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Classification of simple knots by Levine pairings

by SADAYOSHI KOJIMA

§1. Introduction

An n -knot will be a smooth oriented submanifold K of S^{n+2} , where K is homeomorphic to S^n . By the Alexander duality theorem, the complement $X = S^{n+2} - K$ is a homology circle. Abelianization defines an epimorphism $\varepsilon : \pi_1(X) \rightarrow \mathbf{Z}$, and the covering space $\tilde{X} \rightarrow X$ associated to $\text{Ker } \varepsilon \subset \pi_1(X)$ is the universal abelian covering of X . The group of covering translations of \tilde{X} is infinite cyclic, generated by t , and this defines a unique module structure on $H_q(\tilde{X}; \mathbf{Z})$ over the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$. We will use the notation $A_q = H_q(\tilde{X})$ and refer to it as the q -th Alexander module of K .

An n -knot K is *simple* if \tilde{X} is $[(n-1)/2]$ -connected. In this paper, we shall study a classification of some simple knots. If $n = 2q - 1$, there is a linking form $\langle, \rangle : A_q \times A_q \rightarrow Q(\Lambda)/\Lambda$ satisfying the $(-1)^{q+1}$ -Hermitian property, where $Q(\Lambda)$ is the quotient field of Λ . Moreover the isotopy class of the knot K is completely determined by (A_q, \langle, \rangle) when K is simple, [K1], [L2], [T]. In case $n = 2q$, J. Levine has defined a more obscure linking form $[,]$ on the \mathbf{Z} -torsion submodule T_q of A_q with values in \mathbf{Q}/\mathbf{Z} , satisfying the $(-1)^{q+1}$ -symmetric property. A simple $2q$ -knot K is *odd-finite* if $\pi_q(X)$ is 2-torsion free and finite. The main result is

THEOREM 1. *If K_0, K_1 are odd finite simple $2q$ -knots with isometric Levine pairings and $q \geq 4$, then K_0 is isotopic to K_1 .*

Remark 1. A consequence of [B&L] is that a simple $2q$ -knot is fibered if and only if $\pi_q(X)$ is finitely generated. In particular, any finite simple knot is fibered.

Remark 2. Some results are known about the classification of even-dimensional simple knots, [K2], [Ko]. They say that the isotopy class of a knot is in general not determined by the pairing structure on the Alexander module if $\pi_q(X)$ is infinite.

In §4, we shall state some geometrical properties of knots in case that $\pi_q(X)$ is finitely generated and 2-torsion free. Classification in Theorem 2 is established by summing up Theorem 1 and [Ko].

The author would like to express his hearty thanks to Professor J. Levine for introducing him to this problem with a letter. He also would like to thank Professor M. Kato and Professor C. Kearton for many useful comments.

§2. Levine pairings

We recall the definition of the Levine pairing, [L3], [L4]. The duality theorem of [M] implies the isomorphism of right Λ -modules:

$$\bar{H}_q(\tilde{X}) \approx H_e^{n+2-q}(\tilde{X}, \partial\tilde{X}) \tag{1}$$

where $\bar{H}_q(\tilde{X})$ denotes the right Λ -module defined from the original left Λ -structure by the usual means. $H_e^*(\tilde{X}, \partial\tilde{X})$ is the homology of the cochain complex $\text{Hom}_\Lambda(C_*(\tilde{X}, \partial\tilde{X}), \Lambda)$. Then there is an exact sequence for $0 \leq q \leq n$:

$$0 \rightarrow \text{Ext}_\Lambda^2(A_{n-q}, \Lambda) \rightarrow \bar{A}_q \rightarrow \text{Ext}_\Lambda^1(A_{n+1-q}, \Lambda) \rightarrow 0. \tag{2}$$

This follows from the isomorphism (1), the trivial nature of $\partial\tilde{X}$, the universal coefficient spectral sequence, the homological dimension of Λ , and the fact that A_q is a Λ -torsion module. For an Alexander module A_i , $\text{Ext}_\Lambda^2(A_i, \Lambda)$ is a \mathbf{Z} -torsion module and depends only on T_i , while $\text{Ext}_\Lambda^1(A_i, \Lambda)$ is \mathbf{Z} -torsion free and depends only on A_i/T_i ; therefore

$$\bar{T}_q \approx \text{Ext}_\Lambda^2(T_{n-q}, \Lambda) \quad \text{for } 0 \leq q \leq n. \tag{3}$$

Now for any finite Λ -module T , there is a canonical isomorphism of Λ -modules:

$$\text{Ext}_\Lambda^2(T, \Lambda) \approx \text{Hom}_{\mathbf{Z}}(T, \mathbf{Q}/\mathbf{Z}). \tag{4}$$

In fact T_{n-q} is finite, so that one derives the Levine pairing by combining the isomorphism (3), (4):

$$[,]: T_q \times T_{n-q} \rightarrow \mathbf{Q}/\mathbf{Z}.$$

In case $n = 2q$, it satisfies the following four properties:

