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Concordance implies homotopy for classical links in M^3

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Introduction

In this paper I prove that concordance implies homotopy for classical links in any 3-manifold. The notion of concordance was first developed by Fox and Milnor in [2] for knots in $\mathbf{R}^3$, and later extended to links in $\mathbf{R}^3$ by Fox, in Problem 25 of [1]. Homotopy of links in a 3-manifold M^3 was defined and studied by Milnor in [5].

The proof is entirely geometric, and also quite simple. In fact, at this point I would direct the reader's attention to Figure 4, which indicates a homotopy from a particular ribbon link to the trivial link in $\mathbf{R}^3$. The reader might then be led to the proof that all ribbon links are null-homotopic (Lemma 2.3).

The result of this paper has also been obtained by Charles Giffen, independently, and by a different method.

1. Main definitions

All maps and spaces are in the P.L. category. Choose a closed 3-cell in the interior of every 3-manifold M^3, and denote its interior by $\mathbf{R}^3$; let $\mathbf{R}^2_{xy} \subset \mathbf{R}^3$ be the xy-plane in $\mathbf{R}^3$. Recall that a map $g: M \to N$ of manifolds is proper if $g(\partial M) \subseteq \partial N$ and $g(\operatorname{int} M) \subseteq \operatorname{int} N$.

Certain definitions, where indicated, will be taken from [6] (A. J. Tristram).

DEFINITION 1.1. An *oriented link of n-components* in a 3-manifold M^3 is a proper embedding $l: \bigcup_{i=1}^n S_i^1 \to M^3$ of a disjoint union of n-oriented 1-spheres in that 3-manifold. Let L_i denote the oriented image $l(S_i^1)$, and let $L = l(\bigcup_{i=1}^n S_i^1) = \bigcup_{i=1}^n L_i$.

DEFINITION 1.2. Two oriented links $l, l': \bigcup_{i=1}^n S_i^1 \to M^3$ in M^3 are *ambient isotopic*, if there is an isotopy $h_t: M^3 \to M^3$ such that $h_0 = id$ and $h_1 \circ l = l'$.

We will not distinguish between a link and its ambient isotopy class. Since oriented links with the same oriented images are ambient isotopic, let the image $L \subset M^3$ denote the link l, and let the expression $L \equiv L'$ signify that L and L' are ambient isotopic. The trivial link of n disjoint circles in the xy-plane $\mathbf{R}^2_{xy}$, will be denoted $C^n = \bigcup_{i=1}^n C_i^1$.

DEFINITION 1.3. Two oriented links $L, L' \subset M^3$ of n components are *homotopic* if there is a homotopy $h_t : \bigcup_{i=1}^n S_i^1 \to M^3$ from $h_0 = l$ to $h_1 = l'$, such that for all t, and $i \neq j$, $h_t(S_i^1) \cap h_t(S_j^1) = \varnothing$.

DEFINITION 1.4. Two oriented links $L, L' \subset M^3$ of n-components are *concordant* if there is a proper, locally-flat embedding $h : (\bigcup_{i=1}^n S_i^1) \times I \to M^3 \times I$, such that $h[(\bigcup_{i=1}^n S_i^1) \times 0] = L \times 0$ and $h[(\bigcup_{i=1}^n S_i^1) \times 1] = L' \times 1$.

(Note that ambient isotopy implies condordance, but not the reverse). Homotopy and concordance are equivalence relations on oriented, n-component links; let $L \sim L'$ denote "L is homotopic to L'" and let $L \simeq L'$ denote "L is concordant to L'".

DEFINITION 1.5. (Tristram). *Connecting bands and arcs.*

Let $L \subset M^3$ be an oriented link and $b : I \times I \to M^3$ a proper embedding. b is said to be compatible with L if $b(I \times I) \cap L = b(I \times \partial I)$ and if the orientations from L on $b(I \times \partial I)$ induce the same orientation on $b(I \times I)$. In this case the link

$$[L - b(I \times \partial I)] \cup b(\partial I \times I),$$

its orientation inherited from L, will be denoted bL (see Figure 1).

b: $I \times I \to M^3$ is called a *connecting band* for L.

b: $I \times \frac{1}{2} \to M^3$ is called the *associated connecting arc.* (see Figure 2).

