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Simplicity of the projective unitary groups defined by simple factors

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Let $\mathcal{B}$ be a C^*-algebra with unit and let $U(\mathcal{B})$ be the group of all its unitary elements. Assume that the center of $\mathcal{B}$ is reduced to the set of scalar multiples of the identity, and identify the center of $U(\mathcal{B})$ with the group S^1 of complex numbers with modulus $+1$. The *projective unitary group* of $\mathcal{B}$ is the quotient $PU(\mathcal{B})$ of $U(\mathcal{B})$ by S^1 [2]. We want to find conditions on $\mathcal{B}$ for this group to be simple.

Suppose $\mathcal{B}$ has a non trivial two-sided ideal $\mathcal{I}$; it is easy to check that $PU(\mathcal{B})$ is not simple, and the argument runs as follows. First, $\mathcal{I}$ is not dense with respect to the norm topology (because elements near 1 are invertible in $\mathcal{B}$), so that the closure $\bar{\mathcal{I}}$ of $\mathcal{I}$ is a non trivial self-adjoint ideal in $\mathcal{B}$ [8, prop. 1.8.2]. Then the kernel of the natural map $U(\mathcal{B}) \to U(\mathcal{B}/\bar{\mathcal{I}})$ is neither the whole of $U(\mathcal{B})$, because it does not contain all elements near 1, nor a subgroup of S^1, because it contains $(1-x^2)^{1/2}+ix$ if x is self-adjoint in $\bar{\mathcal{I}}$ with small norm. Hence this kernel defines a non trivial normal subgroup of $PU(\mathcal{B})$.

From now on, we shall assume that $\mathcal{B}$ is a *von Neumann factor.* If $\mathcal{B}$ is not countably decomposable $PU(\mathcal{B})$ cannot be simple; see [7, chap. I, §1, exerc. 7]. We shall consequently assume that $\mathcal{B}$ is *countably decomposable.*

If $\mathcal{B}$ is *infinite and semi-finite,* then it has a non trivial two-sided ideal (for example that generated by all finite projections), and $PU(\mathcal{B})$ is not simple. More can be said about normal subgroups of $PU(\mathcal{B})$ in this case: see [11] for type I_∞ and a later note for type II_∞; but this is not our main purpose here. If $\mathcal{B}$ is *finite and discrete,* say $\mathcal{B} = M_n(C)$ with n a positive integer, it is well-known that any normal subgroup of $PU(\mathcal{B})$ contains the simple group $PSU(n)$. The proof follows closely the analogous one for orthogonal groups, which seems to appear first in E. Catan [4]; the best reference is E. Artin [1, chap. V, §2]; there is a discussion of the unitary case in Dieudonné [6, chap. VI].

In the remaining cases, $\mathcal{B}$ is known to be *simple.* Though this will follow from our main theorem, see [7, chap. III, §5, n° 2] for type II_1 and [7, chap. III, §8, exerc. 1] for type III. Kadison has shown that $PU(\mathcal{B})$ is topologically simple in these cases, with the topology defined by the norm [12, th. 2]; but he left open the "algebraic" simplicity of $PU(\mathcal{B})$, though asserting the interest of the problem (see the final remark in [12]). Kaplansky revived the question when he proved that the derived group of the projective general linear group of a factor of type II_1 is algebraically simple; but his methods do not apply to the projective unitary group ([13, appendice IV], and [14]).

The object of the present paper is to show the following

THEOREM. *If* $\mathcal{B}$ *is either of type* II_1 *or of type* III (*and countably decomposable*), *then* $PU(\mathcal{B})$ *is a simple group.*

The proof splits naturally into two parts. Let Γ be a normal subgroup of $U(\mathcal{B})$ which is not contained in the center S^1. The first part consists of checking that Γ contains at least one *involution* (namely a self-adjoint unitary) which is *not trivial* (namely neither $+1$ nor -1); this is an elaboration of the standard proof that $PSU(2) = SO(3)$ is simple. The second and easiest part consists of checking that Γ contains all involutions; this involves playing with the dimension function of the factor $\mathcal{B}$. The conclusion follows since the involutions generate all of $U(\mathcal{B})$ according to a theorem of Broise [3, th. 1], which is due independently to Fillmore in the purely infinite case [10, corollary to th. 3, which applies indeed to any properly infinite von Neumann algebra].

