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The Boolean algebra of spectra

A. K. BOUSFIELD

Introduction

Let \mathbf{Ho}^s denote the stable homotopy category of CW-spectra (cf. [Adams 2]), and for $E \in \mathbf{Ho}^s$ let E_* be the associated homology theory. For $E, G \in \mathbf{Ho}^s$ we say E_* and G_* have the same acyclic spectra if the following equivalent conditions hold:

- (i) For $X \in \mathbf{Ho}^s$, $E_*X = 0 \Leftrightarrow G_*X = 0$.
- (ii) For $f: X \rightarrow Y \in \mathbf{Ho}^s$, $f_*: E_*X \approx E_*Y \Leftrightarrow f_*: G_*X \approx G_*Y$.

This gives a very coarse equivalence relation for spectra, and we let $\mathbf{A}(\mathbf{Ho}^s)$ consist of all the equivalence classes $\langle E \rangle$ for $E \in \mathbf{Ho}^s$, where $\langle E \rangle$ is given by all $G \in \mathbf{Ho}^s$ such that E_* and G_* have the same acyclic spectra. We partially order $\mathbf{A}(\mathbf{Ho}^s)$ by writing $\langle E \rangle \leq \langle G \rangle$ when each G_* -acyclic spectrum is E_* -acyclic. Our purpose in this note is to study the algebraic structure of $\mathbf{A}(\mathbf{Ho}^s)$ when it is equipped with the relation \leq and the operations \vee and \wedge induced from the usual wedge and smash product for spectra.

We say that $\langle E \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ has a *complement* $\langle E \rangle^c \in \mathbf{A}(\mathbf{Ho}^s)$ if $\langle E \rangle \wedge \langle E \rangle^c = \langle 0 \rangle$ and $\langle E \rangle \vee \langle E \rangle^c = \langle S \rangle$ where S is the sphere spectrum, and we note that $\langle E \rangle^c$ is unique when it exists. We let $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$ consist of those $\langle E \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ with complements, and we observe that $\mathbf{BA}(\mathbf{Ho}^s)$ is a Boolean algebra. We prove that $\langle E \rangle \in \mathbf{BA}(\mathbf{Ho}^s)$ whenever E is a (possibly infinite) wedge of finite CW-spectra. It would be most interesting to determine the sublattice of $\mathbf{BA}(\mathbf{Ho}^s)$ given by such $\langle E \rangle$. We show that $\langle S^0 \cup_{\alpha} e^n \rangle = \langle S^0 \rangle$ for each $\alpha \in [S^{n-1}, S^0]$ with $n \neq 1$, and that $\langle DE \rangle = \langle E \rangle$ when E is a finite CW-spectrum and DE is its Spanier-Whitehead dual. This incidentally implies that $G_*E = 0 \Leftrightarrow G^*E = 0$, for $G, E \in \mathbf{Ho}^s$ with E finite. Some other members of $\mathbf{BA}(\mathbf{Ho}^s)$ are $\langle K \rangle$ and $\langle SZ_{(J)} \rangle$ where K is the spectrum of complex K -theory and $SZ_{(J)}$ is the Moore spectrum associated with a subring $Z_{(J)} \subset Q$. Indeed, $\langle K \rangle$ and $\langle SZ_{(J)} \rangle$ are of the form $\langle E \rangle^c$ where E is an appropriate wedge of finite CW-spectra, though the proof for K will be postponed to [Bousfield 3].

We also introduce a distributive lattice $\mathbf{DL}(\mathbf{Ho}^s)$ given by all $\langle E \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ with $\langle E \rangle \wedge \langle E \rangle = \langle E \rangle$, and we show that $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{DL}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$ where both

containments are proper. It turns out that $\langle E \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$ whenever E is a (possibly infinite) wedge of ring spectra and finite CW-spectra. In fact, most familiar spectra represent elements of $\mathbf{DL}(\mathbf{Ho}^s)$.

The class $\mathbf{A}(\mathbf{Ho}^s)$ has applications to the homological localization theory of spectra, cf. [Bousfield 3], [Ravenel]. In particular, the E_* -localization is equivalent to the G_* -localization iff $\langle E \rangle = \langle G \rangle$, and a determination of $\mathbf{A}(\mathbf{Ho}^s)$ would provide an inventory of the possible homological localization functors.

Our results on the structure of $\mathbf{A}(\mathbf{Ho}^s)$, $\mathbf{BA}(\mathbf{Ho}^s)$, and $\mathbf{DL}(\mathbf{Ho}^s)$ are established in §2. Some of our proofs involve $[E,]_*$ -colocalizations of spectra, and we develop the required theory in §1.