Notation. Let $L^1, L^2 \subset M^3$ be oriented links such that $L^1 \cap L^2 = \varnothing$. Put $L = L^1 \cup L^2$. If $b(I \times 0) \subset L^1$ and $b(I \times 1) \subset L^2$, define $L^1 +_b L^2$ to be bL. If the band $b(I \times I)$ is entirely contained in the xy-plane $\mathbf{R}^2_{xy}$, then define $L^1 \#_b L^2$ to be bL.

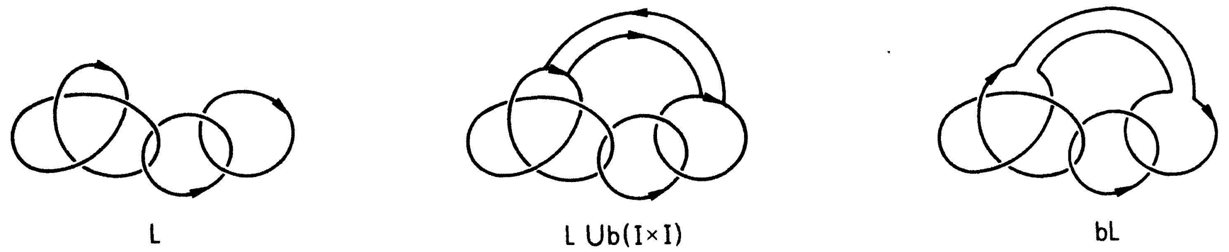


Figure 1

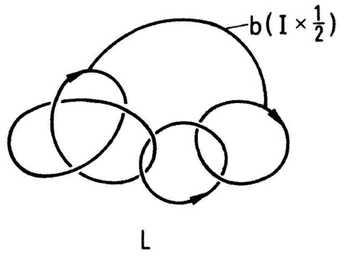


Figure 2

DEFINITION 1.6. The graph associated to a link with connecting bands.

Let $L \subset M^3$ be a link, $b_1, \ldots, b_n : I \times I \to M^3$ be a collection of disjoint connecting bands for L. The graph Γ is constructed as follows: there is a vertex of Γ for each component of L, and for each band b_i between two (possibly identical) components of L, there is an edge joining the corresponding vertices.

DEFINITION 1.7. The link diagram associated to a link with connecting bands.

This is simply $L \cup A$, where A is the collection of connecting arcs associated with the connecting bands for L (see Figure 4A and 4B).

DEFINITION 1.8. A ribbon link.

Let N be a compact, oriented 2-manifold such that every component of N has a non-empty boundary. A *ribbon map* of N into M^3 is a map, g say, with no triple points, satisfying: the doublepoint set consists of mutually disjoint arcs in N which may be paired (I_i, I_i') so that $g(I_i) = g(I_i')$, with I_i properly embedded in N and I_i' contained in int N, for all i in some finite indexing set. It is also assumed that the self-intersections of $g(N)$ at $g(I_i) = g(I_i')$ are transverse.

$g(N)$ will be called a *ribbon of type* N, and $g(\partial N)$, denoted by $\partial(g(N))$, a *ribbon link of type* N. If $N = \bigcup_{i=1}^k B_i = kB$ is a disjoint union of k copies of the 2-disk, then $\partial(g(N))$ is called a ribbon link (see Figure 3A and 3B).

In definition 1.9, let kB be the disjoint union $kB = \bigcup_{i=1}^k B_i$ of k copies of the 2-disk.

DEFINITION 1.9. (Tristram). $L \xrightarrow{r} L'$.

Let L, L' M^3 be oriented links. Then $L \xrightarrow{r} L'$ if for some integer k there exists a ribbon map $g : kB \to M^3 - L$ such that

$$L' \equiv (\ldots((L +_{b_1} \partial\hat{B}_1) +_{b_2} \partial\hat{B}_2) +_{b_3} \partial\hat{B}_3) \ldots) +_{b_n} \partial\hat{B}_n),$$

where $\hat{B}_i = g(B_i)$.