- a) \mathbf{Z} -linear: $[m\alpha, \beta] = [\alpha, m\beta] = m[\alpha, \beta]$ for $m \in \mathbf{Z}$ $\alpha, \beta \in T_q$
- b) conjugate self-adjoint: $[\lambda\alpha, \beta] = [\alpha, \bar{\lambda}\beta]$ for $\lambda \in \Lambda$

- c) non-singular: the adjoint to $[,]$ is bijective as a homomorphism: $T_q \rightarrow \text{Hom}_{\mathbf{Z}}(T_q, \mathbf{Q}/\mathbf{Z})$
- d) $(-1)^{q+1}$ -symmetric: $[\alpha, \beta] = (-1)^{q+1}[\beta, \alpha]$.

Remark 3. This is an almost complete algebraic characterization of the Levine pairing. In fact, for any pairing $[,]$ on T_q satisfying a), b), c) and d), there exists a $2q$ -knot with the Levine pairing $[,]$ provided $q \geq 2$, [L3], [L4].

We shall now give an alternative description of Levine pairings by making use of the τ -Seifert form defined by M. A. Gutiérrez.

A Seifert manifold V of a $2q$ -knot K is a smooth oriented submanifold of S^{2q+2} which is bounded by K . Writing τ for the torsion subgroup, we define the τ -intersection pairing (classically called the linking number) $I: \tau H_q(V) \otimes \tau H_q(V) \rightarrow \mathbf{Q}/\mathbf{Z}$ and the τ -linking pairing $L: \tau H_q(V) \otimes \tau H_q(S^{2q+2} - V) \rightarrow \mathbf{Q}/\mathbf{Z}$ as follows. Let $\alpha \in \tau H_q(V)$ have order d . Represent α by a q -chain ξ , and let ζ be a $(q+1)$ -chain such that $\partial\zeta = d \cdot \xi$. Then if $\beta \in \tau H_q(V)$ is represented by a q -chain η , we have $I(\alpha, \beta) = \text{Int}(\zeta, \eta)/d \pmod{1}$, where $\text{Int}(\cdot, \cdot)$ is the usual intersection number in V . And if $\gamma \in \tau H_q(S^{2q+2} - V)$ is represented by a q -chain μ with $\partial\rho = \mu$ for some $(q+1)$ -chain ρ in S^{2q+2} , we have $L(\alpha, \gamma) = \text{Int}(\zeta, \rho)/d \pmod{1}$, where $\text{Int}(\cdot, \cdot)$ is now the intersection number in S^{2q+2} . Then the τ -Seifert form of V

$$\theta: \tau H_q(V) \otimes \tau H_q(V) \rightarrow \mathbf{Q}/\mathbf{Z}$$

is defined by letting $\theta(\alpha \otimes \beta)$ be the τ -linking number $L(\alpha, \beta_+)$ where β_+ is the translate in the positive normal direction off V of the cycle β .

A finite abelian group G splits as the direct sum of its Sylow subgroups. Since these are clearly orthogonal with respect to any form on G , the whole problem splits also geometrically. This refers to arguments given later in the paper. Now, we assume that $\tau H_q(V)$ is a p -group throughout this paper. Let $\{\alpha_i\}_{i=1}^k$ be a basis of $\tau H_q(V)$ having order p_i with elementary divisor $p_1 | p_2 | \dots | p_k$ and $\{\beta_i\}_{i=1}^k \in \tau H_q(S^{2q+2} - V)$ be the Alexander dual basis of $\{\alpha_i\}$. The embeddings $\kappa, \iota: V \rightarrow S^{2q+2} - V$ are defined by $\pm\varepsilon$ -pushes. Then, for the maps $\kappa_*, \iota_*: \tau H_q(V) \rightarrow \tau H_q(S^{2q+2} - V)$ induced by κ, ι , we shall give the matrix presentations:

$$\begin{aligned} \kappa_*\alpha_j &= \sum_{i=1}^k m_{ij}^+ \beta_i & j = 1, 2, \dots, k \\ \iota_*\alpha_j &= \sum_{i=1}^k m_{ij}^- \beta_i & j = 1, 2, \dots, k. \end{aligned}$$

Indeed, the matrices $M_{\pm} = (m_{ij}^{\pm})$ are not uniquely determined as integral entries,

but are uniquely determined modulo Δ , where $\Delta = P \cdot J$,

$$P = \left(\begin{array}{cccc} p_1 & & & \\ & p_2 & \bigcirc & \\ & & \cdot & \\ \bigcirc & & & \cdot \\ & & & p_k \end{array} \right) \text{ and } J = \left(\begin{array}{cccc} 1, & 1, & \dots, & 1 \\ 1, & \cdot & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \\ 1, & & & 1 \end{array} \right)$$

And if $M_\theta = (p_i \cdot \theta(\alpha_i \otimes \alpha_j))$, then $M_+ = M_\theta \pmod{\Delta}$ and $M_- = M'_\theta \pmod{\Delta}$ where $M'_\theta = P \cdot {}^t M_\theta \cdot P^{-1}$. The result of [G] is

PROPOSITION 1 (M. A. Gutiérrez). *A presentation matrix of T_q is given by $(t \cdot M_\theta + (-1)^{q+1} M'_\theta, P)$. More precisely, there is an exact sequence*

$$\bigoplus^{2k} \Lambda \xrightarrow{d} \bigoplus^k \Lambda \longrightarrow T_q \longrightarrow 0$$

such that a matrix presentation of d with respect to suitable bases is given by the above.

