

We prove 2.3 in §3.

For a graded vector space V over Z_2 , let $K(V)$ denote the Eilenberg-MacLane spectrum such that $\pi_*(K(V)) = V^*$ and $H^*(K(V)) = A \otimes V$.

PROPOSITION 2.4. *There is a sequence of Ω -spectra X_l , $l = 0, 1, 2, \dots$ and maps $\alpha_l : X_{l-1} \rightarrow K(U_l)$ of degree $+1$ such that*

- (i) $X_0 = K(U_0)$
- (ii) X_l is the fibration over X_{l-1} induced by α_l from the contractible fibration over $K(U_l)$.
- (iii) If $i : K(U_l) \rightarrow X_l$ is the inclusion of the fibre of $X_l \rightarrow X_{l-1}$, $(\alpha_{l+1} i)^* = d : A \otimes U_{l+1} \rightarrow A \otimes U_l$.
- (iv) If M is a smooth n -manifold, ν is its normal bundle, $g : MO \rightarrow K(U_0)$ realizes η and $h : T(\nu) \rightarrow MO$ comes from the classifying map of ν , then any lifting of $gh : T(\nu) \rightarrow X_0$ to X_{l-1} lifts to X_l .

Since the X_l 's are constructed from an acyclic complex,

$$\lim H^*(X_l) \approx \text{Coker}(d : A \otimes U_1 \rightarrow A \otimes U_0) \approx H^*(MO)/\Phi(I_n).$$

To construct BO/I_n we essentially construct a tower of spaces

$$\longrightarrow B_l \rightarrow B_{l-1} \rightarrow \cdots \rightarrow B_0 = BO$$

with fibres Eilenberg-MacLane spaces, such that if $T_l = T(\zeta_l)$ where $\zeta_l \rightarrow B_l$ is the pull back of the universal bundle over BO , then $T_l = X_l$ in dimensions $\leq n$. We can then, more or less, define $BO/I_n = \lim B_l$.

We recall how the cohomology of a Thom space of a vector bundle changes, in a stable range, when a cohomology class in the base is killed. Suppose $g : B \rightarrow BO$ is a map such that $g_* : \pi_q(B) \approx \pi_q(BO)$ for $2q \leq n$, V is a graded vector space with $V_q = 0$ for $2q \leq n$ and $p : B' \rightarrow B$ is the fibration induced by a map $\gamma : B \rightarrow K(V)_1$ ($K(V) = \{K(V)_q\}$). Let $T = T(g^*\zeta)$ and $T' = T(p^*g^*\zeta)$. Viewing $B' \subset B$ as the fibre of γ , γ factors as $B \xrightarrow{j} B/B' \xrightarrow{\gamma'} K(V)_1$. Let

$$\Psi : (A(BO) \otimes V)^q \rightarrow H^{q+1}(T/T')$$

be given by $\Psi(a \circ u \otimes v) = a(u\Phi((\gamma')^*(v_1)))$ where $v_1 \in H^*(K(V)_1)$ is the element corresponding to $v \in V$ and Φ is the relative Thom isomorphism. In §6 we show that Ψ is an isomorphism for $q \leq n$. (An equivalent form of this was proved in [1].) Combining this with the exact sequence of the pair (T, T') we obtain an exact sequence,

$$\rightarrow H^q(T) \rightarrow H^q(T') \rightarrow (A(BO) \otimes V)^q \rightarrow H^{q+1}(T) \rightarrow$$

for $q \leq n$.

The cohomology of X_l and X_{l-1} are related by the Serre exact sequence,

$$\rightarrow H^q(X_{l-1}) \rightarrow H^q(X_l) \rightarrow (A \otimes U_l)^q \rightarrow H^{q+1}(X_{l-1}) \rightarrow.$$

Thus if we have constructed B_{l-1} such that $T_{l-1} = X_{l-1}$ in dimensions $\leq n$ and we wish to construct B_l , we should take $B = B_{l-1}$ in the above and choose V_l so that $A(BO) \otimes V_l = A \otimes U_l$ as A modules. Our main algebraic result asserts that this is possible. Let

$$V_l = \{\lambda^l u_\omega \in U_l \mid \omega_r = \{ \} \text{ for } r \geq l\}$$

PROPOSITION 2.5. *There are A linear isomorphisms $\theta : A \otimes U_l \rightarrow A(BO) \otimes V_l$ and $A(BO)$ linear maps $d : A(BO) \otimes V_l \rightarrow A(BO) \otimes V_{l-1}$, $l > 1$ and $d : A(BO) \otimes V_1 \rightarrow H^*(MO)$ such that the following diagram is commutative:*

$$\begin{array}{ccccccc} \longrightarrow & A \otimes U_l & \xrightarrow{d} & A \otimes U_{l-1} & \longrightarrow & \cdots & \longrightarrow & A \otimes U_0 \\ & \downarrow \theta & & \downarrow \theta & & & & \downarrow r \\ \longrightarrow & A(BO) \otimes V_l & \xrightarrow{d} & A(BO) \otimes V_{l-1} & \longrightarrow & \cdots & \longrightarrow & H^*(MO). \end{array}$$

Furthermore, if $u \in V_l \subset U_l$, then $\theta(1 \otimes u) = 1 \otimes u$.

The construction of spaces B_l can now be made, modulo technical problems, using 2.5. Given B_{l-1} and $f_{l-1}: T_{l-1} \rightarrow X_{l-1}$, the k -invariant $\beta_l: B_{l-1} \rightarrow K(V_l)_1$ is defined by:

$$\Phi\beta_l^*(v_1) = f_{l-1}^* \alpha_l^*(v)$$

where $\alpha_l: X_{l-1} \rightarrow K(U_l)$ is the k -invariant for X_l , $v \in V$ and $v_1 \in H^*(K(V)_1)$ corresponds to v . If M is an n -manifold and $h: M \rightarrow BO$ classifies its normal bundle, 2.4(iv) shows that any lifting of h to B_{l-1} lifts to B_l . The $A(BO)$ linearity of d allows one (more or less) to construct $f_l: T_l \rightarrow X_l$. Actually, this straightforward procedure is marred by two technical details which we now describe.

Let $s = [n/2]$. To form B_1 from BO , one kills, among other things, the Wu class v_{s+1} , i.e. $d\lambda^s = \chi(Sq^{s+1})U = v_{s+1}U$, where the U is the Thom class. The map Ψ is zero on

$$\sum_{j>0} (Sq^j \circ v_{s+1-j}) \otimes \lambda^s \in (A(BO) \otimes V_1)^{2s+1}$$

As a result, there is a class $x \in H^{2s+1}(X_1)$ which goes to zero in $H^{2s+1}(T_1)$. The class x is killed in going from X_1 to X_2 . Hence if one were to follow the recipe given by 2.5, one would kill a class in B_1 which is already zero and thus produce a class in $H^{2s}(B_2)$ not coming from $H^{2s}(X_2)$. To avoid this, we omit a basis element from V_2 . This same phenomena occurs in dimension $2s + 2$ so we omit some more elements from V_2 and V_3 . Namely, let $\bar{V}_l \subset V_l$ be spanned by $\lambda^l u_\omega \in V_l$ except $\lambda^{0,0} w_s^2$, $\lambda^{0,-1} w_{s+1}^2$, $\lambda^{-1,-2} w_{s+2}^2$ and for s odd, $\lambda^{-1,-2,-4} w_1^4 w_s^2$ ($w_s = u_{(1,1,\dots,1)}$).

In §3 we define a certain $A(BO)$ linear map

$$r: A(BO) \otimes V_l \rightarrow A(BO) \otimes \bar{V}_l \tag{2.6}$$

such that $r|_{A(BO) \otimes \bar{V}_l}$ is the identity. We then use $r\theta$ in place of θ in our construction of B_l .

The second difficulty arises in the following fashion. Again suppose we have B_{l-1} and $f_{l-1}: T_{l-1} \rightarrow X_{l-1}$ and we construct B_l using \bar{V}_l instead of V_l as above. Let $g_l: T_{l-1}/T_l \rightarrow K(U_l)$ be the map such that $g_l^*(u) = \Psi r\theta(u)$ for $u \in U_l$. In order to construct $f_l: T_l \rightarrow X_l$ we need commutativity of the diagram

$$\begin{array}{ccc} T_{l-1} & \xrightarrow{j} & T_{l-1}/T_l \\ \downarrow f_{l-1} & & \downarrow g_l \\ X_{l-1} & \xrightarrow{\alpha_l} & K(U_l). \end{array}$$

We can only prove that this diagram commutes in dimensions $\leq 2s + 1$. To correct for this we relabel B_l above, B'_l and we form B_l from B'_l by killing the obstructions to commutativity as follows:

Define $\Delta = \Delta(f_{l-1}): U_l \rightarrow H^*(T_{l-1})$ by

$$\Delta(u) = f_{l-1}^* \alpha_l^* u - \sum x_i f_{l-1}^* \alpha_l^* u_i$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$, $u_i \in \bar{V}_l$. Then

$$\begin{aligned} j^* g_l^*(u) &= j^* \Psi r\theta(u) = j^* \Psi \left(\sum x_i u_i \right) = \sum x_i j^* \Phi((\beta'_l)^*(u_i)) \\ &= \sum x_i \Phi(\beta_l^*(u_i)) = \sum x_i f_{l-1}^* \alpha_l^*(u_i) = \Delta(u) + f_{l-1}^* \alpha_l^*(u) \end{aligned}$$

Thus Δ is the deviation from commutativity of our diagram above. Let $W_l = U_l / \ker \Delta$. We kill $\Phi^{-1}(\Delta(W))$ in B'_l to form B_l .