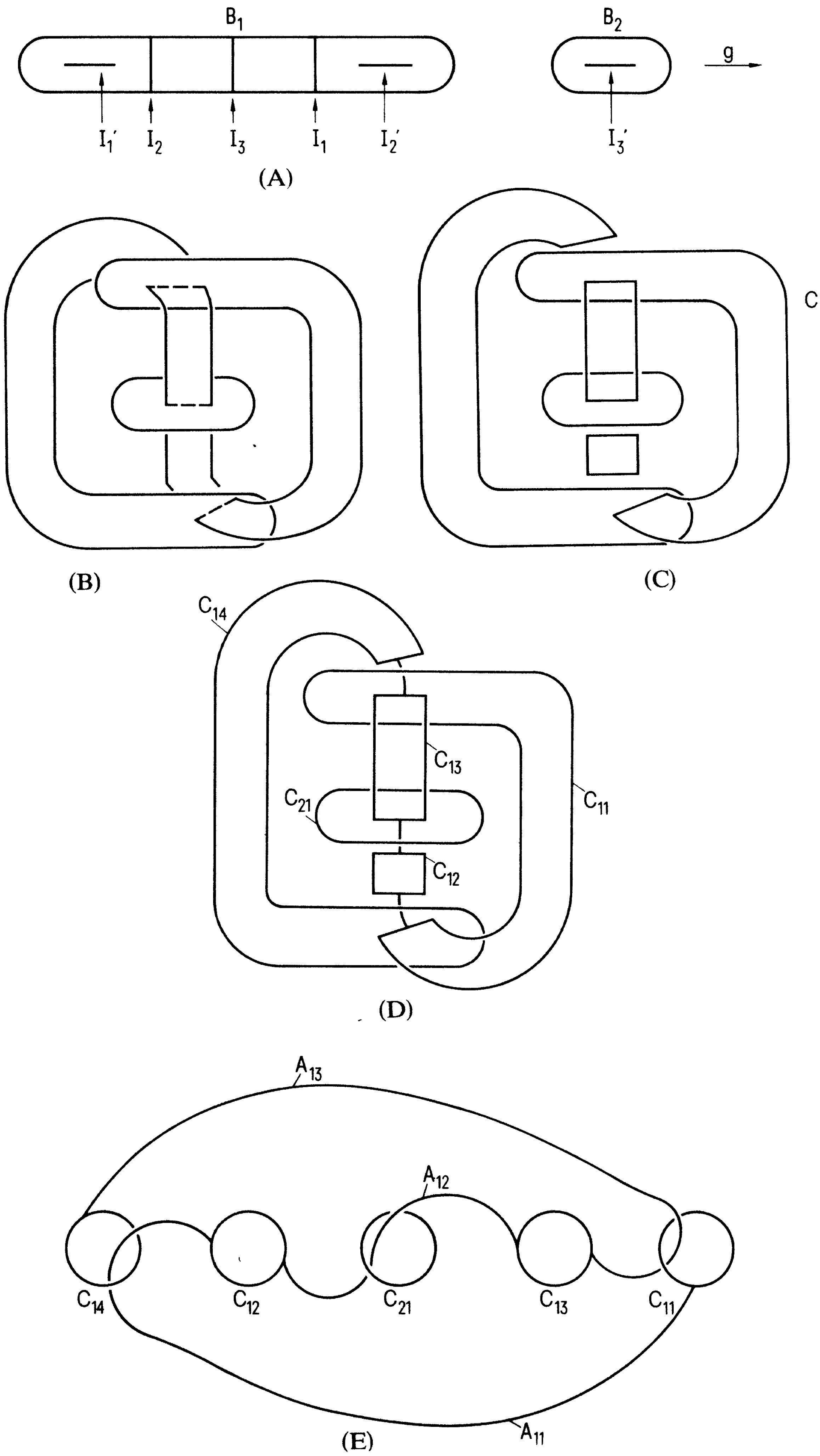


Figure 3. (A) arcs of doublepoints. (B) the Ribbon. (C) the cut ribbon $g(B')$ (D) the trivial link C with connecting arcs A. (E) the trivial link C (deformed into the xy-plane) with connecting arcs of A.