In general, the τ -Seifert form of V may be singular. But in our case, if V is minimal, its Seifert form will be non-singular. Indeed, recall that a Seifert manifold is minimal if $\kappa_*, \iota_*: H_i(V) \rightarrow H_i(S^{2q+2} - V)$ are injective for all i . The following is then implicit in [L1];

PROPOSITION 2 (J. Levine). *Let K be a simple n -knot and $n \geq 4$. Then there exists a minimal Seifert manifold for K . Moreover, we can choose it as being simply connected.*

Let now K be a finite simple $2q$ -knot and let V be a minimal Seifert manifold for K . Then $H_q(V)$ is finite, otherwise $H_q(\tilde{V})$ would not be. As $H_q(S^{2q+2} - V)$ is abstractly isomorphic to $H_q(V)$ and as κ_* and ι_* are injective, they are necessarily isomorphisms. This implies immediately that the τ -Seifert form is non-singular.

Therefore there is an inverse isomorphism $\kappa_*^{-1}: H_q(S^{2q+2} - V) \rightarrow H_q(V)$. Let M_θ^{-1} be the representation matrix of κ_*^{-1} with respect to bases $\{\beta_i\}$ and $\{\alpha_i\}$, then $M_\theta \cdot M_\theta^{-1} = M_\theta^{-1} \cdot M_\theta = E \pmod{\Delta}$ where E is the identity matrix. The square matrix $I_V = (I(\alpha_i, \alpha_j))$ with entries in \mathbf{Q}/\mathbf{Z} is the τ -intersection matrix. By Proposition 1, an element of T_q can be represented by a column vector of $\bigoplus^k \Lambda$. The quotient form $[\cdot, \cdot]_\theta$ on T_q determined by θ with values in \mathbf{Q}/\mathbf{Z} is defined by letting

$$[\alpha, \beta]_\theta = \text{the constant term of } {}^t \tilde{u} \cdot \tilde{U}_\theta \cdot v \pmod{1}$$

where u and v are representatives of α and β . Here \tilde{U}_θ is the square matrix with entries in $\mathbf{Q}/\mathbf{Z} [[t, t^{-1}]]$ defined as follows: let U_θ be the \mathbf{Q}/\mathbf{Z} -matrix $U_\theta = {}^t M_\theta^* \cdot I_V \cdot M_\theta^*$, where $M_\theta^* = (M'_\theta)^{-1}$ and A_θ be the \mathbf{Z} -matrix $A_\theta = (-1)^q M'_\theta \cdot M_\theta^{-1} \pmod{\Delta}$. Then $\tilde{U}_\theta = \sum_{n=-\infty}^{\infty} t^n \cdot ({}^t A_\theta)^n \cdot U_\theta$.

We recall the formula [G]:

$$\theta(\alpha \otimes \beta) + (-1)^{q+1} \theta(\beta \otimes \alpha) = -I(\alpha, \beta) \pmod{1}.$$

So that $M_\theta + (-1)^{q+1} M'_\theta = -P \cdot I_V \pmod{\Delta}$. This implies that the quotient form $[\cdot, \cdot]_\theta$ is well defined and satisfies the four properties a), b), c) and d). And because of Levine's observation, it coincides with the Levine pairing $[\cdot, \cdot]$ of K (see[L3], Proposition 7.1).

Remark 4. The above computation method of the Levine pairing is essentially due to the minimality of a Seifert manifold. And the simplicity of the knot, the existence of a fibration and the finiteness of $\pi_q(X)$ are inessential.

§3. Proof of Theorem 1

From now on, we shall prove Theorem 1 in three steps. The first is devoted to show that an isometry of Levine pairings induces an isometry of τ -Seifert forms. And in the next step, we construct an isotopy between Seifert manifolds when q is odd. The case q even is studied in the last.

3.1. Now, let K_0 and K_1 be odd finite simple $2q$ -knots with isometric Levine pairings. V and W are minimal, simply connected Seifert manifolds of K_0 and K_1 , and θ, η are their τ -Seifert forms.