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On the group of rotations

We recall the standard proof that $SO(3)$ is a simple group. This will be done in a way preparing the introduction below of a continuous parameter.

We view $SO(3)$ as a compact group acting on the *unit sphere* S^2 of Euclidean space. This sphere is endowed with its usual metric, which is invariant by $SO(3)$ and for which diametrically opposite points are at a distance of π from each other. The distance $\delta(P, Q)$ between two points of S^2 is always measured on S^2, never in R^3. Any element $g \in SO(3) - \{1\}$ leaves fixed exactly two points called the *poles* of g; any point on the corresponding equator is then moved to a point at a distance of α_g, which is the *angle* of the rotation g, and which is identified to a real number in $]0, \pi]$. The set Ω of rotations with angle not zero and strictly smaller than π is homeomorphic to the complement of a point in an open 3-cell. The orientation on R^3 makes it possible to select continuously one of the two poles fixed by a rotation in Ω: this will be the *north pole* N_g of $g \in \Omega$, so that the south pole $S_g = -N_g$ is also defined.

Given two points P and Q on S^2 at a distance α from each other with $\alpha \in]0, \pi[$ there is exactly one rotation $g_{P,Q}$ with angle α which maps P onto Q, because P and Q are on a well-defined great circle. It is important to observe that $g_{P,Q}$ depends continuously on the pair (P, Q), and that the conjugacy class of $g_{P,Q}$ depends only on $\delta(P, Q)$.

Consider $g \in \Omega$ and a point P_0 on the equator between N_g and S_g. The *Archimedean property* of real numbers makes it possible to find a finite sequence $(P_j)_{1 \leq j \leq n}$ of points in S^2 with $P_n = -P_0$ and with $\delta(P_{j-1}, P_j) = \alpha_g$ for $j \in \{1, \ldots, n\}$. The following construction of these points fits our purpose.

Chose an odd integer $n = 2k+1$ with $n\alpha_g \geq \pi$. Let L be the half great circle containing P_0, $P_1 = g(P_0)$ and $P_n = -P_0$. Divide the arc of L between P_1 and P_n into k arcs of equal length; this defines $P_3, P_5, \ldots, P_{2k-1}$ with $\delta(P_{2j-1}, P_{2j+1}) = (1/k)(\pi - \alpha_g)$ for $j \in \{1, \ldots, k\}$. Choose such an integer j and let Q_j be the point half way between P_{2j-1} and P_{2j+1}. If $n\alpha_g = \pi$, define P_{2j} to be Q_j, if $n\alpha_g > \pi$, there are exactly two points on the perpendicular bisector M_j of $P_{2j-1}P_{2j+1}$ at a distance α_g from P_{2j-1}, and P_{2j} is going to be one of them. As M_j is a great circle orthogonal to L, each of these points is the image of Q_j by a rotation having M_j as equator and an angle strictly less that π; each of these rotations thus has its poles on the great circle containing L; choose P_{2j} to be the image of Q_j by the rotation which has its north pole nearer P_0 than P_n. The points $P_1, P_2, \ldots, P_n$ are now all defined; they depend only on g, on P_0 and on n.

It is elementary to check that, given two pairs (P', P'') and (Q', Q'') of points on S^2 with $\delta(P', P'') = \delta(Q', Q'')$, there is one rotation mapping P' to Q' and P'' to Q'': consider for example the product of any rotation mapping P' to Q' with a rotation for which Q' is a fixed point. Moreover, if $\delta(P', P'') < \pi$, this rotation is clearly unique.

For each $j \in \{1, \ldots, n\}$, let us describe the rotation k_j which maps P_0 onto P_{j-1} and P_1 onto P_j. There are well-defined segments of great circles on S^2 between P_0 and P_{j-1} on the one hand and between P_1 and P_j on the other hand. These have perpendicular bisectors which intersect at exactly two points of S^2. And there is one rotation k_j with these points as poles, with angle strictly less than π, which maps P_0 onto P_{j-1}. By the existence and unicity result recalled just above, k_j maps also P_1 onto P_j. Define then $h_j = k_j g k_j^{-1}$ (with $k_1 = 1$ and $h_1 = g$). Then h_j is the unique conjugate of g in $SO(3)$ which maps P_{j-1} onto P_j. The product of the h_j's maps P_0 onto $-P_0$, and is thus a *half-turn*.

