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Concordance implies homotopy for classical links in M^3

by DEBORAH L. GOLDSMITH

Introduction

In this paper I prove that concordance implies homotopy for classical links in any 3-manifold. The notion of concordance was first developed by Fox and Milnor in [2] for knots in \mathbf{R}^3 , and later extended to links in \mathbf{R}^3 by Fox, in Problem 25 of [1]. Homotopy of links in a 3-manifold M^3 was defined and studied by Milnor in [5].

The proof is entirely geometric, and also quite simple. In fact, at this point I would direct the reader's attention to Figure 4, which indicates a homotopy from a particular ribbon link to the trivial link in \mathbf{R}^3 . The reader might then be led to the proof that all ribbon links are null-homotopic (Lemma 2.3).

The result of this paper has also been obtained by Charles Giffen, independently, and by a different method.

1. Main definitions

All maps and spaces are in the P.L. category. Choose a closed 3-cell in the interior of every 3-manifold M^3 , and denote its interior by \mathbf{R}^3 ; let $\mathbf{R}_{xy}^2 \subset \mathbf{R}^3$ be the xy -plane in \mathbf{R}^3 . Recall that a map $g: M \rightarrow N$ of manifolds is proper if $g(\partial M) \subseteq \partial N$ and $g(\text{int } M) \subseteq \text{int } N$.

Certain definitions, where indicated, will be taken from [6] (A. J. Tristram).

DEFINITION 1.1. An *oriented link of n -components* in a 3-manifold M^3 is a proper embedding $l: \bigcup_{i=1}^n S_i^1 \rightarrow M^3$ of a disjoint union of n -oriented 1-spheres in that 3-manifold. Let L_i denote the oriented image $l(S_i^1)$, and let $L = l(\bigcup_{i=1}^n S_i^1) = \bigcup_{i=1}^n L_i$.

DEFINITION 1.2. Two oriented links $l, l': \bigcup_{i=1}^n S_i^1 \rightarrow M^3$ in M^3 are *ambient isotopic*, if there is an isotopy $h_t: M^3 \rightarrow M^3$ such that $h_0 = id$ and $h_1 \circ l = l'$.

We will not distinguish between a link and its ambient isotopy class. Since oriented links with the same oriented images are ambient isotopic, let the image $L \subset M^3$ denote the link l , and let the expression $L \equiv L'$ signify that L and L' are ambient isotopic. The trivial link of n disjoint circles in the xy -plane \mathbf{R}_{xy}^2 , will be denoted $C^n = \bigcup_{i=1}^n C_i^1$.

DEFINITION 1.3. Two oriented links $L, L' \subset M^3$ of n components are *homotopic* if there is a homotopy $h_t : \bigcup_{i=1}^n S_i^1 \rightarrow M^3$ from $h_0 = l$ to $h_1 = l'$, such that for all t , and $i \neq j$, $h_t(S_i^1) \cap h_t(S_j^1) = \emptyset$.

DEFINITION 1.4. Two oriented links $L, L' \subset M^3$ of n -components are *concordant* if there is a proper, locally-flat embedding $h : (\bigcup_{i=1}^n S_i^1) \times I \rightarrow M^3 \times I$, such that $h[(\bigcup_{i=1}^n S_i^1) \times 0] = L \times 0$ and $h[(\bigcup_{i=1}^n S_i^1) \times 1] = L' \times 1$.

(Note that ambient isotopy implies concordance, but not the reverse). Homotopy and concordance are equivalence relations on oriented, n -component links; let $L \sim L'$ denote " L is homotopic to L' " and let $L \approx L'$ denote " L is concordant to L' ".