LEMMA 1. *If the quotient form $[\cdot, \cdot]_\theta$ is isometric to $[\cdot, \cdot]_\eta$, then θ is isometric to η .*

Proof. Let T_q and T'_q be q -th Alexander modules of K_0 and K_1 . Then by assumption, there is a Λ -isomorphism $f: T_q \rightarrow T'_q$ such that $[\alpha, \beta]_\theta = [f(\alpha), f(\beta)]_\eta$ for each $\alpha, \beta \in T_q$. For some bases $\{\alpha_i\}, \{\alpha'_i\}$ of T_q, T'_q , f is of the form $f(\alpha_i) = \sum a_{ij} \alpha'_j$. Since it can be regarded as a group isomorphism: $H_q(V) \rightarrow H_q(W)$, $\{f(\alpha_i)\}$ is a basis of $H_q(W)$. If M_η is a representative matrix of η with respect to the basis $\{f(\alpha_i)\}$, then $U_\theta = U_\eta$ and $A_\theta = A_\eta$ because f is a Λ -isomorphism and an isometry. We recall that $M_\theta + (-1)^{q+1} M'_\theta = -P \cdot I_V \pmod{\Delta}$, so that

$$\begin{aligned} A_\theta &= E + P \cdot I_V \cdot M_\theta^{-1} \pmod{\Delta} \\ &= E + (-1)^q M_\theta \cdot P \cdot U_\theta \cdot A_\theta \pmod{\Delta}. \end{aligned}$$

Similarly $A_\eta = E + (-1)^q M_\eta \cdot P \cdot U_\eta \cdot A_\eta \pmod{\Delta}$, thus we obtain the identity $M_\theta \cdot P \cdot U_\theta \cdot A_\theta = M_\eta \cdot P \cdot U_\eta \cdot A_\eta \pmod{\Delta}$. Now, A_θ is invertible in the sense of modulo Δ . And so is $P \cdot U_\theta$ because the τ -intersection form of V is non-singular. Therefore we have that $M_\theta = M_\eta$. This implies that $f: H_q(V) \rightarrow H_q(W)$ is an isometry, completing the proof.

Remark 5. In the argument above, simplicity, oddness and finiteness of knots are not essential, but minimality of V is.

In the following, we shall construct the isotopy between Seifert manifolds V and W , by virtue of the dimensional condition $q \geq 4$. Indeed, if $q \geq 4$ almost all our arguments are in the metastable range and in that range homotopy implies isotopy by Haefliger's theorem [H].

3.2. Now we assume that q is odd. Then the τ -intersection form of V is symmetric and hence diagonal because $H_q(V)$ has no 2-torsion. Namely there is a basis $\{\alpha_i\}$ of $H_q(V)$ such that $I(\alpha_i, \alpha_j) = \delta_{ij} c_i / p_i$, where c_i is an integer co-prime to p_i , and δ_{ij} is Kronecker's delta. A decomposition of V can be obtained as follows. Let $A_j = S^q \times D^{q+1} \cup_{f_j} h$ where h is a $(q+1)$ -handle and f_j is the attaching map of h , i.e., $f_j: \partial D^{q+1} \times D^q \rightarrow \partial(S^q \times D^{q+1})$. The homotopy group $\pi_q(S^q \times \partial D^{q+1})$ is a free abelian group generated by two elements $x = S^q \times \{pt\}$ and $y = \{pt\} \times \partial D^{q+1}$. And the homotopy class of $f_j | \partial D^{q+1} \times \{0\}$ is $p_j x + q_j y$, where $q_j = c_j \pmod{p_j}$. The manifold A_j is fundamental for V . In fact, it follows from the high connectivity and the parallelizability of V that it is diffeomorphic to the boundary connected sum $\natural_{j=1}^k A_j$ (see [W]). The Seifert manifold W also admits a similar decomposition. More precisely, there is a basis $\{\alpha'_i\}$ of $H_q(W)$ such that $\theta(\alpha_i \otimes \alpha_j) = \eta(\alpha'_i \otimes \alpha'_j)$ because θ is isometric to η . Then W has a decomposition $\natural_{j=1}^k A'_j$ associated to the basis $\{\alpha'_i\}$.

We next show how to isotopically deform A_j onto A'_j in S^{2q+2} . For this reason, a number of preliminaries are necessary.

A q -annulus will be a smooth oriented submanifold R of S^{2q+2} , where R is diffeomorphic to $S^q \times D^{q+1}$. A q -annulus has a trivial disc bundle structure on $S^q = S^q \times \{0\}$. Let ν be the positive unit normal field to R on S^q . Then by the tubular neighborhood theorem, R can be considered as the orthogonal complement of ν in a normal disc bundle neighborhood of S^q in S^{2q+2} . Let $R_i = S^q \times D_i^{q+1}$, $i=0, 1$, be q -annuli. R_0 and R_1 are isotopic if there is an ambient isotopy χ_t of R_0 to R_1 such that $\chi_0 = \text{id}$, $\chi_1(S^q_0) = S^q_1$ and χ_1 is a bundle map. However, by the unknotting theorem, the tubular neighborhood theorem and the fact $\pi_q(S^{q+1}) = 0$, all the q -annuli are isotopic if $q \geq 1$. The condition q odd is not necessary for this. Next, we consider a q -annulus with a non-zero section (R, s) , where s is a section of the associated sphere bundle ∂R . We shall say that (R_0, s_0)