To recapitulate, we inductively construct a sequence of spaces B_l , stable vector bundles ζ_l over B_l and maps $f_l: T_l = T(\zeta_l) \rightarrow X_l$ such that $\Delta(f_l) = 0$. We take $B_0 = BO$, $\zeta_0 = \zeta$ the universal bundle and f_0 the map such that $f_0^*(u_\omega) = \Phi(u_\omega)$ for $u_\omega \in U_0$. ($X_0 = K(U_0)$.) Referring to 2.5, $f_0^* = \eta$, $\alpha_1^* = d$ and $\Delta(f_0) = \eta d - d\theta = 0$. Suppose B_{l-1} , ζ_{l-1} and f_{l-1} have been defined and $\Delta(f_{l-1}) = 0$. Let $p': B'_l \rightarrow B_{l-1}$ be the fibration induced by $\beta_l: B_{l-1} \rightarrow K(\bar{V}_l)_1$ where β_l is defined by

$$\Phi(\beta_l^*(v_1)) = f_{l-1}^* \alpha_l^*(v)$$

for $v \in \bar{V}_l \subset U_l$ and $v_1 \in H^*(K(\bar{V}_l)_1)$ the element corresponding to v . Let $\zeta'_l = (p')^* \zeta_{l-1}$ and $T'_l = T(\zeta'_l)$.

Viewing $B'_l \subset B_{l-1}$ as the fibre of β_l , β_l factors through $\beta'_l: B_{l-1}/B'_l \rightarrow K(\bar{V}_l)_1$. Let $\Psi: A(BO) \otimes \bar{V}_l \rightarrow H^*(T_{l-1}/T'_l)$ be the $A(BO)$ linear map such that $\Psi(v) = \Phi((\beta'_l)^*(v_1))$ for $v \in \bar{V}_l$. Let θ be as in 2.5, r as in 2.6, and let $g'_l: T_{l-1}/T'_l \rightarrow K(U_l)$ be defined by $(g'_l)^*(u) = \Psi r\theta(u)$. Since $\Delta(f_{l-1}) = 0$, there is a map f'_l making a commutative diagram

$$\begin{array}{ccccccc} T_{l-1}/T'_l & \longrightarrow & T'_l & \longrightarrow & T_{l-1} & \longrightarrow & T_{l-1}/T'_l \\ \downarrow g'_l & & \downarrow f'_l & & \downarrow f_{l-1} & & \downarrow g'_l \\ K(U_l) & \xrightarrow{i} & X_l & \longrightarrow & X_{l-1} & \xrightarrow{\alpha_l} & K(U_l). \end{array}$$

Let $\Delta(f'_l): U_{l+1} \rightarrow H^*(T_l)$ be given by $\Delta(f'_l)(u) = (f'_l)^* \alpha_{l+1}^* u + \sum x_i (f'_l)^* \alpha_{l+1}^* u_i$ where $r\theta u = \sum x_i u_i$. Let $W_{l+1} = U_{l+1} / \ker \Delta(f'_l)$ and let $p: B_l \rightarrow B'_l$ be the fibration induced

by $\gamma_l : B_l \rightarrow K(W_{l+1})_1$ where $\Phi(\gamma_l^* u_1) = \Delta(f_l)(u)$ for $u \in W_{l+1}$. Finally let $\zeta_l = p^* \zeta'_l$ and $f_l = f'_l T(p)$. Then $\Delta(f_l) = T(p)^* \Delta(f'_l) = 0$ and the inductive step is complete.

In §5 we prove:

LEMMA 2.7. *If $l \geq 3$ and $q \leq n$, $f_l^* : H^q(X_l) \approx H^q(T(\zeta_l))$. Furthermore, if M is a smooth n -manifold and $h : M \rightarrow B_0 = BO$ classifies its normal bundle, then any lifting of h to B_{l-1} lifts to B_l .*

We next examine $H^*(B_l)$ for l large.

LEMMA 2.8. *If $l \geq n$, $V_l^q = U_l^q = 0$ for $q < n - 1$, $W_l^q = 0$ for $q \leq n$ and*

$$V_l^{n-1} = U_l^{n-1} = \{ \lambda^{(0,0,\dots,0)} u_\omega \mid u_\omega \in U_0^{n-1} \}. \text{ Furthermore,}$$

$$\Phi(\beta_l^*(\lambda^{(0,0,\dots,0)} u_\omega)) = \delta_l \tilde{u}_\omega$$

$\tilde{u}_\omega \in H^*(T_{l-1}; Z_{2l})$, $u_\omega U \in H^*(T_{l-1})$ is the mod two reduction of \tilde{u}_ω and δ_l is the Bockstein associated with $Z_2 \rightarrow Z_{2l+1} \rightarrow Z_{2l}$.

Thus for $l \geq n$,

$$H^q(B_l) \approx H^q(BO)/I_n^q \quad q < n$$

$$H^n(B_l)/\Phi^{-1}\{\delta_{l+1} \tilde{u}_\omega\} \approx H^n(BO)/I_n^n$$

We form B_∞ from B_l , $l \geq n$, by killing classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_\omega) \in H^{n+1}(B_l; Z_\tau)$ where Z_τ denotes twisted integer coefficients, twisted by w_1 , $\Phi : H^*(B_l; Z_\tau) \approx H^*(T(\zeta_l); Z)$ is the Thom isomorphism and δ^l is the Bockstein associated with $Z \rightarrow Z \rightarrow Z_{2^l}$. Let \tilde{B}_l be the two sheeted cover of B_l defined by w_1 . The classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_\omega)$ may be represented by Z_2 -equivariant maps $x_\omega : \tilde{B}_l \rightarrow K(Z, n)$ where $K(Z, n)$ has the action defined by the nontrivial action of Z_2 on Z . Let \tilde{B}_∞ be the fibration over \tilde{B}_l induced by

$$x = \prod x_\omega : \tilde{B}_l \rightarrow \prod K(Z, n)$$

Since x is Z_2 -equivariant, Z_2 acts freely on \tilde{B}_∞ . Let $B_\infty = \tilde{B}_\infty/Z_2$. The map $B_\infty = \tilde{B}_\infty/Z_2 \rightarrow \tilde{B}_l/Z_2 = B_l$ has fibre $\prod K(Z, n)$. With Z_2 coefficients, $\pi_1(B_l)$ acts trivially on the cohomology of the fibre. The Serre spectral sequences, with Z_2 coefficients has its usual, nonlocal coefficient form and the usual argument shows

that in dimensions $\leq n$,

$$H^*(BO_\infty) = H^*(B_l) / \{\Phi^{-1}(\delta^{l+1}\tilde{u}_\omega)\}.$$

Thus for $q \leq n$

$$0 \rightarrow I_n^q \rightarrow H^q(BO) \rightarrow H^q(B_\infty) \rightarrow 0$$

is exact. Also if M is an n -manifold and $h: M \rightarrow B$ is covered by a bundle map $g: \nu \rightarrow \zeta$, $T(g)^*(\delta^{l+1}\tilde{u}_\omega) = \delta^{l+1}T(g^*)(\tilde{u}_l) = 0$ since the top homology class of $T(\nu)$ is spherical. Therefore, h lifts to B_∞ .

Finally, assume B_∞ is a CW complex and let

$$BO/I_n = B_\infty^n \cup e_1^{n+1} \cup e_2^{n+1} \cdots e_m^{n+1}$$

where e_i^{n+1} is attached by $f_i | S^n$, $f_i: (D^{n+1}, S^n) \rightarrow (B_\infty^{n+1}, B_\infty^n)$ and $[f_i] \in \pi_{n+1}(B_\infty^{n+1}, B_\infty^n)$ give a Z_2 -basis for the image of

$$\pi_{n+1}(B_\infty^{n+1}, B_\infty^n) \xrightarrow{\rho} H_{n+1}(B_\infty^{n+1}, B_\infty^n) \xrightarrow{\partial^*} H_n(B_\infty^n, B_\infty^{n-1})$$

The maps f_i give an extension of $B_\infty^n \subset B_\infty$, $f: BO/I_n \rightarrow B_\infty$ and

$$\begin{aligned} f^*: H^q(B_\infty) &\simeq H^q(BO/I_n) \quad \text{for } q \leq n \\ H^q(BO/I_n) &= H^q(BO)/I_n = 0 \quad \text{for } q > n \end{aligned}$$

Also any map of an n -manifold into B_∞ is homotopic to a map factoring through f . The proof of Theorem 1 is thus complete, modulo the lemmas and propositions of this section.