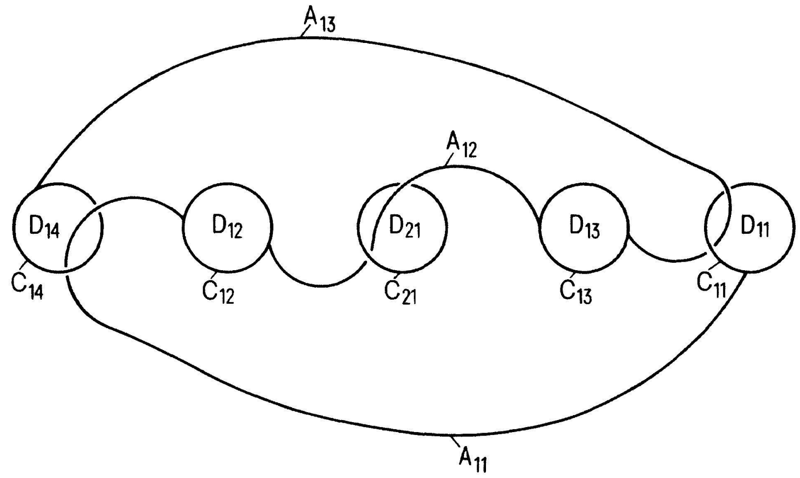


Figure 3F

DEFINITION 1.10. (Tristram). Ribbon equivalence.

L is *ribbon equivalent* to L', denoted $L \stackrel{r}{\equiv} L'$, if there exists a sequence of oriented links $L^1, \ldots, L^m$ such that $L^1 = L$, $L^m = L'$, and for $j = 1, \ldots, m$, either $L^j \stackrel{r}{\longmapsto} L^{j+1}$ or $L^{j+1} \stackrel{r}{\longmapsto} L^j$. (The equivalence relation $\stackrel{r}{\equiv}$ preserves the number of components of L.)

2. The main theorem

The approach will be to prove that ribbon equivalence implies homotopy for oriented links in M^3, since Tristram showed ([6]) that concordance and ribbon equivalence are identical equivalence relations on oriented links in M^3. (He actually shows this for oriented links in $\mathbf{R}^3$; however his proof goes over unchanged for an arbitrary 3-manifold.)

LEMMA 2.1. *Every ribbon link is of the form* $(b_n \cdots (b_2(b_1C)) \cdots)$, *where* C *is a trivial link in the xy-plane* $\mathbf{R}^2_{xy} \subset \mathbf{R}^3$, $b_1, \ldots, b_n : I \times I \to M^3$ *are disjoint connecting bands for* C, *and the graph associated to each component of* $(b_n \cdots (b_2(b_1C)) \cdots)$ *is a tree.*

Proof. Let the ribbon link $\partial(g(N))$ be the boundary of the ribbon $g(N)$, where $N = kB = \bigcup_{i=1}^k B_i$ is a collection of k disjoint 2-disks (see Figure 3A and 3B). Cut kB along each properly embedded arc I_i of doublepoints (i.e., remove the interior $\bigcup_i I_i \times (0, 1) \subset N$ of a closed, regular neighborhood $\bigcup_i I_i \times [0, 1] \subset N$ of the arcs $\bigcup_i I_i$ of doublepoints). Denote the result by B' (see Figure 3C).

Then B' is a union of 2-disks $B_{i1}, \ldots, B_{i,m(i)} \subset B_i$, $1 \leq i \leq k$, and $C = \partial(gB')$ is a trivial link, since $g \mid B'$ is an embedding. Put $C_{ij} = \partial(gB_{ij})$, $C_i = \bigcup_{j=1}^{m(i)} C_{ij}$.