DEFINITION 1.5. (Tristram). *Connecting bands and arcs.*

Let $L \subset M^3$ be an oriented link and $b : I \times I \rightarrow M^3$ a proper embedding. b is said to be compatible with L if $b(I \times I) \cap L = b(I \times \partial I)$ and if the orientations from L on $b(I \times \partial I)$ induce the same orientation on $b(I \times I)$. In this case the link

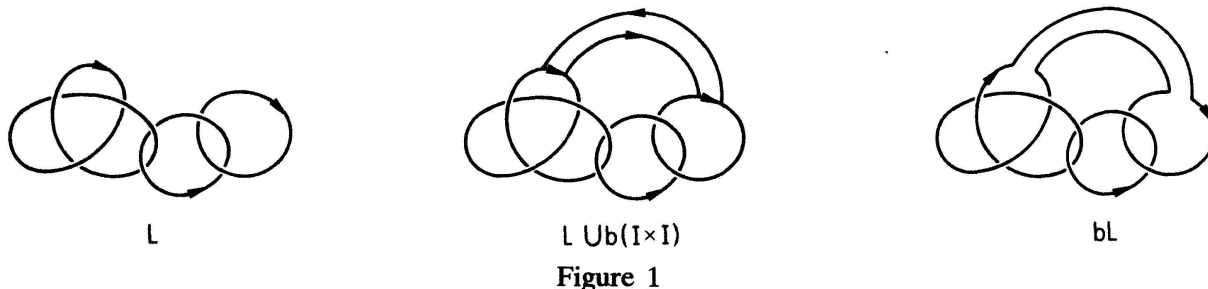
$$[L - b(I \times \partial I)] \cup b(\partial I \times I),$$

its orientation inherited from L , will be denoted bL (see Figure 1).

$b : I \times I \rightarrow M^3$ is called a *connecting band* for L .

$b : I \times \frac{1}{2} \rightarrow M^3$ is called the *associated connecting arc*. (see Figure 2).

Notation. Let $L^1, L^2 \subset M^3$ be oriented links such that $L^1 \cap L^2 = \emptyset$. Put $L = L^1 \cup L^2$. If $b(I \times 0) \subset L^1$ and $b(I \times 1) \subset L^2$, define $L^1 +_b L^2$ to be bL . If the band $b(I \times I)$ is entirely contained in the xy -plane \mathbf{R}_{xy}^2 , then define $L^1 \#_b L^2$ to be bL .



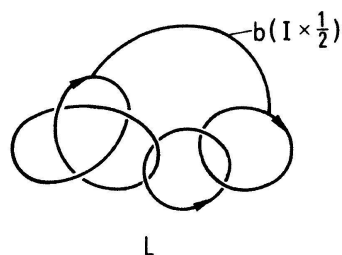


Figure 2

DEFINITION 1.6. The graph associated to a link with connecting bands.

Let $L \subset M^3$ be a link, $b_1, \dots, b_n : I \times I \rightarrow M^3$ be a collection of disjoint connecting bands for L . The graph Γ is constructed as follows: there is a vertex of Γ for each component of L , and for each band b_i between two (possibly identical) components of L , there is an edge joining the corresponding vertices.

DEFINITION 1.7. The link diagram associated to a link with connecting bands.

This is simply $L \cup A$, where A is the collection of connecting arcs associated with the connecting bands for L (see Figure 4A and 4B).

DEFINITION 1.8. A ribbon link.

Let N be a compact, oriented 2-manifold such that every component of N has a non-empty boundary. A *ribbon map* of N into M^3 is a map, g say, with no triple points, satisfying: the doublepoint set consists of mutually disjoint arcs in N which may be paired (I_i, I'_i) so that $g(I_i) = g(I'_i)$, with I_i properly embedded in N and I'_i contained in $\text{int } N$, for all i in some finite indexing set. It is also assumed that the self-intersections of $g(N)$ at $g(I_i) = g(I'_i)$ are transverse.

$g(N)$ will be called a *ribbon of type N* , and $g(\partial N)$, denoted by $\partial(g(N))$, a *ribbon link of type N* . If $N = \bigcup_{i=1}^k B_i = kB$ is a disjoint union of k copies of the 2-disk, then $\partial(g(N))$ is called a ribbon link (see Figure 3A and 3B).

In definition 1.9, let kB be the disjoint union $kB = \bigcup_{i=1}^k B_i$ of k copies of the 2-disk.

DEFINITION 1.9. (Tristram). $L \xrightarrow{r} L'$.