§3. Proofs of 2.3, 2.5, and 2.6

Let Λ_l^k be the Z_2 -subspace of Λ^* generated by λ^I with $l(I) = l$, $t(I) \geq k$, and I admissible. Let

$$d: A \otimes \Lambda_l^k \rightarrow A \otimes \Lambda_{l-1}^k$$

be defined by

$$d(1 \otimes \lambda^I) = \sum \lambda^I(\lambda_j \lambda_j) \chi(Sq^{j+1}) \otimes \lambda^J \tag{3.1}$$

where the sum is over all j and admissible J . Proposition 2.3 follows from 2.1 and 3.2(ii) below:

PROPOSITION 3.2.

- (i) $\{\lambda_I \mid I \text{ admissible}\}$ is a Z_2 -basis for Λ .
- (ii) The following is exact:

$$\longrightarrow A \otimes \Lambda_i^k \xrightarrow{d} A \otimes \Lambda_{i-1}^k \longrightarrow \cdots \longrightarrow A \otimes A_0^k \xrightarrow{\epsilon} A/A\{\chi(Sq^i) \mid i > k\}$$

where $\epsilon(a \otimes \lambda^{(i)}) = \{a\}$.

- (iii) If I and J are admissible, $l(I) = l$, $l(J) = l - 1$, and $I_l = (1, 2, 4, \dots, 2^{l-1})$, then $\lambda^{I+rI}(\lambda_{j+r}\lambda_{J+2rI_{l-1}}) = \lambda^I(\lambda_j\lambda_J)$.

Proof. For any sequence $T = (t_1, t_2, \dots, t_l)$ and integer r , let $h^r(\lambda_T) = \lambda_{T+rI}$. Extending linearly, h^r gives a well defined map $h^r : \Lambda \rightarrow \Lambda$ since for any element of Λ of the form $\alpha = \lambda_{I_1}\beta\lambda_{I_2}$ where β is a relation for Λ as in 2.2, $h^r(\alpha)$ also has this form. Since $h^r h^{-r}$ is the identity, h^r is an isomorphism for all r . Furthermore, $h^r(\lambda_I)$ is admissible if and only if λ_I is admissible.

Let $\bar{\Lambda} \subset \Lambda$ be the subalgebra generated by $\lambda_0, \lambda_1, \lambda_2, \dots$. In [8] it is proved that $\{\lambda_I \mid I \text{ admissible}\}$ is a basis for $\bar{\Lambda}$. For any λ_I , $h^r(\lambda_I) \in \bar{\Lambda}$ for r sufficiently large. Thus $\{\lambda_I \mid I \text{ admissible}\}$ is a basis for Λ .

In [2], 3.2(ii) was proved for $k \geq 0$. From 2.2 one sees that $\lambda_{-1}\lambda_{-1} = 0$ and if $t(J) \geq 0$, $\lambda_{-1}\lambda_J$ is a sum involving $\lambda_{J'}$'s with $t(J') > 0$ and $\lambda_J\lambda_{-1}$. Suppose $J_1 = (j_1, \dots, j_m)$, $J_2 = (j_{m+1}, \dots, j_l)$ and $J = (j_1, \dots, j_l)$ are admissible with J_1 or J_2 possibly the empty sequence $()$. Define $\lambda^{J_1}\lambda^{J_2} = \lambda^J$. Suppose $j_m \geq 0$ and $j_{m+1} < -1$. Then 3.1 yields

$$d(\lambda^{J_1}\lambda^{-1}\lambda^{J_2}) = (d\lambda^{J_1})\lambda^{-1}\lambda^{J_2} + \lambda^{J_1}\lambda^{J_2}$$

$$d(\lambda^{J_1}\lambda^{J_2}) = (d\lambda^{J_1})\lambda^{J_2}.$$

Let

$$D(\lambda^{J_1}\lambda^{J_2}) = \lambda^{J_1}\lambda^{-1}\lambda^{J_2}, D(\lambda^{J_1}\lambda^{-1}\lambda^{J_2}) = 0.$$

Then for $k < 0$, $D : A \otimes \Lambda_i^k \rightarrow A \otimes \Lambda_{i+1}^k$ satisfies $dD + Dd = \text{identity}$. Therefore 3.2(ii) holds for $k < 0$.

Finally we prove 3.2(iii). Note that if I is admissible, $I + rI_l$ is admissible and if $(h^r)^*: \Lambda^* \rightarrow \Lambda^*$ is the dual of h^r , h^r , $(h^r)^*\lambda^I = \lambda^{I-rI_l}$. Therefore

$$\begin{aligned} \lambda^I(\lambda_j \lambda_J) &= (h^r)^*(\lambda^{I+rI_l})(\lambda_j \lambda_J) \\ &= \lambda^{I+rI_l}(h^r(\lambda_j \lambda_J)) = \lambda^{I+rI_l}(\lambda_{j+r} \lambda_{J+2rI_{l-1}}) \end{aligned}$$

Proof of 2.5. Let $C_l = A \otimes U_l$, $D_l = A(BO) \otimes V_l$, $l > 0$, and $D_0 = H^*(MO)$. Denote $a \otimes u \in C_l$ by au and $a \circ v \otimes w \in D_l$, $l > 0$, by $(a \circ v)w$. We filter C_l and D_l as follows: $F_q(C_l)$ is spanned by $a\lambda^I u_l$ with $|u_\omega| \leq q$ and $F_q(D_l)$, $l > 0$, is spanned by all $a \circ v \lambda^I u_l$ with $|u_\omega| + 2^l |v| \leq q$. $F_q(D_0)$ is spanned by all au_ω where $a \in A$, $u_\omega \in U_0 = \{u_\omega \mid 2^i - 1 \notin \omega\}$ and $|u_\omega| \leq q$.

The chain complex (C_l, d) is a direct sum of chain complexes of the form described in 3.2, indexed by the $u_\omega \in U_0$. Hence d is filtration preserving and:

(3.3) The following is exact.

$$\longrightarrow F_q(C_l) \xrightarrow{d} F_q(C_{l-1}) \longrightarrow \cdots \longrightarrow F_q(C_0)$$

Using induction on l we define A linear maps $\theta: C_l \rightarrow D_l$ and $A(BO)$ linear maps $d: D_l \rightarrow D_{l-1}$ such that

- (i) θ is an isomorphism and $\theta: C_0 \rightarrow D_0$ is given by $\theta(a \otimes u_\omega) = a\Phi(u_\omega) \in H^*(MO)$, $u_\omega \in U_0$.
- (ii) $d\theta = \theta d$
- (iii) If $u \in V_l \subset U_l$, $\theta(u) = u$
- (iv) $\theta(F_q(C_l)) = F_q(D_l)$
- (v) Suppose $\lambda^I u_\omega \in U_l$. Let α and β be the partitions

$$\alpha = \bigcup_{r < l} 2^r \omega_r, \quad \beta = \bigcup_{r \geq l} 2^{r-l} \omega_r$$

Note $u_\omega = u_\alpha u_\beta^{2^l}$. Then θ satisfies

$$\theta(\lambda^I u_\omega) = u_\beta \lambda^{I'} u_\alpha \pmod{F_{|u_\omega|-1}(D_l)}$$

where $I' = I + |u_\beta| I_l$.

Note that Proposition 2.5 consists of statements (i), (ii), and (iii) above.

For $l = 0$, θ is defined by (i) and $d = 0$ on D_0 .