We essentially use the notation and terminology of [Adams 2]. However, we let \mathbf{Ho}^s be the category of CW-spectra and homotopy classes of maps of degree 0, cf. [Adams 2, p. 144]. Thus \mathbf{Ho}^s is an additive category equipped with an equivalence $\Sigma : \mathbf{Ho}^s \rightarrow \mathbf{Ho}^s$ induced by the “shift” suspension Σ of CW-spectra. We write $[X, Y]$ for the group of morphisms $X \rightarrow Y \in \mathbf{Ho}^s$, and write $[X, Y]_n$ for $[\Sigma^n X, Y]$ where $n \in \mathbb{Z}$. By a cofibre sequence we mean a sequence in \mathbf{Ho}^s equivalent to $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$ for some cellular map f of degree 0 between CW-spectra, cf. [Adams 2, p. 155]. Recall that \mathbf{Ho}^s has arbitrary coproducts induced by the wedge \vee for CW-spectra, and for $X, Y \in \mathbf{Ho}^s$ there is a natural smash product $X \wedge Y \in \mathbf{Ho}^s$ which is associative, commutative, and unitary (with the sphere spectrum S as unit) up to coherent natural equivalences, cf. [Adams 2, p. 158]. We call $E \in \mathbf{Ho}^s$ a ring spectrum if it is equipped with an associative (but not necessarily commutative) multiplication $E \wedge E \rightarrow E$ and a two sided unit $S \rightarrow E$ in \mathbf{Ho}^s . As usual, we let $X_* Y = \pi_* X \wedge Y = [S, X \wedge Y]_*$ for $X, Y \in \mathbf{Ho}^s$.

§1. $[E,]_*$ -colocalizations of spectra

In preparation for §2 and for [Bousfield 3], we now develop the $[E,]_*$ -colocalization theory of spectra. Some of the concepts here have previously been developed by J. P. May (unpublished) and in [Bousfield 2].

For $E \in \mathbf{Ho}^s$, a map $f : A \rightarrow B \in \mathbf{Ho}^s$ is called an $[E,]_*$ -equivalence if $f_* : [E, A]_* \approx [E, B]_*$, and a spectrum $C \in \mathbf{Ho}^s$ is called $[E,]_*$ -colocal if $g_* : [C, X]_* \approx [C, Y]_*$ whenever $g : X \rightarrow Y$ is an $[E,]_*$ -equivalence. It is easy to check:

- (1.1) E is $[E,]_*$ -colocal.
- (1.2) If $\{X_\alpha\}$ is a set of $[E,]_*$ -colocal spectra, then $\vee_\alpha X_\alpha$ is $[E,]_*$ -colocal.
- (1.3) If $W \rightarrow X \rightarrow Y$ is a cofibre sequence in \mathbf{Ho}^s and any two of W, X, Y are $[E,]_*$ -colocal, then so is the third.
- (1.4) If X is $[E,]_*$ -colocal, then so is $X \wedge Y$ for all $Y \in \mathbf{Ho}^s$.

A map $\varphi : X \rightarrow A \in \mathbf{Ho}^s$ is called an $[E,]_*$ -colocalization of A if X is $[E,]_*$ -colocal and φ is an $[E,]_*$ -equivalence. Note that the $[E,]_*$ -colocalizations of A are initial among the $[E,]_*$ -equivalences with target A , and are terminal among the maps from $[E,]_*$ -colocal spectra to A . $[E,]_*$ -colocalizations are clearly unique up to equivalence and

PROPOSITION 1.5. *Each spectrum $A \in \mathbf{Ho}^s$ has an $[E,]_*$ -colocalization.*

Proof. We inductively construct a transfinite sequence of inclusions of CW-spectra

$$A = B_0 \subset B_1 \subset \cdots \subset B_s \subset B_{s+1} \subset \cdots$$

where $B_\lambda = \bigcup_{s < \lambda} B_s$ for each limit ordinal λ and where $B_s \subset B_{s+1}$ is given by the push-out square

$$\begin{array}{ccc} \bigvee_{\alpha \in I} & \bigvee_{\alpha \in I} M_\alpha & \longrightarrow B_s \\ \downarrow & f : M_\alpha \rightarrow B_s & \downarrow \\ \bigvee_{\alpha \in I} & \bigvee_{\alpha \in I} \text{Cone}(M_\alpha) & \longrightarrow B_{s+1} \end{array}$$