Let b_i be the connecting band $g: I_i \times I \to M^3$ for C, and let $b_{il}, \ldots, b_{i,m(i)-1}$ be the subcollection of connecting bands b_j such that $I_j \subset B_i$. Then the ribbon link $R = \partial(gN)$ is $(b_n \cdots (b_2(b_1C)) \cdots)$, and the ith component of R is $(b_{i,m(i)-1} \cdots (b_{i2}(b_{i1}C_i)) \cdots)$; the graph associated to the latter is clearly a tree (see Figure 3D). An ambient isotopy will deform C into the xy-plane $\mathbf{R}^2_{xy} \subset \mathbf{R}^3$.

COROLLARY 2.2. *If* $L \xrightarrow{r} L'$, *then*

$$L' \equiv (\cdots((L \#_{d_1} R_1) \#_{d_2} R_2) \# \cdots) \#_{d_n} R_n)$$

where $R = \bigcup_{i=1}^{n} R_i$ *is a ribbon link with components* R_i, *each ribbon knot* R_i *is of the form* $(b_{i,m(i)-1} \cdots (b_{i2}(b_{i1}C_i)) \cdots)$ *where* C_i *is a trivial link of* $m(i)$ *components in the* xy-*plane* (*as in Lemma* 2.1), *and the connecting band* d_i *joins the component* L_i *of* L *to the component* C_{i1} *of* C_i, *and is contained in the* xy-*plane*, $1 \le i \le n$.

Proof. By Definition 1.9 we have $L' \equiv (\cdots((L +_{d_1} R_1) +_{d_2} R_2) + \cdots) +_{d_n} R_n)$, where $R = \bigcup_{i=1}^{n} R_i$ is a ribbon link with components R_i, and where R_i is of the form $(b_{i,m(i)} \cdots (b_{i2}(b_{i1}C_i)) \cdots)$ as in Lemma 2.1. After ambient isotopy, we may assume each component L_i of L passes through the 3-cell $\mathbf{R}^3$, and intersects the xy-plane $\mathbf{R}^2_{xy} \subset \mathbf{R}^3$ in a closed subarc; further, we may assume that this subarc is joined by the connecting band d_i to a closed subarc of C_{i1}. Now if $d_i \not\subset \mathbf{R}^2_{xy}$, deform C_{i1} by an ambient isotopy which slides the latter closed subarc across the band d_i, while fixing its endpoints; call the result C'_{i1}. Then $C'_{i1} \equiv C_{i1} +_{d_1} C_{i0}$, where C_{i0} is a tiny circle in the xy-plane. Obviously, $L_i +_{d_i} R_i$ is ambient isotopic to $L'_i \#_{d'_i} R'_i$, in the complement $M^3 - \bigcup_{j \neq i} L_j +_{d_j} R_j$, where R'_i is the ribbon knot

$$(d_i(b_{i,m(i)-1} \cdots (b_{i2}(b_{i1}C'_i)) \cdots)), \quad C'_i = C_i \cup C_{i0},$$

$L'_i \equiv L_i$ is moved just slightly to avoid C_{i0}, and the connecting band $d'_i \subset \mathbf{R}^2_{xy}$ joins L'_i to C_{i0}.

LEMMA 2.3. *Ribbon links are null-homotopic* (*homotopic to a trivial link*).

Proof. Let the ribbon link be $R = \bigcup_{i=1}^{n} R_i \subset M^3$ with components R_i. As in Lemma 2.1, let $R_i = (b_{i,m(i)-1} \cdots (b_{i1}C_i)) \cdots)$, where C_i is a trivial link of $m(i)$ components in $\mathbf{R}^2_{xy}$, and the associated link diagram is a tree. Let a_{ij} be the connecting arc associated to the connecting band b_{ij}, and set $A_i = \bigcup_{j=1}^{m(i)-1} a_{ij}$, $A = \bigcup_{i=1}^{n} A_i$. Thus the link diagram associated to R is $C \cup A$, with components $C_i \cup A_i$. Let C_{ij} bound the disk $D_{ij} \subset \mathbf{R}^2_{xy}$, and set $D_i = \bigcup_{j=1}^{m(i)} D_{ij}$, $D = \bigcup_{i=1}^{n} D_i$. Note that the D_{ij}'s are necessarily disjoint. Finally, let the open 3-cells $B_{ij} \subset \mathbf{R}^3$ be disjoint, regular neighborhoods of the 2-disks D_{ij}. (See Figure 3F). Without loss of generality, we may assume that each arc a_{ij} meets the xy-plane transversely, and $a_{ij} \cap C_{ij} = \partial a_{ij}$.