Suppose θ and d have been defined on C_k and D_k $k < l$, and satisfy (i)–(v). Define $d = d_D: D_l \rightarrow D_{l-1}$ to be the $A(BO)$ linear map such that for $u \in V_l$,

$d_D(u) = \theta(d_C u)$. We next define $\theta : C_l \rightarrow D_l$. Suppose $\lambda^I u_\omega \in U_l$ and $u_\omega = u_\alpha u_\beta^{2^l}$ as in (v). If $u_\beta = 1$, $\lambda^I u_\omega \in V_l$ and we define $\theta(\lambda^I u_\omega) = \lambda^I u_\omega$. In this case (i)–(v) are satisfied. Suppose $u_\beta \neq 1$. Let

$$X = \theta(d(\lambda^I u_\omega)) + u_\beta \theta(d\lambda^I u_\alpha)$$

where $I' = I + |u_\beta| I_l$. By induction, $\theta d = d\theta$ on C_{l-1} and hence $\partial X = 0$. We show that $X \in F_{p-1}(D_l)$ where $p = |u_\omega|$. Decompose u_α into $u_{\alpha_1} u_{\alpha_2}^{2^{l-1}}$ as in (v).

$$\begin{aligned} \theta(d\lambda^I u_\omega) &= \sum \lambda^I(\lambda_j \lambda_K) \chi(Sq^{j+1}) \theta(\lambda^K u_\omega) \\ &= \sum \lambda^I(\lambda_j \lambda_K) (\chi(Sq^{j+1}) \circ u_{\alpha_2} u_\beta^2) \lambda^{K'} u_{\alpha_1} \text{ mod } F_{p-1} \end{aligned}$$

where $K' = K + |u_{\alpha_2} u_\beta^2| I_{l-1}$. On the other hand,

$$u_\beta \theta(d\lambda^I u_\alpha) = \sum \lambda^{I'}(\lambda_j \lambda_J) u_\beta \chi(Sq^{j+1}) \theta(\lambda^J u_\alpha)$$

In $A(BO)$,

$$u_\beta \chi(Sq^{j+1}) = \chi(Sq^{j-q+1}) \circ u_\beta^2 + \sum_{k < q} \chi(Sq^{j-k+1}) \circ Sq^k u_\beta$$

where $q = |u_\beta|$.

$$\theta(\lambda^J u_\alpha) = u_{\alpha_2} \lambda^{J'} u_{\alpha_1} \text{ mod } F_{|u_\alpha|-1}$$

where $J' = J + |u_{\alpha_2}| I_{l-1}$. If $u \lambda^I v$ has filtration less than $|u_\alpha| - 1$ and $k < q$, $Sq^k u_\beta u \lambda^I v$ has filtration less than $p = |u_\omega|$.

Hence

$$u_\beta \theta(d\lambda^I u_\alpha) = \sum_{j,J} \lambda^{I'}(\lambda_j \lambda_J) (\chi(Sq^{j-q+1}) \circ u_{\alpha_2} u_\beta^2) \lambda^{J'} u_{\alpha_1} \text{ mod } F_{p-1}$$

In the above sum, replace j by $j + q$ and J by $K + 2qI_{l-1}$. Then

$$u_\beta \theta(d\lambda^I u_\alpha) = \sum_{j,K} \lambda^{I'}(\lambda_{j+q} \lambda_{K+2qI_{l-1}}) \chi(Sq^{j+1}) \circ u_{\alpha_2} u_\beta^2 \lambda^{K'} u_{\alpha_1} \text{ mod } F_{p-1}$$

where $K' = K + |u_{\alpha_2} u_\beta^2| I_{l-1}$. But $I' = I + qI_l$ and hence by 3.2(iii),

$$\lambda^{I'}(\lambda_{j+q} \lambda_{K+2qI_{l-1}}) = \lambda^I(\lambda_j \lambda_K)$$

Hence $X \in F_{p-1}(D_l)$.

By (iv) there is a $Y \in F_{p-1}(C_{l-1})$ such that $\theta(Y) = X$ and by (i) and (ii), $dY = 0$. Hence for $l > 1$, by 3.3, there is a $Z \in F_{p-1}(C_l)$ such that $dZ = Y$. We verify that there is such a Z for $l = 1$ by showing that when $l = 1$, $X \in \Phi(I_n)$. In this case

$$\begin{aligned} X &= \chi(Sq^{i+1})\Phi(u_\alpha u_\beta^2) + u_\beta \chi(Sq^{i+q+1})\Phi(u_\alpha) \\ &= \sum_{j < q} \chi(Sq^{i+q+1-j})\Phi((Sq^j u_\beta)u_\alpha) \end{aligned}$$

where $2(i+1) > n - q$, $q = |u_\beta|$. But then, $2(i+q-j+1) > n - |(Sq^j u_\beta)u_\alpha|$ and hence $X \in \Phi(I_n)$.

We now define $\theta(\lambda^I u_\omega)$ by induction on $|u_\omega| = \text{filtration degree of } \lambda^I u_\omega$. For $|u_\omega| = 0$, $\theta(\lambda^I 1) = \lambda^I 1$. If θ is defined on $F_{|u_\omega|-1}(C_l)$, let

$$\theta(\lambda^I u_\omega) = u_\beta \lambda^{I'} u_\alpha + \theta(Z)$$

where Z , α , β , and I' are as above. Then $d\theta(Z) = \theta(dZ) = \theta(Y) = X$ and

$$\begin{aligned} d\theta(\lambda^I u_\omega) &= d(u_\beta \lambda^{I'} u_\alpha) + d\theta(Z) \\ &= u_\beta \theta(d(\lambda^{I'} u_\alpha)) + X = \theta(d(\lambda^I u_\omega)) \end{aligned}$$

Note that elements of the form $u_\beta \lambda^{I'} u_\alpha$, as above, together with $F_{p-1}(D_l)$, span $F_p(D_l)$ over A . Thus $\theta : C_l \rightarrow D_l$ is an epimorphism. (It is at this point that we use λ^I where I has negative entries. For each $u_\beta \lambda^{I'} u_\alpha \in H^*(BO)V_l$ we need $\lambda^I u_\alpha u_\beta^{2^l} \in U_l$ such that $I' = I + |u_l| I_l$.) Elements of the form $\lambda^I u_\alpha u_\beta^{2^l}$ are an A basis for C_l and elements of the form $u_\beta \lambda^{I'} u_\alpha$ are an A basis for D_l . Hence $\theta : C_l \rightarrow D_l$ is an isomorphism and the proof of 2.5 is complete.

Proof of 2.6. Let $v_i \in H^*(BO)$ be the Wu classes, that is, $\Phi(v_i) = \chi(Sq^i)\Phi(1)$ where $\Phi : H^*(BO) \rightarrow H^*(MO)$ is the Thom isomorphism.

LEMMA 3.4.

$$v_i = \sum s_\omega$$

where the sum ranges over all ω with entries only of the form $2^j - 1$ and $|s_\omega| = i$.

Proof. We view $H^*(BO) \subset \mathbb{Z}_2[t_1, t_2, \dots]$, $|t_i| = 1$, and $t_1 t_2 \dots$ as the Thom class. Let $Sq = Sq^0 + Sq^1 + \dots$ and $v = v_0 + v_1 + \dots$. Then

$$\chi(Sq)t_i = \sum t_i^{2^j}$$

and

$$\begin{aligned} v(t_1, t_2, \dots)(t_1 t_2 \cdots) &= \chi(Sq)(t_1 t_2 \cdots) \\ &= \prod_i \left(\sum_j t_i^{2^j-1} \right) (t_1 t_2 \cdots) = \left(\sum_{\omega} s_{\omega} \right) (t_1 t_2 \cdots) \end{aligned}$$

where the sum ranges over ω with entries only of the form $2^j - 1$.

Let x_1 and $x_2 \in A(BO)$ be given by

$$x_1 = \sum_{j>0} Sq^j \circ v_{s+1-j}, \quad x_2 = \sum Sq^j \circ v_{s+2-j}$$

Recall $s = [n/2]$ and n is the dimension of the manifolds we are considering. Let $y_i^1 \in D_1$ be defined by

$$y_1^1 = x_1 \lambda^s, \quad y_2^1 = x_2 \lambda^s, \quad y_3^1 = v_{s+1} \lambda^{s+1} + v_{s+2} \lambda^s + x_2 \lambda^s$$

LEMMA 3.5. *There are elements $y_i^2 \in D_2$ such that $dy_i^2 = y_i^1$ and*

$$\begin{aligned} y_1^2 &= \lambda^{0,0} v_s^2 \bmod F_{2s-1} \\ y_2^2 &= \lambda^{0,-1} v_{s+1}^2 \bmod F_{2s+1} \\ y_3^2 &= \lambda^{-1,-2} v_{s+2} \bmod F_{2s+3} \end{aligned}$$

If s is odd, there is an element y_2^3 such that $y_2^3 = (Sq^1 + w_1)y_2^2$ and

$$y_2^3 = \lambda^{-1,-2,-4} w_1^4 v_{s+2}^2 \bmod F_{2s+7}$$

Proof. We first show that $dy_i^1 = 0$, $d: D_1 \rightarrow D_0 = H^*(MO)$. Let $U \in H^0(MO)$ be the Thom class.

$$\begin{aligned} dy_1^1 &= x_1 d\lambda^s = \sum Sq^j (v_{s+1-j} \chi(Sq^{s+1}) U) + v_{s+1} \chi(Sq^{s+1}) U \\ &= (Sq^{s+1} v_{s+1}) U + v_{s+1}^2 U = 0 \\ dy_2^1 &= \sum Sq^j (v_{s+2-j} \chi(Sq^{s+1}) U) \\ &= \sum Sq^j (v_{s+1} \chi(Sq^{s+2-j}) U) = (Sq^{s+2} v_{s+1}) U = 0 \\ dy_3^1 &= v_{s+1} \chi(Sq^{s+2}) U + v_{s+2} \chi(Sq^{s+1}) U + dy_2^1 = 0 \end{aligned}$$

We next show that y_1^2 exists. In $A \otimes \Lambda^*$ one may easily calculate $d\lambda^{0,0} = Sq^1 \lambda^0$.