It follows that any normal subgroup of $SO(3)$ containing more than one element contains one half-turn. It is straightforward that two half-turns are conjugate inside $SO(3)$ and that any rotation in $SO(3)$ is the product of two half-turns. Hence the (abstract) group $SO(3)$ is *simple*.

Let N and S be two diametrically opposite points on S^2, let ε be a real number with $0<\varepsilon\leqslant\pi/2$, and let ω be the subset of $SO(3)$ consisting of those rotations with angle in $[\varepsilon, \pi-\varepsilon]$ and with N as north pole. If n is an odd integer with $n\varepsilon\geqslant\pi$, the construction above can be made simultaneously for all rotations in ω; this provides n-tuples of continuous functions

$$\begin{cases}\omega\to S^2\\ g\mapsto P_j(g)\end{cases}\qquad \begin{cases}\omega\to SO(3)\\ g\mapsto h_j(g)\end{cases}\qquad \begin{cases}\omega\to SO(3)\\ g\mapsto k_j(g)\end{cases}$$

with the following properties: for each $j\in\{1,\ldots,n\}$, the rotation $h_j(g)=k_j(g)gk_j(g)^{-1}$ maps $P_{j-1}(g)$ to $P_j(g)$. Hence the product of the $h_j(g)$'s maps P_0 to $-P_0$ for each $g\in\omega$. We have essentially proved the fact formalized in Lemma 1 below.

Consider the covering $\tau:S^1\to S^1$ which multiplies angles by two. We assume in Lemma 1 that the topological space T has the following property; for any continuous map $f:T\to S^1$, there is a lifting $F:T\to S^1$ with $\tau F=f$. For example, any space with vanishing Cech cohomology group $\check{H}^1(T, Z)$ qualifies.

LEMMA 1. *Let T be a compact space with the property above, let $SO(3, T)$ denote the group of all continuous maps from T to $SO(3)$ with pointwise multiplication, and let Γ be a normal subgroup of $SO(3, T)$. Suppose Γ contains an element γ with the following properties: the angle $\alpha(t)$ of $\gamma(t)$ is in $]0,\pi[$ for each $t\in T$ and the north pole of $\gamma(t)$ does not depend on t. Then Γ contains any constant map.*

Proof. The map α being continuous and the space T compact, there exists $\varepsilon\in]0,\pi/2]$ with $\varepsilon\leqslant\alpha(t)\leqslant\pi-\varepsilon$ for all $t\in T$. The argument above shows that there exists also $\kappa\in SO(3, T)$ with $\kappa(t)$ moving some point P_0 (independent of t) to its opposite for each $t\in T$. In cartesian coordinates with P_0 on the first axis, this is expressed by the fact that

$$\kappa(t)=\begin{pmatrix}-1 & 0 & 0\\ 0 & \cos\theta(t) & \sin\theta(t)\\ 0 & \sin\theta(t) & -\cos\theta(t)\end{pmatrix}$$

for all $t\in T$, where $\theta:T\to S^1$ is some continuous function. (For each $t\in T$, there is one line in the plane spanned by the second and the third axis which is fixed by $\kappa(t)$; if the second axis and this line define the angle $\varphi'(t)$, then $\theta(t)=2\varphi'(t)$; note that there is no a priori choice between $\varphi'(t)$ and $\varphi'(t)\pm\pi$, but that $\theta(t)$ is well-defined.)

Let $\varphi:T\to S^1$ be a continuous function with $2\varphi(t)=\theta(t)$ (here does enter the assumption on T). Define $\rho \in SO(3, T)$ by

$$\rho(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & \sin\varphi(t) \\ 0 & -\sin\varphi(t) & \cos\varphi(t) \end{pmatrix}$$

for all $t \in T$. It is routine to check that $\rho\kappa\rho^{-1}$ is the constant map onto

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

As Γ contains one constant map with value a half-turn, it contains also any constant map with value a half-turn, hence Γ contains all constant maps.