in which $\{M_\alpha\}_{\alpha \in I}$ consists of all cofinal subspectra of the spectra $\sum^n E$ for $n \in \mathbb{Z}$, and f ranges over all cellular functions $M_\alpha \rightarrow B_s$ of degree 0, cf. [Adams, p. 140, 154]. Now let σ be the number of stable cells in E and let γ be the first infinite ordinal of cardinality greater than σ . Then for each $\alpha \in I$, each cellular function $M_\alpha \rightarrow B_\gamma$ of degree 0 extends over $\text{Cone}(M_\alpha)$ because the image of M_α is contained in B_s for some $s < \gamma$. Thus $[E, B_\gamma]_* = 0$. Since A is a closed subspectrum of B_γ (cf. [Adams 2, p. 154]), there is an associated cofibre sequence

$$\Sigma^{-1}(B_\gamma/A) \rightarrow A \rightarrow B_\gamma$$

in \mathbf{Ho}^s . The morphism $\Sigma^{-1}(B_\gamma/A) \rightarrow A$ is clearly an $[E,]_*$ -equivalence, so it suffices to show $\Sigma^{-1}(B_\gamma/A)$ is $[E,]_*$ -colocal. For this it suffices to show inductively that B_s/A is $[E,]_*$ -colocal for all s . If B_s/A is $[E,]_*$ -colocal, then so is B_{s+1}/A because there is a cofibre sequence

$$B_s/A \rightarrow B_{s+1}/A \rightarrow B_{s+1}/B_s \in \mathbf{Ho}^s$$

where B_{s+1}/B_s is equivalent to a wedge of iterated (de)-suspensions of E . If B_s/A

is $[E,]_*$ -colocal for all $s < \lambda$ where λ is a limit ordinal, then B_λ/A is $[E,]_*$ -colocal because there is a cofibre sequence

$$\bigvee_{s < \lambda} B_s/A \xrightarrow{1-g} \bigvee_{s < \lambda} B_s/A \rightarrow B_\lambda/A \in \mathbf{Ho}^s$$

where g is induced by the maps $B_s/A \rightarrow B_{s+1}/A$. This completes the induction and the proof 1.5.

For each $A \in \mathbf{Ho}^s$ let $\varphi: {}^E A \rightarrow A \in \mathbf{Ho}^s$ denote the $[E,]_*$ -colocalization given by $\sum^{-1}(B_\gamma/A) \rightarrow A$ above, and note that it is functorial and idempotent in the obvious sense. To clarify the nature of $[E,]_*$ -colocal spectra, we let *Class- E* denote the smallest class of spectra in \mathbf{Ho}^s such that: (i) $E \in \text{Class-}E$; (ii) if $\{X_\alpha\}$ is a set of spectra in *Class- E* , then $\vee_\alpha X_\alpha \in \text{Class-}E$; and (iii) if $W \rightarrow X \rightarrow Y$ is a cofibre sequence in \mathbf{Ho}^s and any two of W, X, Y are in *Class- E* , then so is the third.

PROPOSITION 1.6. *Class- E equals the class of $[E,]_*$ -colocal spectra in \mathbf{Ho}^s .*

Proof. *Class- E* is contained in the class of $[E,]_*$ -colocal spectra by (1.1)–(1.3). Conversely, if X is $[E,]_*$ -colocal, then $X \in \text{Class-}E$ because ${}^E X \simeq X$ and ${}^E X \in \text{Class-}E$ by the proof of 1.5.

We call a spectrum $W \in \mathbf{Ho}^s$ $[E,]_*$ -trivial if $[E, W]_* = 0$, and we note that $[V, W]_* = 0$ whenever V is $[E,]_*$ -colocal and W is $[E,]_*$ -trivial. Each spectrum A can be canonically built from $[E,]_*$ -colocal and $[E,]_*$ -trivial spectra as follows. Extend $\varphi: {}^E A \rightarrow A$ to the cofibre sequence

$$(1.7) \quad {}^E A \xrightarrow{\varphi} A \xrightarrow{\nu} A^E \in \mathbf{Ho}^s$$

given by $\sum^{-1}(B_\gamma/A) \rightarrow A \rightarrow B_\gamma$ above, and observe that A^E is $[E,]_*$ -trivial. Indeed, ν is clearly the $[E,]_*$ -trivialization of A , i.e. ν is the initial example of a map from A to an $[E,]_*$ -trivial spectrum. It is useful to observe:

(1.8) If $V \rightarrow X \rightarrow W$ is a cofibre sequence in \mathbf{Ho}^s with V $[E,]_*$ -colocal and with W $[E,]_*$ -trivial, then $V \rightarrow X \rightarrow W$ is equivalent to the cofibre sequence

$${}^E X \xrightarrow{\varphi} X \xrightarrow{\nu} X^E$$

It is straightforward to check that the $[E,]_*$ -colocalization and $[E,]_*$ -trivialization functors on \mathbf{Ho}^s commute with suspension and preserve cofibre sequences. In [Bousfield 3] we will show that for each $E \in \mathbf{Ho}^s$ there exists a spectrum $aE \in \mathbf{Ho}^s$ such that the E_* -localization and E_* -acyclization functors are respectively equivalent to the $[aE,]_*$ -trivialization and $[aE,]_*$ -colocalization functors on \mathbf{Ho}^s . Thus, many examples of trivialization and colocalization functors will be implicitly studied in [Bousfield 3].

§2. On the structure of $\mathbf{A}(\mathbf{Ho}^s)$

We now examine the structure of the class $\mathbf{A}(\mathbf{Ho}^s)$ of “acyclicity types” of spectra, and we establish the results mentioned in the introduction concerning the distributive lattice $\mathbf{DL}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$ and the Boolean algebra $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{DL}(\mathbf{Ho}^s)$.

$\mathbf{A}(\mathbf{Ho}^s)$ has the following relations and operations:

(2.1) For $\langle X \rangle, \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$, define $\langle X \rangle \leq \langle Y \rangle$ if each Y_* -acyclic spectrum is X_* -acyclic. This is a partial order relation. Clearly $\langle 0 \rangle$ is the smallest element of $\mathbf{A}(\mathbf{Ho}^s)$ and $\langle S \rangle$ is the largest. Note that if X is $[Y,]_*$ -colocal (or equivalently, if $X \in \text{Class-}Y$), then $\langle X \rangle \leq \langle Y \rangle$.

(2.2) For a set $\{\langle X_\alpha \rangle\}$ of elements in $\mathbf{A}(\mathbf{Ho}^s)$, define $\vee_\alpha \langle X_\alpha \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ by $\vee_\alpha \langle X_\alpha \rangle = \langle \vee_\alpha X_\alpha \rangle$. Note that $\vee_\alpha \langle X_\alpha \rangle$ is the least upper bound of $\{\langle X_\alpha \rangle\}$ in $\mathbf{A}(\mathbf{Ho}^s)$, and \vee is associative, commutative, and idempotent. Of course, $\langle 0 \rangle \vee \langle X \rangle$ and $\langle S \rangle \vee \langle X \rangle = \langle S \rangle$.

(2.3) For $\langle X \rangle, \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ define $\langle X \rangle \wedge \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ by $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. This is well-defined: if $\langle X \rangle = \langle X_1 \rangle$ and $\langle Y \rangle = \langle Y_1 \rangle$, then clearly $\langle X \wedge Y \rangle = \langle X_1 \wedge Y \rangle = \langle X_1 \wedge Y_1 \rangle$. Note that $\langle X \rangle \wedge \langle Y \rangle$ is a lower bound of $\langle X \rangle, \langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$, and that if $\langle X \rangle \leq \langle X_1 \rangle$ and $\langle Y \rangle \leq \langle Y_1 \rangle$ then $\langle X \rangle \wedge \langle Y \rangle \leq \langle X_1 \rangle \wedge \langle Y_1 \rangle$. Clearly \wedge is associative and commutative, with $\langle S \rangle \wedge \langle X \rangle = \langle X \rangle$ and $\langle 0 \rangle \wedge \langle X \rangle = \langle 0 \rangle$. Also the distributive law $\langle X \rangle \wedge (\vee_\alpha \langle Y_\alpha \rangle) = \vee_\alpha (\langle X \rangle \wedge \langle Y_\alpha \rangle)$ and absorption law $\langle X \rangle \vee (\langle X \rangle \wedge \langle Y \rangle) = \langle X \rangle$ hold.