Hence, by the arguments in the proof of 2.5,

$$\begin{aligned} d\lambda^{0,0}v_s^2 &= \theta(d\lambda^{0,0}v_s^2) = \theta(Sq^1\lambda^0v_s^2) \\ &= Sq^1 \circ v_s\lambda^s \text{ mod } F_{2s-1} \\ &= \sum_{j>0} Sq^j \circ v_{s+1-j}\lambda^s \text{ mod } F_{2s-1} = y_1^1 \text{ mod } F_{2s-1} \end{aligned}$$

Thus $u = d\lambda^{0,0}v_s^2 + y_1^1 \in F_{2s-1}$ and $du = 0$. Therefore there is a $z \in F_{2s-1}(D_2)$ such that $dz = u$. Let $y_1^2 = \lambda^{0,0}v_s^2 + z$. The existence of y_2^2 , y_3^2 , and y_3^3 are proven in an analogous fashion.

We now define $r : A(BO) \otimes V_l \rightarrow A(BO) \otimes \bar{V}_l$. For $l \neq 2$ and $l \neq 3$, s odd, $\bar{V}_l = V_l$ and r is the identity; $\bar{V}_l \subset V_l$ and $r|_{A(BO) \otimes \bar{V}_l}$ is the identity. \bar{V}_2 is formed from V_2 by omitting the basis elements $\lambda^{0,0}w_s^2$, $\lambda^{0,-1}w_{s+1}^2$ and $\lambda^{-1,-2}w_{s+2}^2$. By 3.4, v_i involves $w_i = s_{(1,1,\dots,1)}$ when v_i is expressed in the u_ω basis. Let

$$\begin{aligned} r(\lambda^{0,0}w_s^2) &= y_1^2 - \lambda^{0,0}w_s^2 \\ r(\lambda^{0,-1}w_{s+1}^2) &= y_2^2 - \lambda^{0,-1}w_{s+1}^2 \\ r(\lambda^{-1,-2}w_{s+2}^2) &= y_2^3 - \lambda^{-1,-2}w_{s+2}^2 \end{aligned}$$

We define r on $A(BO) \otimes V_3$ analogously. Then $r(y_i^2) = r(y_i^3) = 0$.

We conclude this section with an algebraic lemma about the y_j^i 's. Let $L_l \subset A(BO) \otimes V_l$ be defined as follows: $L_l = 0$ for $l = 0$, $l = 3$ and s even, and $l > 3$.

$$L_1 = A(BO)(\{y_i^1\} + S_1)$$

where $S_1 = \{v_3Sq^2\lambda^2\}$ when $s = 2$ and $S_1 = 0$ for $s \neq 2$.

$$L_2 = A(BO)(\{y_i^2\} + S)$$

where $S_2 = \{v_3\lambda^{1,2}\}$ when $s = 2$ and $S_2 = 0$, $s \neq 2$.

$$L_3 = A(BO)\{y_3^2\}$$

$$(d(v_3\lambda^{1,2}) = v_3Sq^2\lambda^2).$$

LEMMA 3.6. $d(L_l) \subset L_{l-1}$, $r(L_l) = 0$ for $l > 1$ and the sequence

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \longrightarrow \cdots \longrightarrow L_0$$

is exact at L_l^q for all l and $q \leq 2s + 2$.

Proof. The first part of 3.6 is clear from the definition of L_l . One easily checks that if $x \in A(BO)$, $|x| \leq 1$ and $d(xy_2^3) = 0$, then $x = 0$ and therefore $d : L_3^q \rightarrow L_2^{q+1}$ is an injection for $q \leq 2s + 2$. $d : L_2 \rightarrow L_1$ is clearly onto. To check exactness at L_2^q , $q \leq 2s + 2$ one must verify that if $y = x_1y_1^1 + x_2y_2^1 + x_3y_3^1 + x_4v_3Sq^2\lambda^2 = 0$, $x_i \in A(BO)$ and $|y| \leq 2s + 3$, then $x_1 = x_3 = x_4 = 0$ and $x_2 = 0$ or s is odd and $x_2 = Sq^1 + w_1$. This is a tedious but straightforward calculation, made somewhat simpler by the following observation. Let

$$F : A(BO) \otimes \{\lambda^s\} \rightarrow H^*(MO \wedge K(Z_2, N))$$

be given by

$$F(a \circ u\lambda^s) = a(u\chi(Sq^{s+1})U \otimes \iota_N)$$

Then

$$F(y_1^1) = v_{s+1}U \otimes \iota_N + U \otimes Sq^{s+1}\iota_N$$

$$F(y_2^1) = U \otimes Sq^{s+2}\iota_N$$

$$F(v_3Sq^2\lambda^2) = v_3^2U \otimes Sq^2\iota_N$$

We leave the details to the reader.

§4. Proofs of 2.4 and 2.8

Let $\{A \otimes \Lambda_l^k, d\}$ be the chain complex described in Proposition 3.2.

PROPOSITION 4.1. *For each integer k , there are Ω -spectra $Y_l = Y_l(k)$ and maps $\rho_l = \rho_l(k) : Y_{l-1} \rightarrow K(\Lambda_l^k)$ of degree one, $l = 0, 1, 2, \dots$ such that*

(i) $Y_0 = K(\Lambda_0^k)$. Y_l is a fibration over Y_{l-1} induced by ρ_l from the contractible fibration over $K(\Lambda_l^k)$.

(ii) If $i : K(\Lambda_{l-1}^k) \rightarrow Y_{l-1}$ is the inclusion of the fibre,

$$(\rho_l i)^* = d : A \otimes \Lambda_l^k \rightarrow A \otimes \Lambda_{l-1}^k$$

where d is as in 3.2.

(iii) If M is a smooth, compact n -manifold and ν is its normal bundle, then

$$[T(\nu), Y_l]_p \rightarrow [T(\nu), Y_{l-1}]_p$$

is an epimorphism for $p < 2k + 2$.

(iv) Suppose $k = 0$. Let $I(l, 0) = (0, \dots, 0)$ have length l .

$$\rho_l^* \lambda^{I(l,0)} = \delta_l \tilde{\iota}$$

where $\iota \in H^0(Y_{l-1}; Z_{2l})$, $\tilde{\iota}$ reduced modulo two is the generator $\iota \in H^0(Y_{l-1}) \approx Z_2$ and δ_l is the Bockstein associated to $Z_2 \rightarrow Z_{2^{l+1}} \rightarrow Z_{2^l}$.

Proof. For $k \geq 0$, 4.1(i), (ii), and (iii) were proved in [5]. For $k < 0$, $\{A \otimes \Lambda_l^k, d\}$ is a free acyclic resolution of the zero A module so that the existence of Y_l and ρ_l easily follow by induction on l . If M is as in (iii), $v: T(\nu) \rightarrow Y_{l-1}$ has degree p , $p < 2k + 2$ and $k < 0$, then $|(\rho_l v)^*(\lambda^I)| > n$ and (iii) follows.

Finally we prove (iv). The formula for d in 3.1 shows that $d\lambda^{I(l,0)} = Sq^1 \lambda^{I(l-1,0)}$. The complex,

$$\longrightarrow A \otimes \{\lambda^{I(l,0)}\} \xrightarrow{d} A \otimes \{\lambda^{I(l-1,0)}\} \longrightarrow \dots \longrightarrow A \otimes \{\lambda^{I(0,0)}\}$$

is realized by the tower

$$\rightarrow K(Z_{2^l}) \rightarrow K(Z_{2^{l-1}}) \rightarrow \dots \rightarrow K(Z_2)$$

with k -invariants, $\delta_l: K(Z_{2^l}) \rightarrow K(Z_2)$. Except for $\lambda^{I(l,0)}$, the generators of Λ_l^0 have dimension > 0 and hence kill classes of dimension > 1 . Thus $Y_l = K(Z_{2^{l+1}})$ in dimensions ≤ 1 . Therefore (iv) holds.

Proof of 2.4: We wish to realize the complex $\{A \otimes U_l, d\}$ by a tower of spectra, X_i . Let $Y_l(k)$ and $\rho_l(k)$ be as in 4.1. For a spectrum Z , let SZ denote the shift suspension, i.e., $(SZ)_q = Z_{q+1}$. Define X_i and $\alpha_i: X_{i-1} \rightarrow K(Y_i)$ by

$$X_i = \prod_{u_\omega \in U_0} S^{|u_\omega|} Y_i([(n - |u_\omega|)/2])$$

$$\alpha_i = \prod S^{|u_\omega|} \rho_i([(n - |u_\omega|)/2])$$

The map α_i takes X_{i-1} into $K(U_i)$ since

$$\prod S^k K(\Lambda_i^k) = K(U_i)$$

where k ranges over $[(n - |u_\omega|)/2], |u_\omega| \in U_0$. Proposition 2.4 now follows directly from 4.1.