Let $L, L' \subset M^3$ be oriented links. Then $L \xrightarrow{r} L'$ if for some integer k there exists a ribbon map $g : kB \rightarrow M^3 - L$ such that

$$L' \equiv (\dots ((L +_{b_1} \partial \hat{B}_1) +_{b_2} \partial \hat{B}_2) +_{b_3} \partial \hat{B}_3) \dots) +_{b_n} \partial \hat{B}_n,$$

where $\hat{B}_i = g(B_i)$.

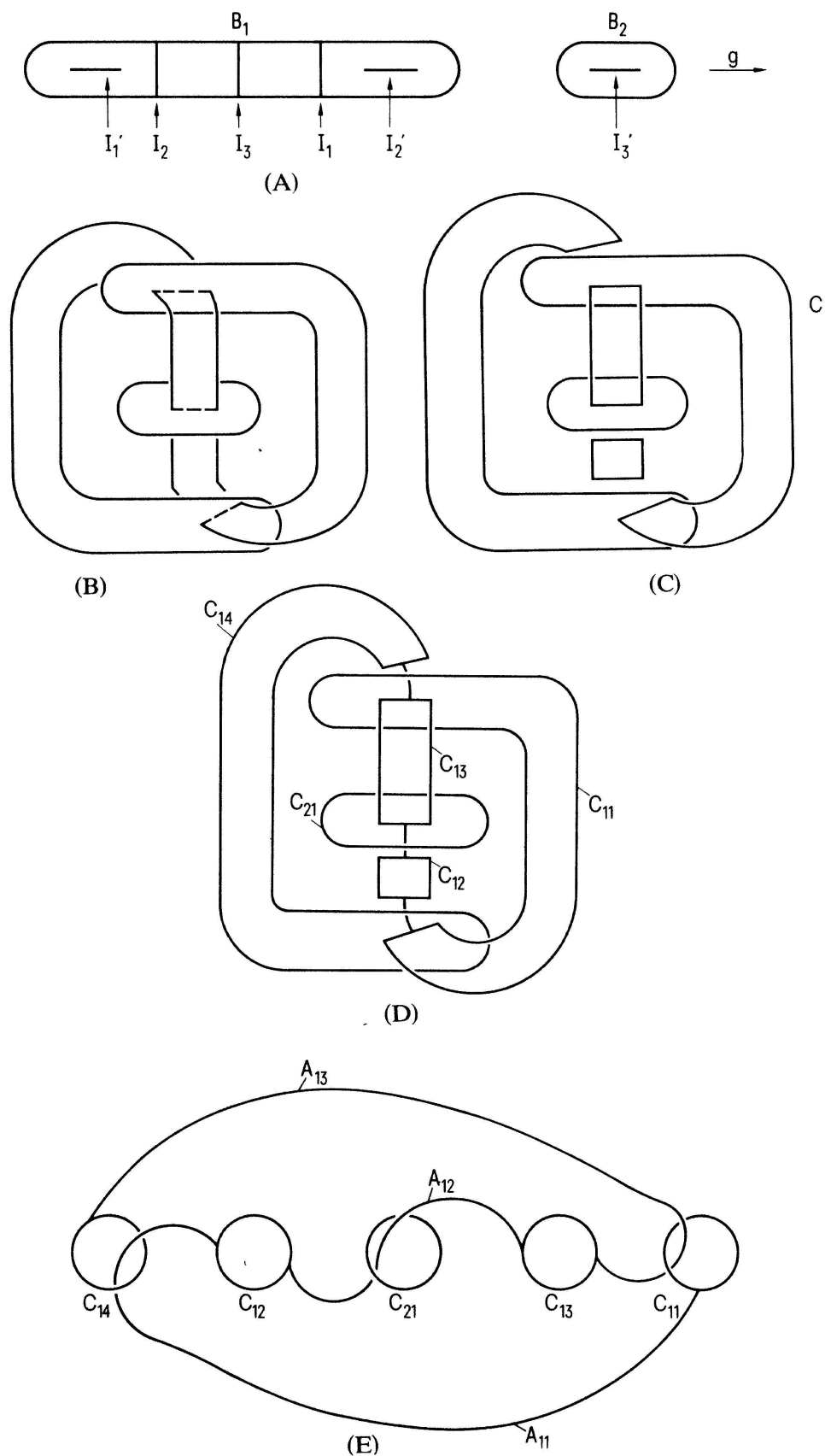


Figure 3. (A) arcs of doublepoints. (B) the Ribbon. (C) the cut ribbon $g(B')$ (D) the trivial link C with connecting arcs A . (E) the trivial link C (deformed into the xy -plane) with connecting arcs of A .