(2.4) For each $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ there is an element $a\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ such that $a\langle X \rangle$ is the greatest member of $\mathbf{A}(\mathbf{Ho}^s)$ with $\langle X \rangle \wedge a\langle X \rangle = \langle 0 \rangle$. Moreover, $aa\langle X \rangle = \langle X \rangle$ for each $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$, and $\langle X \rangle \leq \langle Y \rangle$ if and only if $a\langle Y \rangle \leq a\langle X \rangle$. This will be shown in [Bousfield 3], and we remark that $a\langle X \rangle = \langle aX \rangle$ where aX is the spectrum mentioned at the end of §1. It turns out that $\mathbf{DL}(\mathbf{Ho}^s)$ is not closed under $a()$, although $a()$ gives the complement in $\mathbf{BA}(\mathbf{Ho}^s)$. We won't use $a()$ in this paper.

So far, $\mathbf{A}(\mathbf{Ho}^s)$ resembles a Boolean algebra with complement $a()$, but the following lemma shows that \wedge is not idempotent in $\mathbf{A}(\mathbf{Ho}^s)$.

LEMMA 2.5. *Let $X \in \mathbf{Ho}^s$ be a finite CW-spectrum with H_*X finite, and let $cX \in \mathbf{Ho}^s$ be the Brown-Comenetz dual of X . If $X \neq 0$, then $\langle cX \rangle \wedge \langle cX \rangle = \langle 0 \rangle \neq \langle cX \rangle$.*

Proof. Using [Brown-Comenetz, 1.14] it is easy to show $H_*(cX; \mathbb{Z}) = 0$, and thus $\langle H \rangle \wedge \langle cX \rangle = \langle 0 \rangle$ where H is the spectrum for integral homology. Since $\pi_i cX$ is the Pontrjagin dual of $\pi_{-i} X$, it vanishes for sufficiently large i . Hence

$(cX)(n, \infty) \in \text{Class-}H$ for each n where $(cX)(n, \infty)$ is the $(n-1)$ -connected section of cX . The cofibre sequence

$$\bigvee_{n \leq 0} (cX)(n, \infty) \rightarrow \bigvee_{n \leq 0} (cX)(n, \infty) \rightarrow cX \in \mathbf{Ho}^s$$

now shows $cX \in \text{Class-}H$, and thus $\langle cX \rangle \leq \langle H \rangle$. The lemma now follows since $\langle cX \rangle \wedge \langle cX \rangle \leq \langle H \rangle \wedge \langle cX \rangle = \langle 0 \rangle$ and since $(cX)_*(S) \neq 0 = 0_*(S)$.

To avoid the pathological spectra revealed by 2.5, we introduce

2.6 The distributive lattice of spectra $\mathbf{DL}(\mathbf{Ho}^s)$

Let $\mathbf{DL}(\mathbf{Ho}^s)$ consist of all $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ with $\langle X \rangle \wedge \langle X \rangle = \langle X \rangle$. For instance, if E is a ring spectrum, then $\langle E \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$ because E is a retract of $E \wedge E$ in \mathbf{Ho}^s . Also, if E a Moore spectrum or a finite CW spectrum, then $\langle E \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$ by 2.9 and 2.13 below. Many other examples can be derived from the preceding, since $\mathbf{DL}(\mathbf{Ho}^s)$ is closed under the operation \vee (with any number of summands) and under \wedge ; the proof for \vee uses the equality $\langle X \rangle \vee (\langle X \rangle \wedge \langle Y \rangle) = \langle X \rangle$. With the operations \vee and \wedge , $\mathbf{DL}(\mathbf{Ho}^s)$ is clearly a distributive lattice with $0, 1$ as defined in the next paragraph.

We refer the reader to [Dwinger] or [Grätzer] for an exposition of distributive lattice theory, but for convenience we recall that a class L with binary operations \vee, \wedge is a *distributive lattice* with $0, 1$ if:

- (i) $x \wedge x = x$ and $x \vee x = x$ for $x \in L$.
- (ii) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ for $x, y \in L$.
- (iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ for $x, y, z \in L$.
- (iv) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ for $x, y \in L$.
- (v) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for $x, y, z \in L$.

(vi) There exist elements $0, 1 \in L$ such that $0 \vee x = x$ and $1 \wedge x = x$ for all $x \in L$. (Clearly, 0 and 1 are unique.) Now let L be a distributive lattice with $0, 1$. For $x, y \in L$ one writes $x \leq y$ if the equivalent conditions $x \wedge y = x$ and $x \vee y = y$ are satisfied. Then \leq is a partial order relation on L , and $x \vee y$ (resp. $x \wedge y$) is the l.u.b. (resp. g.l.b.) of $x, y \in L$, cf. [Dwinger, p. 44] or [Grätzer, p. 6]. We also recall that L is called a *Boolean algebra* if for each $x \in L$ there exists $y \in L$ with $x \wedge y = 0$ and $x \vee y = 1$.