Proof of 2.8: Using induction on l , one easily proves that if I is admissible and $l = l(I)$,

$$|\lambda^I| \geq 2t(I) \left(1 - \frac{1}{2^l}\right)$$

Suppose $l \geq n$ and $\lambda^I u_\omega \in U_l$. Then $2(t(I) + 1) > n - |u_\omega|$. Therefore

$$|\lambda^I u_\omega| \geq 2t(I) \left(1 - \frac{1}{2^l}\right) + |u_\omega| \geq n - 1 - \frac{n - |u_\omega| - 1}{2^l} > n - 2$$

Also if $|u_\omega| > n - 1$, $|\lambda^I u_\omega| > n - 1$. If $|u_\omega| < n - 1$, $t(I) \geq 1$ and hence $|\lambda^I| \geq l \geq n$. Therefore $U_l^q = 0$ for $q < n - 1$ and $U_l^{n-1} = \{\lambda^{I(l,0)} u_\omega \mid u_\omega \in U_0^{n-1}\}$ since $\lambda^{I(l,0)}$ is the only λ^I with $t(I) \geq 0$ and $|\lambda^I| = 0$. If $r > l$ and $\omega_r \neq \{\}$, $|u_\omega| \geq |u_{\omega_r}^{2^r}| \geq 2^r > n$. Hence $V_l^q = U_l^q$ for $q \leq n - 1$.

By the definition of $\beta_l : B_{l-1} \rightarrow K(V_l)$,

$$\Phi(\beta_l^*(\lambda^{I(l,0)} u_\omega)) = f_{l-1}^* \alpha_l^*(\lambda^{I(l,0)} u_\omega)$$

By 4.1(iv) $\alpha_l^*(\lambda^{I(l,0)} u_\omega) = \delta_l \tilde{t}$ where $\tilde{t} \in H^*(X_{l-1}; Z_{2^l})$ comes from the factor of X_{l-1} , $Y([n - |u_\omega|/2])$. Since the diagram

$$\begin{array}{ccc} T_{l-1} & \xrightarrow{f_{l-1}} & X_{l-1} \\ \downarrow p_1 & & \downarrow p_2 \\ T_0 & \xrightarrow{f_0} & X_0 \end{array}$$

commutes, $\tilde{u} = f_{0-1}^* \tilde{t}$ reduced modulo two is $p_1^* f_0^* u_\omega = p_1^* u_\omega U_0 = u_\omega U_{l-1}$, where U_l is the Thom class of T_l and the proof of 2.8 is complete.

§5. Proof of 2.7

If G_1 and G_2 are graded groups and $h : G_1 \rightarrow G_2$ is a homomorphism of degree i , we will say that h is k connected if $h : G_1^q \rightarrow G_2^{q+i}$ is an epimorphism for $q < k$ and a monomorphism if $q \leq k$. We will say that a sequence of graded groups and homomorphisms,

$$\cdots \rightarrow G_l \rightarrow G_{l-1} \rightarrow \cdots$$

is k -exact if

$$G_{l+1}^{q-i} \rightarrow G_l^q \rightarrow G_{l-1}^{q+j}$$

is exact for all l and $q \leq k$.

In §3 we constructed isomorphisms $\theta : A \otimes U_l \rightarrow A(BO) \otimes V_l$ and a subcomplex $\{L_l, d\} \subset \{A(BO) \otimes V_l, d\}$ such that

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \xrightarrow{d} \cdots \longrightarrow L_0 = 0$$

is $2s+2$ exact, $s = [n/2]$. In §4 we constructed a tower of fibrations $\rightarrow X_l \rightarrow X_{l-1} \rightarrow$ with k -invariants $\alpha_l : X_{l-1} \rightarrow K(U_l)$ associated to the complex $\{A \otimes U_l, d\}$. Let

$$\bar{H}^*(K(U_l)) = H^*(K(U_l)) / \theta^{-1}(L_l)$$

$$\bar{H}^*(X_l) = H^*(X_l) / \alpha_{l-1}^* \theta^{-1}(L_{l-1})$$

LEMMA 5.1: *The maps*

$$K(U_l) \xrightarrow{i} X_l \xrightarrow{p} X_{l-1} \xrightarrow{\alpha_l} K(U_l)$$

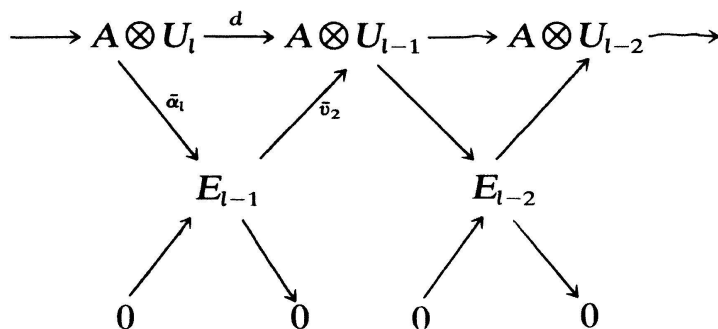
induce a $2s+2$ -exact sequence

$$\rightarrow \bar{H}^*(K(U)) \rightarrow \bar{H}^*(X_{l-1}) \rightarrow \bar{H}^*(X_l) \rightarrow$$

Proof: Let E_l be the kernel of

$$H^*(X_l) \rightarrow \lim_{k \rightarrow \infty} H^*(X_k)$$

Then $H^*(X_l) \approx H^*(MO) / \Phi(I_n) \oplus E_l$ and E_l and $A \otimes U_l$ are related by the diagram



where the $\bar{\alpha}_l$ and \bar{i}_l are defined by α_l^* and i_l^* and each pair of composable arrows is exact. Dividing $A \otimes U_l$ and E_{l-1} by $\theta^{-1}(L_l)$ and $\bar{\alpha}_l \theta^{-1}(L_{l-1})$, respectively, produces the same type of diagram with exactness replaced by $2s + 2$ -exactness. The desired result then follows.

In §2 we defined maps

$$g'_l : K(U_l) \rightarrow T_{l-1}/T'_l$$

In §6 we prove:

LEMMA 5.2. *The map g'_l induces a $2s + 2$ -connected map*

$$F_l : \bar{H}^*(K(U_l)) \rightarrow H^*(T_{l-1}/T'_l)$$

for $l \geq 1$.

Proof of 2.7: We first prove 2.7(ii). Suppose M is a smooth n -manifold, $h : M \rightarrow B_0 = BO$ classifies ν , the normal bundle of M and $\tilde{h} : M \rightarrow B_{l-1}$ is a lifting of h . Let $T(\tilde{h}) : T(\nu) \rightarrow T_{l-1}$ denote the associated Thom space map. Then $f_{l-1}T(\tilde{h}) : T(\nu) \rightarrow X_{l-1}$ is a lifting of $f_0T(h) : T(\nu) \rightarrow X_0$ and hence by 2.4(iv), $f_{l-1}T(\tilde{h})$ lifts to X_l and therefore $\alpha_l f_{l-1}T(\tilde{h}) = 0$. Thus for $v \in \bar{V}_l$

$$\Phi h^* \beta_l^*(v_1) = T(\tilde{h})^* \Phi(\beta_l^*(v_1)) = T(\tilde{h})^* f_{l-1}^* \alpha_l^*(v) = 0$$

Thus $\beta_l \tilde{h} = 0$ and \tilde{h} lifts to $h' : M \rightarrow B'_l$

If $u \in U_{l+1}$, $\bar{u} = \{u\} \in W_{l+1} = U_{l+1}/\ker \Delta$ and $\nu\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$, then

$$\Phi((h')^* \gamma_l^*(\bar{u}_1)) = T(h')^* \Phi(\gamma_l^* \bar{u}_1) = T(h')^* \Delta(u).$$

Recall,

$$\Delta(u) = (f'_l)^* \alpha_{l+1}^* u - \sum x_i (f'_{l+1})^* \alpha^* u_i$$

But $T(h')^*$ is $A(BO)$ linear and $\alpha_{l+1} f'_l T(h') = 0$ as above. Thus $T(h')^* \Delta(u) = 0$ and hence $\gamma_l h' = 0$. Therefore h' lifts to B_l and the proof of 2.7(ii) is complete. We note for further reference:

LEMMA 5.3: $T(h')^* \Delta(u) = 0$ for $u \in U_{l+1}$.