For clarity, I will indicate the homotopy from R to a trivial link, by describing a homotopy of the link diagram $C \cup A$. It will be sufficient to move $C \cup A$ to a homeomorph $C' \cup A' \subset \mathbf{R}^2_{xy}$ by an appropriate kind of homotopy. The proof that this can be done goes by induction on the components of $C \cup A$:

Induction Hypothesis. For all $i < k$, $C_i \cup A_i \subset \mathbf{R}^2_{xy}$.

Now assume that the induction hypothesis is satisfied for $k = m$.

Proof sketch. We will first perform a homotopy $h_t(C \bigcup A)$ to eliminate points of intersection of A_m with $\operatorname{int} D_m$. During this homotopy, the components $C_i \bigcup A_i$, $1 \le i \le n$, must remain disjoint. Then an ambient isotopy will suffice to untangle $A_m \cup D_m$ from $\bigcup_{i<m} C_i \cup A_i$, and carry it into the xy-plane $\mathbf{R}^2_{xy}$, thereby proving the I.H. for $k = m+1$. In so doing, the arcs A_i, $i > m$, may become more entangled with C_j, $j < m$.

There exists an isotopy $h_t : M^3 \to M^3$ which has support on the 3-cell $\mathbf{R}^3$, which leaves the xy-plane invariant, which fixes $\bigcup_{i<m} D_i$ and $\bigcup_{i<m} A_i$, which is the identity outside of $B_m = \bigcup_j B_{mj}$ and which fixes the endpoints ∂A_m, such that $h_1(A_m) \cap \operatorname{int} D_m = \varnothing$; then $h_t(A) \cup C$ is a homotopy of $A \cup C$ to a homeomorph $A' \cup C' = h_1(A) \cup C$, which satisfies $A'_m \cap \operatorname{int} D'_m = \varnothing$, in addition to all of the properties attributed to A, C and D. We will assume that $A \cup C$ has been replaced by $A' \cup C'$.

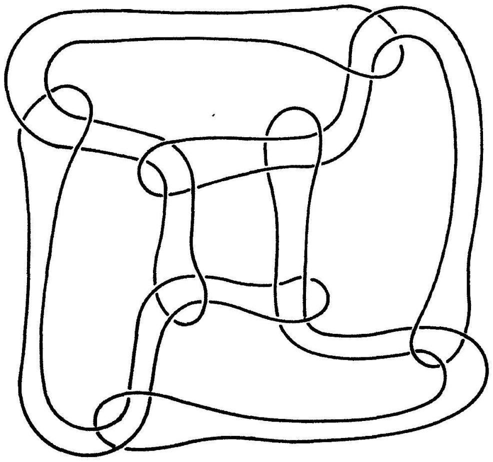
Now $D_m \cup A_m$ is a simply-connected 2-complex, since $A_m \cap \operatorname{int} D_m = \varnothing$. There exist disjoint regular neighborhoods U of $D_m \cup A_m$ and V of $(\bigcup_{i \ne m} C_i) \cup (\bigcup_{i<m} A_i)$, such that U is a 3-cell, and $B_{m1} \subset U$. There is then an isotopy $h_t : M^3 \to M^3$ with support in U (hence fixing V), which fixes D_{m1}, such that $h_0 = id$ and $h_1(D_m \cup A_m) \subset B_{m1}$. There is a further isotopy whose support is in B_{m1}, which is the identity on D_{m1}, and carries $h_1(D_m \cup A_m)$ to a homeomorph $D'_m \cup A'_m \subset \mathbf{R}^2_{xy}$ (The details of this are omitted; however the proof is easy, and involves an application or two of the Schoenflies theorem.) Thus, the I.H. has been verified for $k = m+1$, which completes the proof.