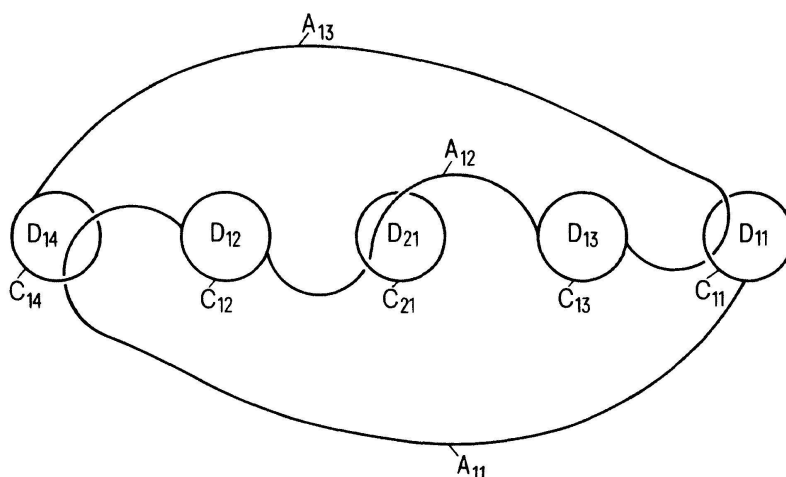


Figure 3F

DEFINITION 1.10. (Tristram). Ribbon equivalence.

L is *ribbon equivalent* to L' , denoted $L \equiv L'$, if there exists a sequence of oriented links L^1, \dots, L^m such that $L^1 = L$, $L^m = L'$, and for $j = 1, \dots, m$, either $L^j \xrightarrow{r} L^{j+1}$ or $L^{j+1} \xrightarrow{r} L^j$. (The equivalence relation \equiv preserves the number of components of L .)

2. The main theorem

The approach will be to prove that ribbon equivalence implies homotopy for oriented links in M^3 , since Tristram showed ([6]) that concordance and ribbon equivalence are identical equivalence relations on oriented links in M^3 . (He actually shows this for oriented links in \mathbf{R}^3 ; however his proof goes over unchanged for an arbitrary 3-manifold.)

LEMMA 2.1. *Every ribbon link is of the form $(b_n \cdots (b_2(b_1 C)) \cdots)$, where C is a trivial link in the xy -plane $\mathbf{R}_{xy}^2 \subset \mathbf{R}^3$, $b_1, \dots, b_n : I \times I \rightarrow M^3$ are disjoint connecting bands for C , and the graph associated to each component of $(b_n \cdots (b_2(b_1 C)) \cdots)$ is a tree.*

Proof. Let the ribbon link $\partial(g(N))$ be the boundary of the ribbon $g(N)$, where $N = kB = \bigcup_{i=1}^k B_i$ is a collection of k disjoint 2-disks (see Figure 3A and 3B). Cut kB along each properly embedded arc I_i of doublepoints (i.e., remove the interior $\bigcup_i I_i \times (0, 1) \subset N$ of a closed, regular neighborhood $\bigcup_i I_i \times [0, 1] \subset N$ of the arcs $\bigcup_i I_i$ of doublepoints). Denote the result by B' (see Figure 3C).

Then B' is a union of 2-disks $B_{i1}, \dots, B_{i,m(i)} \subset B_i$, $1 \leq i \leq k$, and $C = \partial(gB')$ is a

trivial link, since $g|B'$ is an embedding. Put $C_{ij} = \partial(gB_{ij})$, $C_i = \bigcup_{j=1}^{m(i)} C_{ij}$.