LEMMA 5.4. If $\delta^* : H^*(T'_l) \rightarrow H^*(T_{l-1}/T'_l)$, $\delta^* \Delta(u) = 0$ for $u \in U_{l+1}$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} H^*(X_l) & \xrightarrow{i^*} & H^*(K(U_l)) \\ \downarrow f_l & & \downarrow (g_l) \\ H^*(T_l) & \xrightarrow{\alpha^*} & H^*(T_{l-1}/T_l) \end{array}$$

Recall, g'_0 realizes $\Psi r\theta$, $i^*\alpha_{l+1}^* = d$ and Ψ , r , and $d : A(BO) \otimes V_{l-1} \rightarrow A(BO) \otimes V_{l-1}$ are $A(BO)$ linear. Hence,

$$\begin{aligned} \delta^* \Delta(u) &= \delta^*((f'_l)^* \alpha_{\alpha+1}^* u + \sum x_i (f'_l)^* \alpha_{l+1}^* u_i) \\ &= (g'_l)^* i^* \alpha_{l+1}^* u + \sum x_i (g'_l)^* i^* \alpha_{l+1}^* u_i \\ &= \Psi r \theta du + \sum x_i \Psi r \theta du_i = \Psi r d\theta u + \sum \Psi r dx_i \theta(u_i) \end{aligned}$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$. But for $v \in V_{l+1}$, $\theta(v) = v$. Thus

$$\sum x_i \theta(u_i) = \sum x_i u_i = r\theta u = \theta u + z$$

where $z \in L_{l+1}$. Furthermore $dz \in L_l$. Hence $\delta^* \Delta(u) = \Psi r dz = \Psi r \theta \theta^{-1} dz = (g'_l)^* \theta^{-1} dz$.

But by 5.2, $\theta^{-1}(L_l)$ is the kernel of $(g'_l)^*$.

We now prove that f_l induces a $2s+2$ -connected map $\bar{f}_l : \bar{H}(X_l) \rightarrow H^*(T_l)$ by induction on $l \geq 0$. We first show that \bar{f}_l is well defined.

$$\bar{H}^*(X_l) = H^*(X_l) / \alpha_{l+1}^*(\theta^{-1}(L_{l+1}))$$

From the commutative diagram:

$$\begin{array}{ccc} T_l & \xrightarrow{j} & T_l/T'_{l+1} \\ \downarrow f_l & & \downarrow g'_{l+1} \\ X_l & \xrightarrow{\alpha_{l+1}} & K(U_{l+1}) \end{array}$$

we see that

$$f_l^* \alpha_{l+1}^*(\theta^{-1}(L_{l+1})) = j^*(g'_{l+1})^*(\theta^{-1}(L_{l+1}))$$

By 5.2, $\theta^{-1}(L_{l+1})$ is in the kernel of $(g'_{l+1})^*$.

Since f_0^* is an isomorphism, $\bar{f}_0 = f_0^*$ and \bar{f}_0 is an isomorphism.

Suppose \bar{f}_{l-1} is $2s+2$ connected. If $u \in U_{l+1}$, $\Delta(u) \in H^q(T'_l)$ pulls back to $H^q(T_{l-1})$ since, by 5.4, $\delta^* \Delta(u) = 0$ and it pulls back to $H^q(X_{l-1})$ if $q < 2s+2$, that is, if $|u| < 2s+1$, $\Delta(u) = (f'_l)^* p^* x$ where $p: X_l \rightarrow X_{l-1}$. But since the X_l 's are constructed from an acyclic complex, image $p^* = \text{image } (H^*(X_0) \rightarrow H^*(X_l))$. Therefore image $(f'_l)^* p^* = \text{image } (H^*(T_0) \rightarrow H^*(T'_l)) = H^*(MO)/\Phi(I_n)$. But by 5.3, $\Delta(u)$ is zero on all n -manifolds. Hence $\Delta(u) = 0$ and we have shown that $W_{l+1}^q = (U_{l+1}/\ker \Delta)^q = 0$ for $q < 2s+1$. Therefore $H^q(B'_l) \rightarrow H^q(B_l)$ is an isomorphism for $q \leq 2s+2$ since B_l is a fibration over B'_{l-1} induced by $\gamma_l: B'_l \rightarrow K(W_{l+1})_1$. Then $H^q(T'_l/T_l) = H^q(B'_l, B_l) = 0$ for $q < 2s+2$ and hence

$$H^*(T_{l-1}/T'_l) \rightarrow H^*(T_{l-1}/T_l)$$

is $(2s+2)$ -connected. Let g_l be the composition

$$T_{l-1}/T_l \longrightarrow T_{l-1}/T'_l \xrightarrow{g_l} K(U_l)$$

and let $\bar{g}_l: \bar{H}^*(K(U_l)) \rightarrow H^*(T_{l-1}/T_l)$ be induced by g_l . Then \bar{g}_l is $(2s+2)$ -connected by 5.2. Consider the commutative diagram:

$$\begin{array}{ccccccc} \longrightarrow & \bar{H}^*(K(U_l)) & \longrightarrow & \bar{H}^*(X_{l-1}) & \longrightarrow & \bar{H}^*(X_l) & \longrightarrow & \bar{H}^*(K(U)) & \longrightarrow \\ & \downarrow \bar{g}_l & & \downarrow \bar{f}_{l-1} & & \downarrow \bar{f} \sim & & \downarrow & \\ \longrightarrow & H^*(T_{l-1}/T_l) & \longrightarrow & H^*(T_{l-1}) & \longrightarrow & H^*(T_l) & \longrightarrow & H^*(T_{l-1}/T_l) & \longrightarrow \end{array}$$

A five lemma argument and the fact that \bar{f}_{l-1} and \bar{g}_l are $(2s+2)$ -connected shows that \bar{f}_l is $2s+2$ -connected.

Since $L_l = 0$ for $l > 3$, $\bar{H}^*(X_l) = H^*(X_l)$ for $l \geq 3$ and therefore $f_l^*: H^q(X_l) \rightarrow H^q(T_l)$ is an isomorphism for $q \leq n < 2s+2$. This completes the proof of 2.7.

§6. Proof of 5.2

LEMMA 6.1.

$$H^q(B_{l-1}) \rightarrow H^q(B'_l)$$

is an isomorphism for $l > 1$ and $q \leq s+1$. For $l = 1$ it is an epimorphism for $q \leq s+1$ and $v_{s+1}, w_1 v_{s+1}, Sq^1 v_{s+1}$ and v_{s+2} generate the kernel for $q \leq s+2$.

Proof. As we saw in the proof of 2.8, if $\lambda^I u_\omega \in V_l$, $|\lambda^I u_\omega| \cong (n-1) - (n - |u_\omega| - 1)/2^l$. Hence the lowest dimensional element in V_l is of the form λ^I with $t(I) = s$. For such an I , $|\lambda^I| \cong s+2$ except for $l=1$ or $l=2$ and $s=1$ and 2. The space B'_l is a fibration over B_{l-1} induced by $\beta_l: B_{l-1} \rightarrow K(V_l)_1$ and for $l > 1$, $K(V_l)_1$ is $s+2$ connected except when $l=2$ and $s=1$ or 2. For $s=1$ or 2, the lowest dimensional elements in V_2 are $\lambda^{1,1}$ and $\lambda^{1,2}$ respectively; $d\lambda^{1,1} \neq 0$ and $d\lambda^{1,2} \neq 0$ so these elements kill nonzero classes in B_1 . Thus for $l > 1$, $H^q(B_{l-1}) \approx H^q(B'_l)$ for $q \leq s+1$.

Suppose $l=1$. From 3.1 one sees that $d\lambda^i = \chi(Sq^{i+1})U = \Phi(v_{i+1})$ where U is the Thom class and v_{i+1} is the Wu class. Hence $\beta_1: B_0 \rightarrow K(V_1)_1$ takes λ^i into v_{i+1} . One easily checks that $V_1^q = 0$ for $q < s$, $V_1^s = \{\lambda^s\}$ and $V_1^{s+1} = \{\lambda^{s+1}\}$. The remainder of 6.1 now follows by a simple Serre spectral sequence argument.

Let $K_l = K(V_l)_1$. Viewing $\beta_l: B_{l-1} \rightarrow K_l$ as a fibre map with fibre B'_l , consider the pair of fibrations p_1 and p_2 :

$$\begin{array}{ccc} (B_{l-1}, B'_l) & \xrightarrow{c} & (B_{l-1} \times K_l, B_{l-1} \times \{*\}) \\ & \searrow p_1 & \swarrow p_2 \\ & & (K_l, *) \end{array}$$

where p_1 is defined by β_l , p_2 is projection on the second factor and $c = id \times p$. Note c is a fibre preserving map so we may use it to compare the Serre spectral sequences of p_1 and p_2 .