is isotopic to (R_1, s_1) if there is an isotopy χ_t of R_0 to R_1 such that $\chi_1 \circ s_0 = s_1 \circ \chi_1|_{S^q}$. Isotopy \sim is an equivalence relation on Ω , the set of q -annuli with a non-zero section. Clearly it is a necessary condition for (R_i, s_i) , $i = 0, 1$, to be isotopic that the characteristic class of the normal bundle of s_0 in ∂R_0 coincides with that of s_1 . If q is odd, $\neq 1, 3, 7$, the image of $\partial: \pi_q(S^q) \rightarrow \pi_{q-1}(SO_q)$ is \mathbf{Z}_2 . So that Ω/\sim has at least two classes in this case. In fact:

LEMMA 2. *There is a bijective map $\phi: \Omega/\sim \rightarrow \{0, 1\}$ if q is odd and $q \geq 1$.*

Proof. First of all, ϕ will be defined as follows. We set $S^{2q+2} = \{(x_0, x_1, \dots, x_{2q+2}) \in R^{2q+3} \mid x_0^2 + x_1^2 + \dots + x_{2q+2}^2 = 1\}$, the equator $S^{2q+1} = \{x \in S^{2q+2} \mid x_{2q+2} = 0\}$ and $S^q = \{x \in S^{2q+2} \mid x_{q+1} = x_{q+2} = \dots = x_{2q+2} = 0\}$. R is the normal unit disc bundle of S^q in S^{2q+1} . For any q -annulus R' , there is an isotopy χ_t of R' to R . Then $\phi((R', s))$ is the modulo two reduction of the linking number between S^q and $\chi_1 \circ s \circ \chi_1^{-1}(S^q)$ in S^{2q+1} .

We must prove two points: that ϕ is well-defined and that it is bijective. Suppose that there is an isotopy f_t of (R_0, s_0) to (R_1, s_1) . Without loss of generality, we may assume that $(R_0, s_0) = (R, s)$ where s is a suitable section of R . Let g_t be the isotopy of R_1 to R with which we define the number $\phi((R_1, s_1))$. Combining these,

$$\chi_t = \begin{cases} f_{2t} & \text{for } 0 \leq t \leq 1/2 \\ g_{2t-1} & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

is an isotopy of the q -annulus R to itself. Now, the following diagram will be studied.

$$\begin{array}{ccc} \pi_q(SO_{q+1}) & \xrightarrow{i_*} & \pi_q(SO_{q+2}) \longrightarrow 0 \\ & \searrow p_* & \\ & & \pi_q(S^q) \approx \mathbf{Z} \end{array}$$

The framing of R differs from that of $\chi_1(R)$ by an element of $\pi_q(SO_{q+1})$. But it is null homotopic in SO_{q+2} . In other words, it is an element of $\text{Ker } i_*$. If q is odd, a generator of $\text{Ker } i_*$ is mapped to 2 by p_* , so that the difference between the linking number of $s(S^q)$ and $g_1 \circ s_1 \circ g_1^{-1}(S^q)$ with S^q in S^{2q+1} is even. Thus ϕ is well-defined.

Conversely, we shall suppose that $\phi((R_0, s_0)) = \phi((R_1, s_1))$. Then by using a generator of $\text{Ker } i_*$, the isotopy between them can be easily constructed. Therefore ϕ is bijective. This completes the proof.

Now, we shall define two isotopies which play a key role to deform a $(q + 1)$ -handle h to h' . Let a q -annulus R be as in Lemma 2 and N be the normal disc bundle of S^q in S^{2q+2} . Then we take a trivialization $T: N \rightarrow S^q \times D^{q+2}$ such that $p \circ T(R) = D^{q+1}$ where p denotes the projection onto the second factor. Here $D^{q+2} = \{(y_1, y_2, \dots, y_{q+2}) \in \mathbb{R}^{q+2} \mid y_1^2 + y_2^2 + \dots + y_{q+2}^2 \leq 1\}$ and $D^{q+1} = \{y \in D^{q+2} \mid y_{q+2} = 0\}$. As was seen before, a q -annulus R is the orthogonal complement of a positive unit normal field ν to R on S^q in N . When we regard ν as a map of S^q to ∂D^{q+2} by composing T and p , it is a constant map to the base point $* = (0, 0, \dots, 0, 1)$ of ∂D^{q+2} . A homotopical deformation of ν as a section induces an ambient isotopy of R . Let $F_i: S^q \times I \rightarrow \partial D^{q+2}$, $i = 0, 1$, be homotopical deformations of ν which satisfy two properties.

- (1) $F_i(x, 0) = F_i(x, 1) = F_i(b, t) = *$ for each $x \in S^q$, $t \in [0, 1]$ and $i = 0, 1$, where b is the base point of S^q .