Let b_i be the connecting band $g: I_i \times I \rightarrow M^3$ for C_i , and let $b_{i1}, \dots, b_{i, m(i)-1}$ be the subcollection of connecting bands b_j such that $I_j \subset B_i$. Then the ribbon link $R = \partial(gN)$ is $(b_n \cdots (b_2(b_1 C)) \cdots)$, and the i th component of R is $(b_{i, m(i)-1} \cdots (b_{i2}(b_{i1} C_i)) \cdots)$; the graph associated to the latter is clearly a tree (see Figure 3D). An ambient isotopy will deform C into the xy -plane $\mathbf{R}_{xy}^2 \subset \mathbf{R}^3$.

COROLLARY 2.2. *If $L \xrightarrow{r} L'$, then*

$$L' \equiv (\cdots ((L \#_{d_1} R_1) \#_{d_2} R_2) \# \cdots) \#_{d_n} R_n$$

where $R = \bigcup_{i=1}^n R_i$ is a ribbon link with components R_i , each ribbon knot R_i is of the form $(b_{i, m(i)-1} \cdots (b_{i2}(b_{i1} C_i)) \cdots)$ where C_i is a trivial link of $m(i)$ components in the xy -plane (as in Lemma 2.1), and the connecting band d_i joins the component L_i of L to the component C_{i1} of C_i , and is contained in the xy -plane, $1 \leq i \leq n$.

Proof. By Definition 1.9 we have $L' \equiv (\cdots ((L +_{d_1} R_1) +_{d_2} R_2) + \cdots) +_{d_n} R_n$, where $R = \bigcup_{i=1}^n R_i$ is a ribbon link with components R_i , and where R_i is of the form $(b_{i, m(i)} \cdots (b_{i2}(b_{i1} C_i)) \cdots)$ as in Lemma 2.1. After ambient isotopy, we may assume each component L_i of L passes through the 3-cell \mathbf{R}^3 , and intersects the xy -plane $\mathbf{R}_{xy}^2 \subset \mathbf{R}^3$ in a closed subarc; further, we may assume that this subarc is joined by the connecting band d_i to a closed subarc of C_{i1} . Now if $d_i \not\subset \mathbf{R}_{xy}^2$, deform C_{i1} by an ambient isotopy which slides the latter closed subarc across the band d_i , while fixing its endpoints; call the result C'_{i1} . Then $C'_{i1} \equiv C_{i1} +_{d_i} C_{i0}$, where C_{i0} is a tiny circle in the xy -plane. Obviously, $L_i +_{d_i} R_i$ is ambient isotopic to $L'_i \#_{d'_i} R'_i$, in the complement $M^3 - \bigcup_{j \neq i} L_j +_{d_j} R_j$, where R'_i is the ribbon knot

$$(d_i(b_{i, m(i)-1} \cdots (b_{i2}(b_{i1} C'_i)) \cdots)), \quad C'_i = C_i \cup C_{i0},$$

$L'_i \equiv L_i$ is moved just slightly to avoid C_{i0} , and the connecting band $d'_i \subset \mathbf{R}_{xy}^2$ joins L'_i to C_{i0} .

LEMMA 2.3. *Ribbon links are null-homotopic (homotopic to a trivial link).*

Proof. Let the ribbon link be $R = \bigcup_{i=1}^n R_i \subset M^3$ with components R_i . As in Lemma 2.1, let $R_i = (b_{i, m(i)-1} \cdots (b_{i1} C_i)) \cdots$, where C_i is a trivial link of $m(i)$ components in \mathbf{R}_{xy}^2 , and the associated link diagram is a tree. Let a_{ij} be the connecting arc associated to the connecting band b_{ij} , and set $A_i = \bigcup_{j=1}^{m(i)-1} a_{ij}$, $A = \bigcup_{i=1}^n A_i$. Thus the link diagram associated to R is $C \cup A$, with components $C_i \cup A_i$. Let C_{ij} bound the disk $D_{ij} \subset \mathbf{R}_{xy}^2$, and set $D_i = \bigcup_{j=1}^{m(i)} D_{ij}$, $D = \bigcup_{i=1}^n D_i$. Note that the D_{ij} 's are necessarily disjoint. Finally, let the open 3-cells $B_{ij} \subset \mathbf{R}^3$ be disjoint, regular neighborhoods of the 2-disks D_{ij} . (See Figure 3F). Without loss of

generality, we may assume that each arc a_{ij} meets the xy -plane transversely, and $a_{ij} \cap C_{ij} = \partial a_{ij}$.