LEMMA 6.2. For $l > 1$, $c^*: H^q(B_{l-1} \times K_l, B_{l-1} \times \{*\}) \rightarrow H^q(B_{l-1}, B'_l)$ is an isomorphism for $q \leq 2s+3$. For $l=1$, c^* is an epimorphism for $q \leq 2s+2$ and for $q \leq 2s+3$ the kernel is generated by

$$\begin{aligned} & v_{s+1} \otimes \lambda_1^s + 1 \otimes (\lambda_1^s)^2 \\ & v_{s+1} \otimes Sq^1 \lambda_1^s + 1 \otimes \lambda_1^s Sq^1 \lambda_1^s \\ & v_{s+1} \otimes \lambda_1^{s+1} + 1 \otimes \lambda_1^s \lambda_1^{s+1} \\ & w_1 v_{s+1} \otimes \lambda_1^s + w_1 \otimes (\lambda_1^s)^2 \\ & Sq^1 v_{s+1} \otimes \lambda_1^s + 1 \otimes \lambda_1^s Sq^1 \lambda_1^s \\ & v_{s+2} \otimes \lambda_1^s + 1 \otimes \lambda_1^s \lambda_1^{s+1} \end{aligned}$$

Proof. Let $E_r^{p,q}$ and $\bar{E}_r^{p,q}$ denote the Serre spectral sequences for p_1 and p_2 respectively.

$$E_2^{p,q} = H^p(K_l, *) \otimes H^q(B_{l-1})$$

$$\bar{E}_2^{p,q} = H^p(K_l, *) \otimes H^q(B')$$

As we saw above, for $l > 1$, K_l is $s + 2$ connected and $H^q(B_{l-1}) \approx H^q(B'_1)$ for $q \leq s + 1$. Therefore c induces an isomorphism at the E_2 level for $p + q \leq 2s + 3$ and the differentials are trivial for p_2 because it is a product fibration. This proves 6.2 for $l > 1$.

For $l = 1$, 6.2 is true at the E_2 level with the first summands in the above list of elements as a basis for the kernel; the second summands are of lower filtration. The same is true at the E_∞ level, so to complete the proof, we must show that these elements are in the kernel of c^* .

Under the map $H^*(B_0, B'_1) \rightarrow H^*(B_0)$, $c^*(1 \otimes \lambda_1^s)$ goes to v_{s+1} . Hence

$$c^*(v_{s+1} \otimes \lambda_1^s + 1 \otimes (\lambda_1^s)^2) = v_{s+1}c^*(1 \otimes \lambda_1^s) + c^*(1 \otimes \lambda_1^s)^2 = 0$$

(If $j: X \subset (X, A)$ and $x \in H^*(X, A)$, $x^2 = (j^*x)x$.) The same argument applies to the other five elements.

Let

$$\phi: (A(BO) \otimes \bar{V}_l)^q \rightarrow H^{q+1}(T_{l-1} \wedge K_l)$$

be defined by

$$\phi((a \otimes w)u) = a(wU \otimes u_1)$$

where U is the Thom class, $a \in A$, $w \in H^*(BO)$ and $u \in \bar{V}_l$.

LEMMA 6.3. For $q \leq 2s + 1$, ϕ is an epimorphism. For $q \leq 2s + 2$ the kernel of ϕ is zero for $l > 1$ and $(l, s) \neq (2, 2)$, is $\{v_3\lambda^{1,2}\}$ for $(l, s) = (2, 2)$ and is $\{\sum Sq^i \circ v_{s+2-i}\lambda^s\}$ for $l = 1$.

Proof. Let $\mu, \mu': A(BO) \rightarrow A(BO)$ be defined by

$$\begin{aligned} \mu(a \circ w) &= \sum a'_i \circ w\sigma(a''_i) \\ \mu'(a \circ w) &= \sum a'_i \circ w\chi(a''_i) \end{aligned}$$

(Recall, wa is defined by $(wa/U = \chi(a))(wU)$.)

Where $a \rightarrow \sum a'_i \otimes a''_i$ in the diagonal in A . Then $\mu\mu' = \mu'\mu = \text{identity}$ and thus μ is a Z_2 -isomorphism. Let $\phi' = \phi(\mu \otimes id)$. Then

$$\phi'((a \circ w)u) = \sum a'_i(\chi(a''_i)(wU) \otimes u_1) = wU \otimes au_1$$

Let λ^l be the lowest dimensional element in \bar{V}_l ; $|\lambda^l| > s$ for $l = 1$. The lowest

dimensional element in $H^*(T_{l-1} \wedge K_l)$ not in the image of ϕ' is $U \otimes (\lambda_1^l \cup Sq^i \lambda_1^l)$, an element of dimension $\geq 2s + 3$. Hence ϕ is an epimorphism for $q < 2s + 2$. The lowest dimensional elements in the kernel of ϕ' are $1 \circ v_{s+1} \lambda_1^l$ or $(Sq^m \circ 1) \lambda_1^l$ where $m = |\lambda_1^l| + 1$. For $l > 2$, $(l, s) \neq (2, 2)$, $\lambda_1^l > s + 1$ and hence these elements occur in dimensions $> 2s + 3$. For $(l, s) = (2, 2)$, $\phi(v_3 \lambda^{1,2}) = \phi'(v_3 \lambda^{1,2}) = 0$. For $l = 1$

$$0 = \phi'((Sq^{s+2} \circ 1) \lambda^s) = \phi\left(\left(\sum Sq^i \circ v_{s+2-i}\right) \lambda^s\right)$$

This proves the last part of 6.3.

Proof of 5.2: We must show that

$$(g_l')^* = \Psi r \theta : (A \otimes U_l)^q \rightarrow H^{q+1}(T_{l-1}/T_l')$$

is an epimorphism for $q \leq 2s + 2$ and $(L_l)^q$ is the kernel for $q \leq 2s + 2$. By 2.5, θ is an isomorphism. Let ϕ be the map in 6.3 and c the map in 6.2. Lifting c to the Thom space level we obtain a map

$$T(c) : T_{l-1}/T_l' \rightarrow T_{l-1} \wedge K_l$$

Furthermore $\Psi = T(c)^*$. Thus by 6.2 and 6.3, Ψ is an epimorphism for $q \leq 2s + 1$ and since r is an epimorphism, $(g_l')^*$ is an epimorphism for $q \leq 2s + 1$. For $l > 1$ and $(l, s) \neq (2, 2)$, $T(c)^*$ and ϕ are monomorphisms for $q \leq 2s + 2$ and L_l^q is the kernel of r . When $(l, s) = (2, 2)$ $r(L_l) = \{v_3 \lambda^{1,2}\}$. This completes the proof of 5.2 for $l > 1$.

Suppose $l = 1$. Then $r = \text{identity}$. We wish to show that $L_1 = \phi^{-1}(\ker T(c)^*)$. In 6.2 a basis for $\ker c^*$ was given for $q \leq 2s + 2$. Since image $\phi = \text{image } \phi'$ cannot involve cup products (except squares) in $H^*(K_l)$, the above basis shows that the following is a basis for image $\phi \cap \ker T(c)^*$:

$$\begin{aligned} &v_{s+1} U \otimes \lambda_1^s + U \otimes Sq^{s+1} \lambda_1^s \\ &w_1 v_{s+1} U \otimes \lambda_1^s + w_1 U \otimes Sq^{s+1} \lambda_1^s \\ &v_{s+1} U \otimes Sq^1 \lambda_1^s + (Sq^1 v_{s+1}) U \otimes \lambda_1^s \\ &v_{s+1} U \otimes \lambda_1^{s+2} + v_{s+2} U \otimes \lambda_1^s \end{aligned}$$

Thus a basis for $\phi^{-1}(\ker c^*)$ is ϕ^{-1} of these elements and $(\sum Sq^i \circ v_{s+2-i}) \lambda^s$ from the kernel of ϕ . A simple calculation shows that these elements form a basis for L_1^q , $q \leq 2s + 2$, completing the proof of 5.2.

BIBLIOGRAPHY

- [1] BROWDER, W. *The Kervaire Invariant of Frammed Manifolds and its Generalizations*, Ann. of Math. (2) 90 (1969), 157–186.
- [2] BROWN, E. and GITLER, S. *A Spectrum Whose Cohomology is a Certain Cyclic Module Over the Steenrod Algebra*, Topology 12, (1973) 283–295.
- [3] BROWN, E. and PETERSON, F. *Relations Among Characteristic Classes I*, Topology (3) (1964), 39–52.
- [4] —, *On Immersions of n -manifolds*, Adv. in Math. 24 (1977), 74–77.
- [5] —, *On Stable Decomposition of $\Omega^2 S^{n+2}$* , Trans. Amer. Math. Soc., to appear.
- [6] MASSEY W. and PETERSON, F. *The Cohomology Structure of Certain Fibre Spaces*, Topology 5 (1965) 47–65.
- [7] MILNOR J. and STASHEFF, J. *Characteristic Classes*, Ann. of Math. Studies 76 (1974).
- [8] BOUSFIELD A. *et al*, *The mod- p Lower Central Central Series and the Adams Spectral Sequence*, Topology 5 (1966) 331–342.
- [9] HIRSCH, M. *Immersions of Manifolds*, Trans. Amer. Math. Soc. 93 (1959) 242–279.

Received June 30, 1978