For $\langle X \rangle, \langle Y \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$ we conclude that $\langle X \rangle \wedge \langle Y \rangle$ is the g.l.b. of $\langle X \rangle$ and $\langle Y \rangle$, where $\mathbf{DL}(\mathbf{Ho}^s)$ has the partial ordering inherited from $\mathbf{A}(\mathbf{Ho}^s)$. Of course, we previously observed that $\langle X \rangle \vee \langle Y \rangle$ is the l.u.b. of $\langle X \rangle$ and $\langle Y \rangle$. Thus the algebraic structure of $\mathbf{DL}(\mathbf{Ho}^s)$ is contained in its partial ordering.

We call $\langle Y \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ the *complement* of $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ if $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$ and $\langle X \rangle \vee \langle Y \rangle = \langle S \rangle$. Note that if $\langle Y_1 \rangle$ is also the complement of $\langle X \rangle$, then $\langle Y \rangle = \langle Y_1 \rangle$

because

$$\langle Y \rangle = \langle Y \rangle \wedge (\langle X \rangle \vee \langle Y_1 \rangle) = \langle Y \rangle \wedge \langle Y_1 \rangle = (\langle X \rangle \vee \langle Y \rangle) \wedge \langle Y_1 \rangle = \langle Y_1 \rangle.$$

If $\langle X \rangle$ has a complement $\langle Y \rangle$, then $\langle X \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$ because $\langle X \rangle = \langle X \rangle \wedge (\langle X \rangle \vee \langle Y \rangle) = \langle X \rangle \wedge \langle X \rangle$, but the members of $\mathbf{DL}(\mathbf{Ho}^s)$ need not have complements.

LEMMA 2.7. $\langle H \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$, but $\langle H \rangle$ does not have a complement.

Proof. $\langle H \rangle \in \mathbf{DL}(\mathbf{Ho}^s)$ since H is a ring spectrum. Suppose $\langle H \rangle$ has a complement $\langle L \rangle$. Let $\langle cX \rangle$ be as in 2.5, and recall that $\langle cX \rangle \neq \langle 0 \rangle = \langle H \rangle \wedge \langle cX \rangle$ and $\langle cX \rangle \leq \langle H \rangle$. Thus

$$\langle cX \rangle = (\langle H \rangle \vee \langle L \rangle) \wedge \langle cX \rangle = \langle L \rangle \wedge \langle cX \rangle \leq \langle L \rangle \wedge \langle H \rangle = \langle 0 \rangle$$

and this contradicts $\langle cX \rangle \neq \langle 0 \rangle$. Therefore $\langle H \rangle$ cannot have a complement.

We now introduce

2.8 The Boolean algebra of spectra $\mathbf{BA}(\mathbf{Ho}^s)$

Let $\mathbf{BA}(\mathbf{Ho}^s)$ consist of all $\langle X \rangle \in \mathbf{A}(\mathbf{Ho}^s)$ such that $\langle X \rangle$ has a complement (written $\langle X \rangle^c$), and note that $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{DL}(\mathbf{Ho}^s)$. If E is a Moore spectrum or a (possibly infinite) wedge of finite CW spectra, then $\langle E \rangle \in \mathbf{BA}(\mathbf{Ho}^s)$ by 2.9 and 2.13 below. Many other members of $\mathbf{BA}(\mathbf{Ho}^s)$ can be derived from the preceding, since $\mathbf{BA}(\mathbf{Ho}^s)$ is clearly closed under $(\)^c$ and the binary operations \vee , \wedge ; indeed, for $\langle X \rangle, \langle Y \rangle \in \mathbf{BA}(\mathbf{Ho}^s)$

$$\begin{aligned} \langle X \rangle^{cc} &= \langle X \rangle \\ (\langle X \rangle \vee \langle Y \rangle)^c &= \langle X \rangle^c \wedge \langle Y \rangle^c \\ (\langle X \rangle \wedge \langle Y \rangle)^c &= \langle X \rangle^c \vee \langle Y \rangle^c. \end{aligned}$$

With these operations, $\mathbf{BA}(\mathbf{Ho}^s)$ is clearly a Boolean algebra.