For clarity, I will indicate the homotopy from R to a trivial link, by describing a homotopy of the link diagram $C \cup A$. It will be sufficient to move $C \cup A$ to a homeomorph $C' \cup A' \subset \mathbf{R}_{xy}^2$ by an appropriate kind of homotopy. The proof that this can be done goes by induction on the components of $C \cup A$:

Induction Hypothesis. For all $i < k$, $C_i \cup A_i \subset \mathbf{R}_{xy}^2$.

Now assume that the induction hypothesis is satisfied for $k = m$.

Proof sketch. We will first perform a homotopy $h_t(C \cup A)$ to eliminate points of intersection of A_m with $\text{int } D_m$. During this homotopy, the components $C_i \cup A_i$, $1 \leq i \leq n$, must remain disjoint. Then an ambient isotopy will suffice to untangle $A_m \cup D_m$ from $\bigcup_{i < m} C_i \cup A_i$, and carry it into the xy -plane \mathbf{R}_{xy}^2 , thereby proving the I.H. for $k = m + 1$. In so doing, the arcs A_i , $i > m$, may become more entangled with C_j , $j < m$.

There exists an isotopy $h_t: M^3 \rightarrow M^3$ which has support on the 3-cell \mathbf{R}^3 , which leaves the xy -plane invariant, which fixes $\bigcup_{i < m} D_i$ and $\bigcup_{i < m} A_i$, which is the identity outside of $B_m = \bigcup_j B_{mj}$ and which fixes the endpoints ∂A_m , such that $h_1(A_m) \cap \text{int } D_m = \emptyset$; then $h_t(A) \cup C$ is a homotopy of $A \cup C$ to a homeomorph $A' \cup C' = h_1(A) \cup C$, which satisfies $A'_m \cap \text{int } D'_m = \emptyset$, in addition to all of the properties attributed to A , C and D . We will assume that $A \cup C$ has been replaced by $A' \cup C'$.

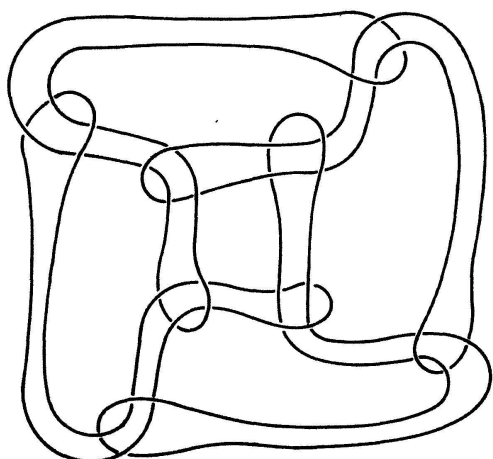
Now $D_m \cup A_m$ is a simply-connected 2-complex, since $A_m \cap \text{int } D_m = \emptyset$. There exist disjoint regular neighborhoods U of $D_m \cup A_m$ and V of $(\bigcup_{i \neq m} C_i) \cup (\bigcup_{i < m} A_i)$, such that U is a 3-cell, and $B_{m1} \subset U$. There is then an isotopy $h_t: M^3 \rightarrow M^3$ with support in U (hence fixing V), which fixes D_{m1} , such that $h_0 = \text{id}$ and $h_1(D_m \cup A_m) \subset B_{m1}$. There is a further isotopy whose support is in B_{m1} , which is the identity on D_{m1} , and carries $h_1(D_m \cup A_m)$ to a homeomorph $D'_m \cup A'_m \subset \mathbf{R}_{xy}^2$ (The details of this are omitted; however the proof is easy, and involves an application or two of the Schoenflies theorem.) Thus, the I.H. has been verified for $k = m + 1$, which completes the proof.

Figure 4 indicates a homotopy from a particular ribbon link in \mathbf{R}^3 to a trivial link.