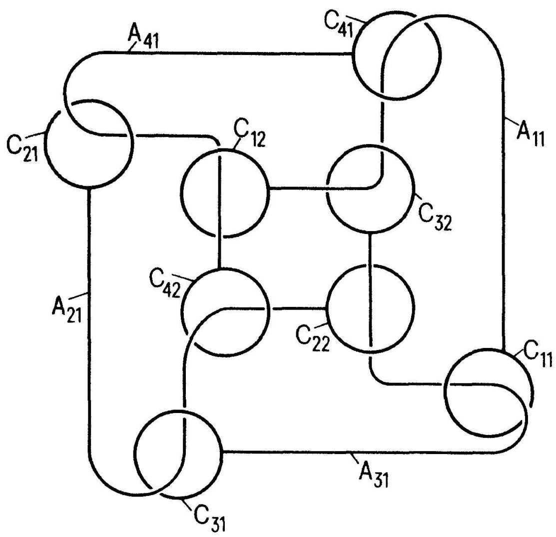
Figure 4 indicates a homotopy from a particular ribbon link in $\mathbf{R}^3$ to a trivial link.

COROLLARY 2.4. *If $L \xrightarrow{r} L'$, then $L \sim L'$.*

Proof. By Corollary 2.2,

$L' \equiv (\cdots((L \#_{d_1} R_1) \#_{d_2} R_2) \# \cdots) \#_{d_n} R_n)$, where $R = \bigcup_{i=1}^n R_i$


(A)