As promised, we now prove

PROPOSITION 2.9. *If $E \in \mathbf{Ho}^s$ is a (possibly infinite) wedge of finite CW-spectra, then $\langle E \rangle \in \mathbf{BA}(\mathbf{Ho}^s)$. Moreover, $\langle E \rangle = \langle {}^E S \rangle$ and $\langle E \rangle^c = \langle S^E \rangle$.*

Proof. Assume $E = \vee_\alpha B_\alpha$ where each B_α is a finite CW spectrum. A spectrum $Y \in \mathbf{Ho}^s$ is $[E,]_*$ -trivial iff $(DB_\alpha) \wedge Y \simeq 0 \in \mathbf{Ho}^s$ for all α , where DB_α is the

Spanier-Whitehead dual of B_α . Thus if Y is $[E,]_*$ -trivial, then so is $X \wedge Y$ for all $X \in \mathbf{Ho}^s$. In particular, ${}^E S \wedge S^E$ is $[E,]_*$ -trivial as well as $[E,]_*$ -colocal (by 1.4), and thus ${}^E S \wedge S^E \simeq 0 \in \mathbf{Ho}^s$. Using the cofibering ${}^E S \rightarrow S \rightarrow S^E$ of (1.7), we conclude that $\langle {}^E S \rangle \vee \langle S^E \rangle = \langle S \rangle$, so $\langle {}^E S \rangle \in \mathbf{BA}(\mathbf{Ho}^s)$ with $\langle {}^E S \rangle^c = \langle S^E \rangle$. It remains to show $\langle E \rangle = \langle {}^E S \rangle$. Applying (1.8) to the cofibre sequence

$$X \wedge {}^E S \rightarrow X \wedge S \rightarrow X \wedge S^E,$$

we find that $X \wedge {}^E S \simeq {}^E X$ and $X \wedge S^E \simeq X^E$ for all $X \in \mathbf{Ho}^s$. Since E is $[E,]_*$ -colocal, this implies $E \wedge {}^E S \simeq E$, and thus $\langle E \rangle \leq \langle {}^E S \rangle$. Since ${}^E S$ is $[E,]_*$ -colocal, we know $\langle {}^E S \rangle \leq \langle E \rangle$, and therefore $\langle E \rangle = \langle {}^E S \rangle$.

We remark that the above spectra ${}^E S$ and S^E satisfy the strong idempotency conditions ${}^E S \wedge {}^E S \simeq {}^E S$ and $S^E \wedge S^E \simeq S^E$. Indeed S^E is a commutative ring spectrum whose multiplication map $S^E \wedge S^E \rightarrow S^E$ is an equivalence. We next observe

PROPOSITION 2.10. *If $E \in \mathbf{Ho}^s$ is a finite CW spectrum, then $\langle E \rangle = \langle DE \rangle$. Consequently, for any $G \in \mathbf{Ho}^s$, $G_*(E) = 0 \Leftrightarrow G^*(E) = 0$.*

Proof. Since $[E, S^E]_* = 0$, we have $(DE) \wedge S^E \simeq 0$ and thus $(DE) \wedge {}^E S \simeq DE$. Since $\langle {}^E S \rangle = \langle E \rangle$, this implies $\langle DE \rangle \wedge \langle E \rangle = \langle DE \rangle$. Dually one shows $\langle E \rangle \wedge \langle DE \rangle = \langle E \rangle$, and therefore $\langle DE \rangle = \langle E \rangle$. The last statement is deduced using $G^*(E) = G_*(DE)$.

We next prove “triangle (in)equalities” for cofibre sequences. Call a map $f: A \rightarrow X \in \mathbf{Ho}^s$ *smash nilpotent* if the m -fold smash product

$$f \wedge \cdots \wedge f: A \wedge \cdots \wedge A \rightarrow X \wedge \cdots \wedge X \in \mathbf{Ho}^s$$

is the 0 map for some $m \geq 1$. Note that the smash nilpotent maps form a subgroup of $[A, X]$, and a composite fg is smash nilpotent if either f or g is. Moreover, if $i \neq 0$ then each map $S^i \rightarrow S^0 \in \mathbf{Ho}^s$ is a smash nilpotent by [Nishida].

PROPOSITION 2.11. *If $A \xrightarrow{f} X \rightarrow B$ is a cofibre sequence in \mathbf{Ho}^s , then:*

- (i) $\langle A \rangle \leq \langle X \rangle \vee \langle B \rangle$, $\langle X \rangle \leq \langle A \rangle \vee \langle B \rangle$, and $\langle B \rangle \leq \langle A \rangle \vee \langle X \rangle$.
- (ii) If $A, X \in \mathbf{DL}(\mathbf{Ho}^s)$ and f is smash nilpotent, then $\langle B \rangle = \langle A \rangle \vee \langle X \rangle$.