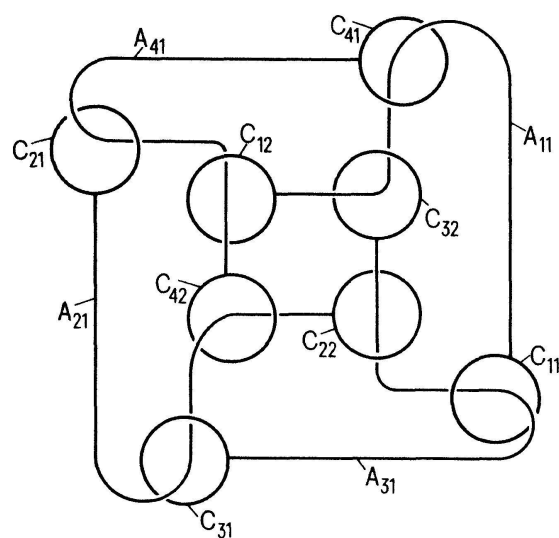
COROLLARY 2.4. If $L \xrightarrow{\cdot} L'$, then $L \sim L'$.

Proof. By Corollary 2.2,

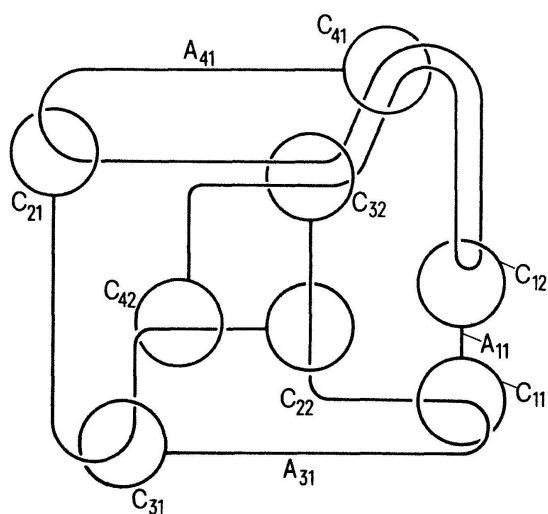
$$L' \equiv (\cdots ((L \#_{a_1} R_1) \#_{a_2} R_2) \# \cdots) \#_{a_n} R_n, \quad \text{where } R = \bigcup_{i=1}^n R_i$$



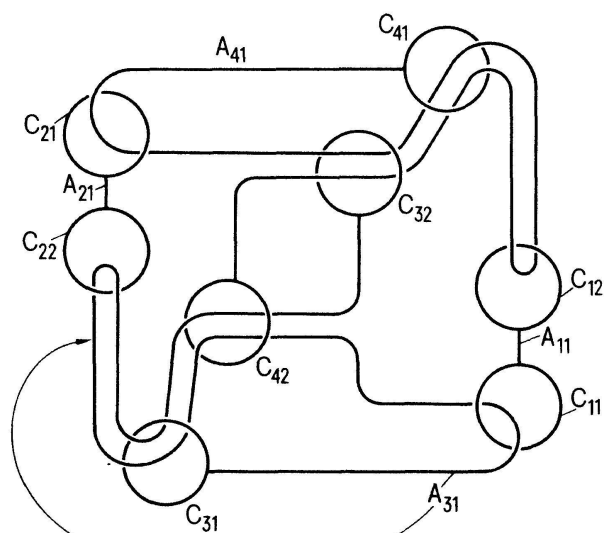
(A)