This property implies that F_i can be considered as a map of S^{q+1} to ∂D^{q+2} .

- (2) F_0 is a degree one map of S^{q+1} . $\text{Im } F_1$ is included in $S^q = \{y \in D^{q+2} \mid y_1 = 0\}$ and F_1 represents the non-trivial element of $\pi_{q+1}(S^q) \approx \mathbb{Z}_2$, ($q \geq 3$).

Let J_i be the isotopy of R induced by F_i . In case q is odd, J_0 is just the one which is induced by a generator of $\text{Ker } i_*$.

Coming back to our previous program, we shall isotopically deform A_1 onto A'_1 , provided $q \geq 5$. Using q -annuli, $A_1 = R_1 \cup_{f_1} h_1$ and $A'_1 = R'_1 \cup_{f'_1} h'_1$. There always exists an isotopy from R_1 to R'_1 . Then the attaching sphere of h_1 represents a homotopy class $p_1x + q_1y \in \pi_q(\partial R_1)$. By hypothesis, the attaching sphere of h'_1 represents $p'_1x + q'_1y$, where $p_1 = p'_1$ and $q_1 \equiv q'_1 \pmod{p_1}$. Since the normal bundle of $f_1(\partial D^{q+1} \times \{0\})$ is trivial, $\partial(\text{Int}(p_1x, q_1y)) = 0$. Here, the intersection number is regarded as an element of $\pi_q(S^q)$ and ∂ is the boundary map: $\pi_q(S^q) \rightarrow \pi_{q-1}(SO_q)$. If q is odd and $\neq 7$, then the image of ∂ is \mathbb{Z}_2 , so that q_1 is even because p_1 is odd. Therefore $q_1 \equiv q'_1 \pmod{2p_1}$. If $q = 7$, this is true by a general argument: namely Lemma 2 shows that the modulo two reduction of q_1 is the primary obstruction to thickening the core of h_1 in S^{2q+2} . Now, $q_1 \equiv q'_1 \pmod{2p_1}$ implies that there is an integer n such that $q_1 + 2np_1 = q'_1$. By Lemma 2, there is an isotopy $(J_0)^n$ from (R_1, x) to $(R_1, x + 2ny)$. Thus it changes the homotopy class of $f_1(\partial D^{q+1} \times \{0\})$ to $p_1(x + 2ny) + q_1y = p'_1x + q'_1y$. After all, we can isotopically deform R_1 onto R'_1 so that the attaching sphere of h_1 coincides with that of h'_1 .

Next, we set the manifold M as being S^{2q+2} -interior N . Then there is a homotopy exact sequence:

$$0 \rightarrow \pi_{q+1}(M) \rightarrow \pi_{q+1}(M, \tilde{S}^q) \rightarrow \pi_q(\tilde{S}^q) \rightarrow 0$$

where \tilde{S}^q is the attaching sphere of h_1 . M has the same homotopy type as S^{q+1} , so

that $\pi_{q+1}(M, \tilde{S}^q)$ is free abelian of rank two. And the core $C_1(C'_1)$ of $h_1(h'_1)$ represents an element $c(c')$ of $\pi_{q+1}(M, \tilde{S}^q)$. Rounding the edge, $C_1 \cup C'_1$ is considered as an immersed $(q+1)$ -sphere with some orientation in interior M . Then the self-intersection number of it is identified with $p_1 \cdot (c - c') \in \pi_{q+1}(M) \approx \mathbf{Z}$ as an integer. On the other hand, the normal bundle of $C_1 \cup C'_1$ is trivial because C_1 and C'_1 are the core of $(q+1)$ -handles. Therefore the self-intersection number of it must be zero. Thus c is in fact the same element as c' . This shows that C_1 is isotopically deformed onto C'_1 .

Let $\mu(\mu')$ be a positive unit normal field to $h_1(h'_1)$ on $C_1(C'_1)$. The $(q+1)$ -handle h_1 is the orthogonal complement of μ in a normal disc bundle of C_1 in S^{2q+2} . Since $\mu = \mu'$ along ∂C_1 , μ differs from μ' by an element of $\pi_{q+1}(S^q) \approx \mathbf{Z}_2$. If $\mu - \mu' = 0$, there is a homotopy of μ to μ' relative to ∂C_1 , so that h_1 is isotopically deformed to h'_1 . If not so, we deform R_1 with an isotopy J_1 . It does not change the homotopy class of ∂C_1 , but changes the homotopy class of μ into the other element if p_1 is odd. And then, for the new field μ , $\mu - \mu' = 0$. Thus we obtain the required isotopy from A_1 to A'_1 .

Remark 6. While constructing the above isotopy, we only used the information that $I_V(\alpha_1, \alpha_1) = I_W(\alpha'_1, \alpha'_1)$. But in fact, the identity $2 \cdot \theta(\alpha_1 \otimes \alpha_1) = I_V(\alpha_1, \alpha_1)$ follows from Gutiérrez's formula, and this implies that the diagonal entry of the τ -Seifert form is completely determined by its self τ -intersection number, if p_1 is odd.