Proof. Part (i) is obvious. For (ii) we assume $f \wedge \cdots \wedge f = 0$ in \mathbf{Ho}^s and form a cofibre sequence

$$A \wedge \cdots \wedge A \xrightarrow{f \wedge \cdots \wedge f} X \wedge \cdots \wedge X \rightarrow C \in \mathbf{Ho}^s$$

Then

$$\langle C \rangle = \langle A \wedge \cdots \wedge A \rangle \vee \langle X \wedge \cdots \wedge X \rangle = \langle A \rangle \vee \langle X \rangle$$

and also $\langle C \rangle \leq \langle B \rangle$ because C is $[B,]_*$ -colocal (i.e. $C \in \text{Class-}B$). Thus $\langle A \rangle \vee \langle X \rangle \leq \langle B \rangle$ and the opposite inequality is given by (i).

COROLLARY 2.12. *If $n \neq 1$ and $\alpha : S^{n-1} \rightarrow S^0$ in \mathbf{Ho}^s , then $\langle S^0 \cup_\alpha e^n \rangle = \langle S \rangle$.*

Because of Nishida's theorem, this follows from 2.11; or instead of using 2.11, we could have used the easy result that if $\sum^m A \xrightarrow{f} A \rightarrow B$ is a cofibre sequence in \mathbf{Ho}^s with f nilpotent in $[A, A]_*$, then $\langle B \rangle = \langle A \rangle$. By combining 2.12 with R. Wood's result $K \simeq (S^0 \cup_\eta e^2) \wedge KO$, we recover the result $\langle K \rangle = \langle KO \rangle$, cf. [Meier], [Ravenel].

Of course, 2.12 fails when $n = 1$, and we now consider $\langle SG \rangle$ where G is an abelian group and $SG \in \mathbf{Ho}$ is a Moore spectrum of type $(G, 0)$ (i.e. $\pi_i SG = 0$ for $i < 0$, $H_0 SG = G$, and $H_i SG = 0$ for $i > 0$). There is a short exact sequence

$$0 \rightarrow G \otimes \pi_n X \rightarrow \pi_n(SG \wedge X) \rightarrow \text{Tor}(G, \pi_{n-1} X) \rightarrow 0.$$

Thus

$$\langle S \rangle = \langle SQ \rangle \vee \bigvee_{p \text{ prime}} \langle SZ/p \rangle$$

$$\langle SQ \rangle \wedge \langle SZ/p \rangle = \langle 0 \rangle = \langle SZ/p \rangle \wedge \langle SZ/q \rangle \text{ for primes } p \neq q.$$

It follows that **BA(Ho^s)** has a sub-Boolean algebra **MBA(Ho^s)** whose members are the wedges of subsets of

$$I = \{\langle SQ \rangle, \langle SZ/2 \rangle, \langle SZ/3 \rangle, \langle SZ/5 \rangle, \dots\}.$$

Moreover, there is an obvious Boolean algebra isomorphism between **MBA(Ho^s)** and the power set $P(I)$. Note that for any set J of primes

$$\langle SZ_{(J)} \rangle = \langle SQ \rangle \vee \bigvee_{p \in J} \langle SZ/p \rangle$$

where $Z_{(J)}$ is the localization of Z at J . More generally,

PROPOSITION 2.13. *For each abelian group G , $\langle SG \rangle \in \mathbf{MBA}(\mathbf{Ho}^s)$.*

Proof. Let C be the class of all abelian groups A such that $G \otimes A = 0 = \text{Tor}(G, A)$, and note that $(SG)_* X = 0$ if $\pi_i X \in C$ for all i . The result now follows easily from [Bousfield 1,2,3] since C is a “special” class.

We conclude by noting that $\mathbf{BA}(\mathbf{Ho}^s)$ contains many elements outside $\mathbf{MBA}(\mathbf{Ho}^s)$. For p prime let

$$A(p): \Sigma^m SZ/p \rightarrow SZ/p \in \mathbf{Ho}^s$$

be the K_* -equivalence of [Adams 1, §12] where $m = 2p - 2$ for p odd and $m = 8$ for $p = 2$. It is easy to check that the cofibre of $A(p)$ represents an element of $\mathbf{BA}(\mathbf{Ho}^s)$ outside $\mathbf{MBA}(\mathbf{Ho}^s)$. In [Bousfield 3] we will show that $\langle K \rangle = \langle E \rangle^c$ where E is the wedge of the cofibres of all the $A(p)$.

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