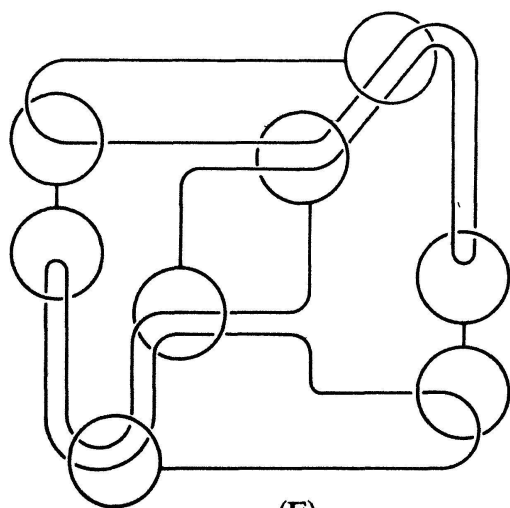
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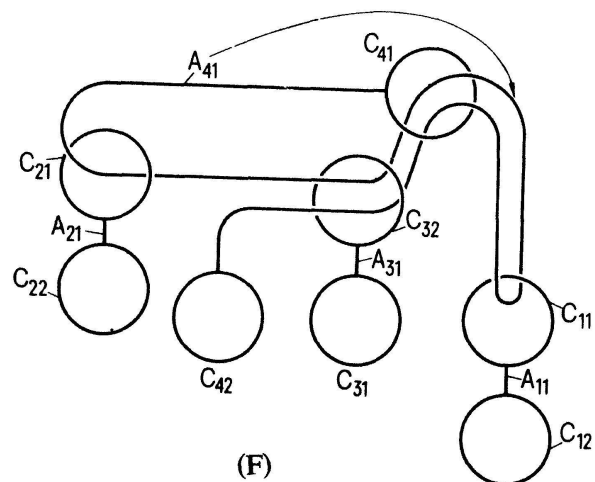
(C)



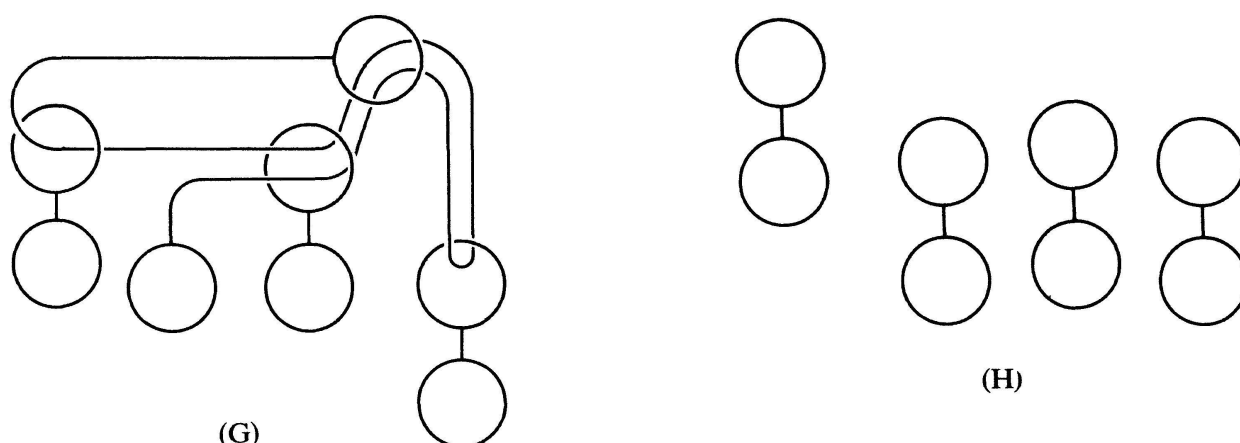
(D)



(E)



(F)

Figure 4. (A) The link R . (B) The link diagram CUA.

is a ribbon link with components R_i , the connecting bands d_i lie in the xy -plane $\mathbf{R}_{xy}^2 \subset \mathbf{R}^3$, and $R_i = (b_{i,m(i)-1} \cdots (b_{i2}(b_{i1}C_i)) \cdots)$ as in Lemma 2.1. Now an inspection of the proof of Lemma 2.3 quickly reveals that the homotopy from R to a trivial link $C^n \subset \mathbf{R}_{xy}^2$ can be made to avoid both L and the connecting bands $\bigcup_{i=1}^n d_i \subset \mathbf{R}_{xy}^2$. Hence $L' \sim (\cdots ((L \#_{d_1} C_1^n) \#_{d_2} C_2^n) \# \cdots) \#_{d_n} C_n^n \equiv L$.

THEOREM 2.5. *Concordance implies homotopy for oriented links in M^3 .*

Proof. It follows from Corollary 2.4 that ribbon equivalence implies homotopy. However, by Tristram (Corollary 1.33, [6]), ribbon equivalence and concordance are identical equivalence relations on oriented links in M^3 .

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