Next we shall construct the deformation of A_2 onto A'_2 in $S^{2q+2} - A_1$. The space $S^{2q+2} - A_1$ has the same homotopy type as A_1 , so that $\pi_q(S^{2q+2} - A_1) \approx \mathbf{Z}_{p_1}$. The core of the q -annulus $R_2(R'_2)$ of $A_2(A'_2)$ represents an element $r(r')$ of $\pi_q(S^{2q+2} - A_1)$ and $r - r'$ can be identified with $\theta(\alpha_1 \otimes \alpha_2) - \eta(\alpha'_1 \otimes \alpha'_2) = 0$. Therefore there is an isotopy from R_2 to R'_2 by which the attaching sphere of h_2 coincides with that of h'_2 . Put $M = S^{2q+2}$ -interior $A_1 \vee N$, where N is the normal disc bundle of R_2 and \vee denotes the one point union. Then there is an exact sequence:

$$0 \rightarrow \pi_{q+1}(M) \rightarrow \pi_{q+1}(M, \tilde{S}^q) \rightarrow \pi_q(\tilde{S}^q) \rightarrow 0$$

where \tilde{S}^q is the attaching sphere of h_2 . I have $\pi_{q+1}(A_1) = 0$, because $A_1 \simeq S^q \cup_{p_1} D^{q+1}$ and p_1 is odd (the left distributivity law applies). So $\pi_{q+1}(M) \approx \mathbf{Z}$ and $\pi_{q+1}(M, \tilde{S}^q) \approx \mathbf{Z} \oplus \mathbf{Z}$. Using the same argument as before we can isotopically deform h_2 onto h'_2 and so that A_2 onto A'_2 in $S^{2q+2} - A_1$.

Proceeding inductively, we finally obtain the required isotopy of V to W . This completes the proof of Theorem 1 in case q is odd.

3.3 Here, we shall give a proof of Theorem 1 when q is even. If so, the

τ -intersection form of V is skew-symmetric and $H_q(V)$ has a basis $\{\alpha_i\}_{i=1}^{2k}$ such that the order $p_{2i-1} = p_{2i}$ and for $i < j$,

$$I(\alpha_i, \alpha_j) = \begin{cases} 1/p_i & \text{if } i = 2n - 1 \text{ and } j = 2n \\ 0 & \text{otherwise.} \end{cases}$$

A decomposition of V can be obtained as follows. Let a fundamental manifold B_j be $S^q \times D^{q+1} \natural S^q \times D^{q+1} \cup_{f_{2j-1}} h \cup_{f_{2j}} h$, where h is a $(q+1)$ -handle and f_{2j-1} (f_{2j}) is an attaching map. Now, $\pi_q(\partial(S^q \times D^{q+1} \natural S^q \times D^{q+1}))$ is generated by the natural four elements x_1, y_1, x_2 and y_2 of the first and the second factors. Choose the attaching sphere of h_{2j-1} (resp. h_{2j}) to represent the element $p_{2j-1}x_1 + y_2$ (resp. $p_{2j}x_2 - y_1$) of $\pi_q(\partial(S^q \times D^{q+1} \natural S^q \times D^{q+1}))$. Then V is diffeomorphic to the boundary connected sum $\natural_{j=1}^k B_j$ (see [W]).

To prove Theorem 1, we shall consider the fundamental manifold B_j from an alternative viewpoint. As a submanifold of S^{2q+2} , $S^q \times D^{q+1} \natural S^q \times D^{q+1}$ is just a q -annulus R_j with a q -handle H_j . Moreover, we may assume that the image of f_{2j-1} is on ∂R_j and the attaching sphere of H_j is on the complement $\partial R_j - \text{Im } f_{2j-1}$. After all, $B_j = R_j \cup H_j \cup_{f_{2j-1}} h_{2j-1} \cup_{f_{2j}} h_{2j}$ as a submanifold of S^{2q+2} . The manifold W also admits a similar decomposition $\natural_{j=1}^k B'_j$ associated to an isometric basis $\{\alpha'_i\}_{i=1}^{2k}$.

We next show how to deform B_1 onto B'_1 . As was seen in 3.2, there always exists an isotopy from R_1 to R'_1 . The attaching sphere of h_1 (resp. h'_1) represents a homotopy class $p_1x + q_1y$ (resp. $p'_1x + q'_1y$) of $\pi_q(\partial R_1)$, where $x = S^q \times \{pt\}$ and $y = \{pt\} \times \partial D^{q+1}$. If q is even, the boundary homomorphism $\partial: \pi_q(S^q) \rightarrow \pi_{q-1}(SO_q)$ is injective. Since the normal bundle of the attaching sphere of h_1 (h'_1) is trivial, $\partial(\text{Int}(p_1x, q_1y)) = 0$ (resp. $\partial(\text{Int}(p'_1x, q'_1y)) = 0$). This implies $q_1 = q'_1 = 0$. Therefore, under this isotopy the image of f_1 coincides with that of f'_1 .