The special unitary group in a homogeneous von Neumann algebra of type I_2

In what follows, T is a compact space which has the property stated just before Lemma 1, and $\mathcal{A}$ is the abelian C^*-algebra of continuous maps from T to the complex numbers. The C^*-algebra $\mathcal{M}$ of continuous maps from T to the matrix algebra $M_2(C)$ will be identified with the algebra of (2×2)-matrices with entries in $\mathcal{A}$. We shall consider the subgroup $SU(2, T)$ of the unitary group of $\mathcal{M}$ which consists of all continuous maps from T to $SU(2)$. The maps with values in $\{+1, -1\}$ define a central subgroup of $SU(2, T)$; we do not assume that T is connected and this group may have more than two elements. We identify the associated quotient with the group $SO(3, T)$ defined in Lemma 1 (this is possible since any continuous map from T to $SO(3)$ lifts to $SU(2)$ by hypothesis on T). The canonical epimorphisms $SU(2) \to SO(3)$ and $SU(2, T) \to SO(3, T)$ are both denoted by p.

We assume moreover that T is a *stonean space*; this means that the closure of any open set is again an open set. This happens for example if T is the Gelfand spectrum of an abelian von Neumann algebra $\mathcal{A}$; in this case, $\mathcal{M}$ is also a von Neumann algebra which is called *homogeneous of type* I_2. It is elementary to check that T being stonean implies $\check{H}^1(T, Z) = \{0\}$, so that Lemma 1 applies.

LEMMA 2. *Let $\tilde{\Gamma}$ be a normal subgroup of $SU(2, T)$. Suppose $\tilde{\Gamma}$ contains an element $\tilde{\gamma}$ such that $\gamma = p(\tilde{\gamma})$ maps any $t \in T$ to a rotation $\gamma(t)$ of angle in $]0, \pi[$. Then $\tilde{\Gamma}$ contains the constant map with value -1.*

Proof. As T is stonean, theorem 2 in [9] shows that $\tilde{\gamma}$ is conjugate within $SU(2, T)$ to an element which maps any $t \in T$ to a diagonal matrix. It follows then from Lemma 1 that the image Γ of $\tilde{\Gamma}$ by p contains any constant map, and in particular that which applies T onto

$$p\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = p\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO(3).$$

Hence there is an element $\tilde{\kappa} \in \tilde{\Gamma}$ and a partition $T' \cup T''$ of T in two disjoint open sets such that

$$\tilde{\kappa}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

if $t \in T'$ and

$$\tilde{\kappa}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $t \in T''$. Lemma 2 follows because $\tilde{\kappa}^2$ is in $\tilde{\Gamma}$.

LEMMA 3. *Let $\tilde{\Gamma}$ be a normal subgroup of $SU(2, T)$ which contains more than one element. Then there exist $\tilde{\rho} \in \tilde{\Gamma}$ and $X \in \mathcal{M} - \{0\}$ with $\tilde{\rho}X = -X$.*

Proof. Let $\tilde{\gamma} \in \tilde{\Gamma}$ with $\tilde{\gamma} \neq 1$ and let $\gamma = p(\tilde{\gamma})$.

Suppose first that $\gamma = 1$. Then there is a partition $T' \cup T''$ of T in disjoint open sets such that $\tilde{\gamma}(t) = 1$ if $t \in T'$ and $\tilde{\gamma}(t) = -1$ if $t \in T''$; as $\tilde{\gamma} \neq 1$ the set T'' is not empty. Define $\tilde{\rho} = \tilde{\gamma}$ and $X \in \mathcal{M}$ by $X(t) = 0$ if $t \in T'$ and $X(t) = 1$ if $t \in T''$.

Suppose next that $\tilde{\gamma}$ is such that $\gamma(t)$ is a half-turn for t in some non empty (open and closed) subset T_1 of T and is the identity for $t \notin T_1$. One shows as at the end of the prooof of Lemma 1 that $\tilde{\Gamma}$ contains a map $\tilde{\kappa}$ with $\kappa = p(\tilde{\kappa})$ having the following properties: $\kappa(t)$ is a constant half-turn when $t \in T_1$ and is the identity if $t \notin T_1$. Define $\tilde{\rho} = \tilde{\kappa}^2$, so that

$$\tilde{\rho}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

if $t \in T_1$, and chose for X any non zero map which restricts to zero outside T_1.