A_{41}
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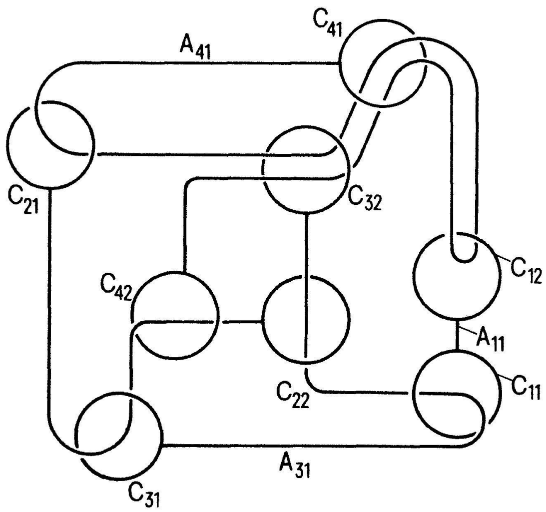
(B)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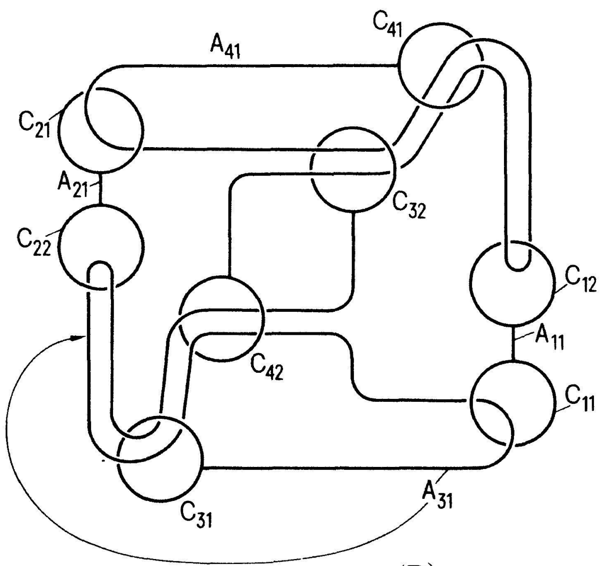
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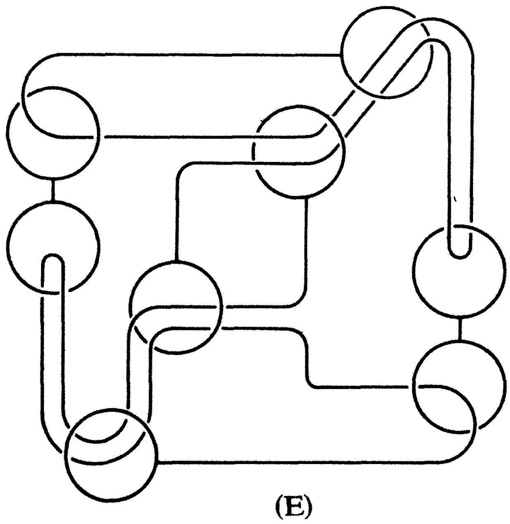
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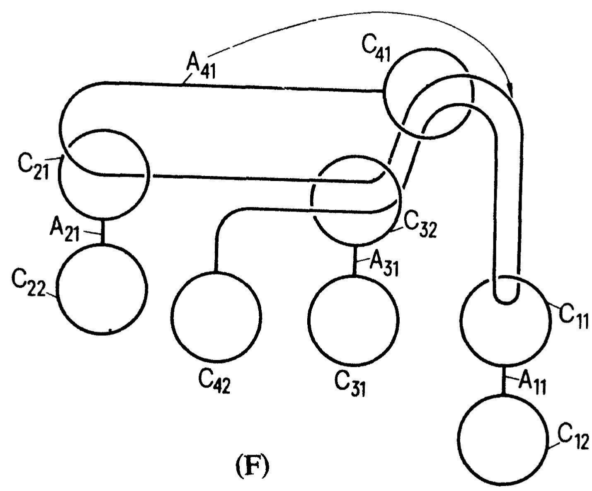
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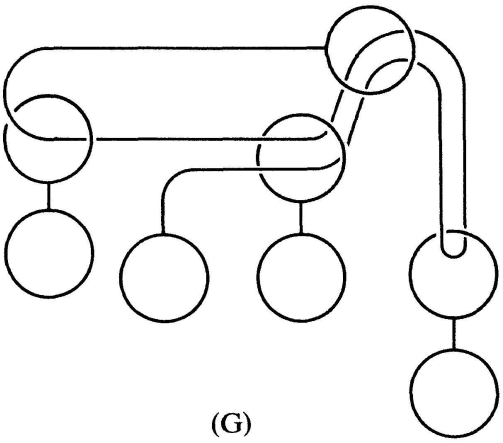
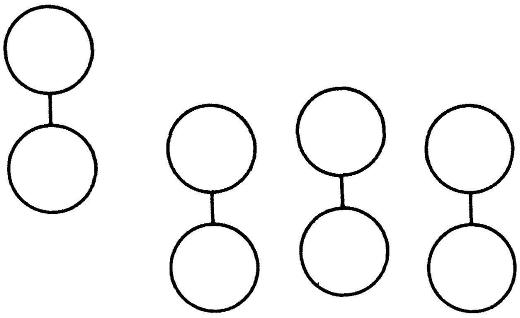


(G)



(H)

Figure 4. (A) The link R. (B) The link diagram CUA.

is a ribbon link with components R_i, the connecting bands d_t lie in the xy-plane $\mathbf{R}^2_{xy} \subset \mathbf{R}^3$, and $R_i = (b_{i,m(i)-1} \cdots (b_{i2}(b_{i1}C_i))\cdots)$ as in Lemma 2.1. Now an inspection of the proof of Lemma 2.3 quickly reveals that the homotopy from R to a trivial link $C^n \subset \mathbf{R}^2_{xy}$ can be made to avoid both L and the connecting bands $\bigcup_{i=1}^n d_i \subset \mathbf{R}^2_{xy}$. Hence $L' \sim (\cdots((L \#_{d_1} C^n_1) \#_{d_2} C^n_2) \# \cdots) \#_{d_n} C^n_n) \equiv L$.

THEOREM 2.5. *Concordance implies homotopy for oriented links in M^3.*

Proof. It follows from Corollary 2.4 that ribbon equivalence implies homotopy. However, by Tristram (Corollary 1.33, [6]), ribbon equivalence and concordance are identical equivalence relations on oriented links in M^3.

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