Let N be a normal disc bundle of R_1 in S^{2q+2} , and put $M = S^{2q+2}$ -interior N . Then there is a split exact sequence:

$$0 \rightarrow \pi_{q+1}(M) \rightarrow \pi_{q+1}(M, \tilde{S}^q) \rightarrow \pi_q(\tilde{S}^q) \rightarrow 0$$

where \tilde{S}^q is the attaching sphere of h_1 (h'_1). Let u, v be a basis of $\pi_{q+1}(M, \tilde{S}^q)$, corresponding to the above splitting. The core C_1 (C'_1) of h_1 (h'_1) represents an element $u + q_1v$ ($u + q'_1v$) of $\pi_{q+1}(M, \tilde{S}^q)$, where $q_1 \equiv p_1 \cdot \theta(\alpha_1 \otimes \alpha_1) \pmod{p_1}$ ($q'_1 \equiv p'_1 \cdot \eta(\alpha'_1 \otimes \alpha'_1) \pmod{p'_1}$). This shows that there is an integer n such that $q'_1 = q_1 + np_1$. Then we deform R_1 by $(J_0)^n$, so that it changes the homotopy class of C_1 into $u + (q_1 + np_1)v = u + q'_1v$. Thus we obtain an isotopy of $R_1 \cup C_1$ to $R'_1 \cup C'_1$. The rest of the procedure to deform $R_1 \cup h_1$ onto $R'_1 \cup h'_1$ is the same as in 3.2.

The attaching sphere of H_1 is on $\partial R_1 - \text{Im } f_1$ and represents an element of

$\pi_{q-1}(\partial R_1 - \text{Im } f_1) \approx \mathbf{Z}_{p_1}$. It differs from that of H'_1 by $I_V(\alpha_1, \alpha_2) - I_W(\alpha'_1, \alpha'_2) = 0$. So that we may assume that these coincide. Using the previous decomposition of $B_1(B'_1)$, the core $C_1(C'_1)$ of $H_1(H'_1)$ is naturally extended to a q -sphere $S(S')$. Let $S_+(S'_+)$ be the translate in the positive normal direction off $V(W)$ of $S(S')$. It represents an element of $\pi_q(S^{2q+2} - R_1 \cup h_1) \approx \mathbf{Z}_{p_1}$, and the difference between those is identified with $\theta(\alpha_1 \otimes \alpha_2) - \eta(\alpha'_1 \otimes \alpha'_2) = 0$. Therefore S_+ is isotopic to S'_+ in $S^{2q+2} - R_1 \cup h_1$, and moreover C_1 is isotopic to C'_1 keeping ∂C_1 fixed. Afterward, using the same procedure as the one we have used in deforming A_2 onto A'_2 , we obtain an isotopy of B_1 to B'_1 .

The rest of the proof is done inductively as in the case q odd. Thus we finally obtain the required isotopy of V to W . This completes the proof of Theorem 1.

§4. Addendum

Here, we shall expose without proof some geometrical properties of some fibered knots. By [B & L], a necessary and sufficient condition for a simple $2q$ -knot K to be fibered is that $\pi_q(X)$ is finitely generated, where X is the complement of K . Throughout this section, a knot K is always fibered. Definitions can be found in our previous article [Ko]. Then our result is

THEOREM 2. *Let K be an odd simple fibered $2q$ -knot, and $q \geq 4$. Then the isotopy class of K is completely determined by the first, the second and the τ -Seifert forms.*

The crux of the proof is how to choose a basis of $H_q(V)$, where V is a minimal Seifert manifold of K . Namely, setting $\{\alpha_i\}$ as being a basis of $\tau H_q(V)$, we can choose a basis $\{\beta_j\}$ of the torsion free part of $H_q(V)$ such that they are null-homologous in the complement of the chains $\{\zeta_i\}$ in S^{2q+2} where $\partial \zeta_i = p_i \cdot \alpha_i$. The reason why we can do this is the non-singularity of the τ -Seifert form of V . Choosing a nice decomposition of V with respect to the basis $\{\alpha_i, \beta_j\}$ of $H_q(V)$, we shall use the same procedure as in the proof of Theorem 1 and [Ko]. This supplies the proof of Theorem 2. Now, the following corollaries are immediately obtainable from Theorem 2.

COROLLARY 1. *If a $2q$ -knot K is as above, then there is a unique splitting $K = K_F \# K_T$ such that $\pi_q(X_F)$ is torsion free and $\pi_q(X_T)$ is finite, where $\#$ denotes the knot connected sum and $X_F(X_T)$ is the complement of a simple knot $K_F(K_T)$.*

COROLLARY 2. *Let K be as above and $p, p' : X \rightarrow S^1$ be fibrations of it. Then there exists a diffeomorphism f of S^{2q+2} such that $p' = p \circ f|X$.*

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