Suppose finally that there exists $t_0 \in T$ with the angle of $\gamma(t_0)$ neither 0 nor π. Then there exists $\varepsilon \in]0, \pi/2[$ and an open and closed neighbourhood T_1 of t_0 such that the angle of $\gamma(t)$ is in $[\varepsilon, \pi-\varepsilon]$ for each $t \in T_1$. One may then apply Lemma 2 above T_1. As there is no obstruction to extend maps defined on T_1 to all of T, the assertion to be proved is again correct in this case.

Involutions in non central normal subgroups of $U(\mathcal{B})$

We shall now connect what we have established about $SU(2, T)$ with unitary groups defined by factors.

Consider an infinite dimensional factor $\mathcal{B}$ and its unitary group $U(\mathcal{B})$. The following fact is an easy corollary of the spectral theorem: let $g \in U(\mathcal{B})$ and let n be a positive integer; then there exist k orthogonal equivalent projections $P_1, \ldots, P_n$ in $\mathcal{B}$ commuting with g and adding up to 1.

Indeed, let $g = \int_0^{2\pi} \exp(i\varphi)\, dE_\varphi$ be the spectral decomposition of g[15, n° 109]. Say first that $\mathcal{B}$ is finite. Let ψ be the smallest number in $[0, 2\pi]$ with the dimension of E_ψ in $\mathcal{B}$ being at least $1/n$. If $\dim(E_\psi) = 1/n$, let $P_1 = E_\psi$. If $\dim(E_\psi) > 1/n$, let F be any projection in $\mathcal{B}$ of dimension $(1/n) - \dim(E_{\psi-0})$ which is majorized by $E_\psi - E_{\psi-0}$ and let $P_1 = E_{\psi-0} + F$. Then P_1 commutes with g and has dimension $1/n$. Define similarly $P_2, \ldots, P_n$, orthogonal and commuting with g. As $P_1, \ldots, P_n$ have the same dimension, they are equivalent in $\mathcal{B}$; as their dimensions add up to 1, their sum is the identity. One may proceed similarly when $\mathcal{B}$ is infinite.

Suppose moreover that g is not a multiple of the identity and that $n \geq 2$; it is important to notice that $P_1, \ldots, P_n$ are not all associated to the same portion of the spectrum of g, so that $P_1 g, \ldots, P_n g$ are not all unitarily equivalent. This construction of the P_j's overlaps partly with lemmas 3 and 4 in [3].

LEMMA 4. *Let Γ be a normal subgroup of $U(\mathcal{B})$ which is not contained in the center S^1. Then there exist $k \in \Gamma$ and $X, Y \in \mathcal{B} - \{0\}$ with $kX = X$ and $kY - Y$.*

Proof. Choose $g \in \Gamma$ with $g \notin S^1$. Let P_1, P_2, P_3 be three equivalent orthogonal projections commuting with g and adding up to the identity. Define $g_j = gP_j$ $(j = 1, 2, 3)$; as g is not central, one may assume that g_2 and g_3 are not unitarily equivalent. It may help to think of g as being the matrix

$$\begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}.$$

Let W be a partial isometry in $\mathcal{B}$ with initial projection P_3 and with final projection P_2, which corresponds to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $V = P_1 + W + W^*$, then V is in $U(\mathcal{B})$ and $h = g^* V g V^*$ is an element in Γ which commutes with the P_j's. Let $h_2 = g_2{}^* W g_3 W^*$ and $h_3 = g_3{}^* W^* g_2 W$ then $h_2 \neq P_2$ and $h_3 \neq P_3$ since g_2 and g_3 are not unitarily equivalent; notice that $h_3 = W^* h_2{}^* W$. One may think of h as being the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}.$$

Let $\mathcal{A}$ be the (abelian) von Neumann algebra generated by h and let $\mathcal{M} = \mathcal{A} \otimes M_2(C)$ be as before Lemma 2. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto aP_2 + bW + cW^* + dP_3$$

defines a normal isomorphism from $\mathcal{M}$ onto a subalgebra of the reduction of $\mathcal{B}$ to $\mathcal{B}_{P_2+P_3}$ (notations as in [7, chap. I, §2, n° 1]). We identify $\mathcal{M}$ with its image; if T is the spectrum of $\mathcal{A}$, this identifies $SU(2, T)$ to a subgroup of $U(\mathcal{B})$.

Now $\{\tilde{\gamma} \in SU(2, T) \mid P_1 + \tilde{\gamma} \in \Gamma\}$ is a normal subgroup of $SU(2, T)$ which contains h, and the conclusion follows from Lemma 3 (with, for example, $X = P_1$).

PROPOSITION 1. *Let $\mathcal{B}$ be a factor (not of dimension 1 or 4), let $U(\mathcal{B})$ be the group of all unitary elements of B, and let Γ be a normal subgroup of $U(\mathcal{B})$ which is not contained in the center S^1. Then Γ contains a non trivial involution.*

Proof. Notice that the proposition is classical for $\mathcal{B} = M_n(C)$ with $n \geq 3$, and assume from now on that $\mathcal{B}$ is infinite dimensional.

Let H be the Hilbert space associated to some faithful finite state on $\mathcal{B}$ by the Gelfand–Naimark–Segal construction. As H is a completion of $\mathcal{B}$, Lemma 4 shows that Γ contains some k with both $+1$ and -1 in its point spectrum. The projections from H onto $\operatorname{Ker}(k-1)$ and $\operatorname{Ker}(k+1)$ are thus non zero, orthogonal elements of $\mathcal{B}$. It follows that there exist an integer $n \geq 3$ and a family

$P_1, \ldots, P_n$ of orthogonal equivalent projections commuting with k, adding up to the identity, with $P_1(H) \subset \operatorname{Ker}(k-1)$ and $P_2(H) \subset \operatorname{Ker}(k+1)$.

One may furthermore find matrix units $(E_{i,j})_{1 \leq i,j \leq n}$ in $\mathscr{B}$ with $E_{j,j} = P_j$ $(j = 1, \ldots, n)$, so that each element in $\mathscr{B}$ can be identified with a $(n \times n)$-matrix having its entries in $P_1 \mathscr{B} P_1$. In particular

$$k = \begin{pmatrix} 1 & & & & & \\ & -1 & & & 0 & \\ & & k_3 & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & & \cdot \\ & & & & & & k_n \end{pmatrix}$$

Now permutation matrices are in $U(\mathscr{B})$. As Γ is normal, the product

$$\begin{pmatrix} 1 & & & & & & \\ & -1 & & & & & \\ & & k_3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & k_n \end{pmatrix} \begin{pmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & k_3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & k_n \end{pmatrix}^* = \begin{pmatrix} -1 & & & & & & \\ & -1 & & & & & \\ & & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{pmatrix}$$

is also in Γ.

This ends the first part of the proof of the main theorem, as described in the introduction.

End of proof of the main result

Let $\mathscr{B}$ be a factor and let D be a normalized relative dimension on $\mathscr{B}$; see [7, chap. III, §2, prop. 14]. Let J be an involution in $\mathscr{B}$; it can be written $J = 1 - 2E$

with E a well-defined projection. The *type* of J is the pair (p, q) with $p=D(1-E)$ and $q=D(E)$. If $\mathscr{B}$ is continuous and finite, $p+q=1$; if $\mathscr{B}$ is infinite and semi-finite, $p+q=\infty$; if $\mathscr{B}$ is purely infinite and if J is not trivial, $p=q=\infty$.

LEMMA 5. *Let $\mathscr{B}$ be a countably decomposable factor and let J, K be two involutions in $\mathscr{B}$. Then J and K are conjugate in $U(\mathscr{B})$ if and only if they are of the same type.*

Proof. This follows from well-known facts on projections. See [7, chap. III, §2 and corollary 5 of §8].

PROPOSITION 2. *The projective unitary group of a purely infinite and countably decomposable factor is simple.*

Proof. Let $\mathscr{B}$ be a factor of type III and let Γ be a normal subgroup of $U(\mathscr{B})$ which is not contained in S^1. Then Γ contains a non trivial involution by proposition 1, so that Γ contains all involutions by Lemma 5. It follows that $\Gamma=U(\mathscr{B})$: see Broise [3, th. 1] or Fillmore [10, corollary to th. 3].

LEMMA 6. *Let $\mathscr{B}$ be a factor of type* II *and let E be a projection in $\mathscr{B}$ with $E\neq 0$ and $E\neq 1$. Let r be a real number with $0<r\leq D(E)$ and $r\leq D(1-E)$. Then there exists $V\in U(\mathscr{B})$ such that $F=EVEV^*$ is a projection with $D(F)=D(E)-r$ and $D(1-F)=D(1-E)+r$.*

Proof. Let P be a projection in $\mathscr{B}$ with $D(P)=r$ and $P\leq E$ (such a P exists by [7, chap. III, §2]). Let Q be a projection in $\mathscr{B}$ with $D(Q)=r$ and $Q\leq 1-E$. As P and Q are equivalent, there exists a partial isometry S in $\mathscr{B}$ with $S^*S=P$ and $SS^*=Q$; as P and Q are orthogonal, one has $S^2=SQ=PS=0$.

Define $W=E-P+S+S^*=W^*$. It is routine to check that $W^2=E+Q$, so that $V=W+(1-E-Q)$ is an involution in $\mathscr{B}$. It is again routine to check that $VEV=E-P+Q$, so that $F=EVEV$ is a projection of the desired type.

Notice that Lemma 6 is empty if $\mathscr{B}$ is of type II_∞ and if both E and $1-E$ have infinite dimension. But the same trick shows in this case that one can find $V\in U(\mathscr{B})$ with $F=EVEV$ a projection of any desired type.

PROPOSITION 3. *The projective unitary group of a finite continuous factor is simple.*

Proof. Let Γ be a normal subgroup of $U(\mathscr{B})$ not contained in S^1, with $\mathscr{B}$ of type II_1. Then Γ contains a non trivial involution, hence an involution of any given type by Lemma 6, hence all involutions by Lemma 5. It follows from Broise's theorem that $\Gamma=U(\mathscr{B})$.

COROLLARY 1. *The unitary group $U(\mathcal{B})$ of a finite continuous factor admits no non trivial finite dimensional unitary representation.*

Proof. Consider commutative sets of involutions. These sets have at most 2^n elements in $U(n)$ but their cardinals are not bounded in $U(\mathcal{B})$. It follows that any homomorphism $\varphi : U(\mathcal{B}) \to U(n)$ has a non trivial kernel, and so is the trivial homomorphism. When φ is moreover assumed to be uniformly continuous, see [12, th. 1].

COROLLARY 2. *Let $\mathcal{B}$ be a continuous, infinite and semi-finite factor; let Γ be a normal subgroup of $U(\mathcal{B})$ which is not contained in S^1. Then Γ contains all unitaries g for which there exists a finite projection $E_g \in \mathcal{B}$ satisfying $g-1=E_g(g-1)E_g$.*

Proof. The argument used above shows that Γ contains an involution of type (p, q) in $\mathcal{B}$ as soon as $p<\infty$. If E is any finite projection in $\mathcal{B}$, it is easy to check that the reduction of $\mathcal{B}$ to $\mathcal{B}_E$ it a factor (this follows for example from [7, chap. I, §1, prop. 7, cor. 3]). As Γ contains an involution of $\{g \in U(\mathcal{B}) \mid g-1 \in \mathcal{B}_E\}$ which is neither 1 nor $1-2E$, Proposition 3 shows that Γ contains this group.

The analogous statement for a discrete, infinite and semi-finite factor is proposition 3(i) of [11]. A similar statement holds with $\mathcal{B}$ a factor of type III which is not countably decomposable (we are grateful to M. Broise for this remark).

COROLLARY 3. *Countably decomposable factors of types* II_1 *and* III *are simple.*

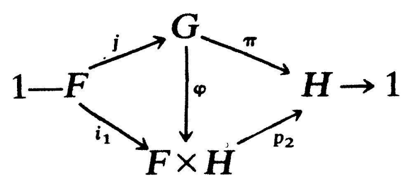
Proof. See the introduction.

COROLLARY 4. *Let $\mathcal{R}$ be the hyperfinite factor of type* II_1. *The group of *-automorphisms of $\mathcal{R}$ has exactly one non trivial normal subgroup, which is the group of inner *-automorphisms.*

Proof. Let us call a short exact sequence

$$1 \longrightarrow F \xrightarrow{\ j\ } G \xrightarrow{\ \pi\ } H \longrightarrow 1$$

of groups and homomorphisms trivial if there exists an isomorphism φ such that



commutes (with i_1 and p_2 the canonical injection and projection respectively). The following is an exercise for pedestrians in group theory: in a non trivial short exact sequence as above with F and H simple, the only non trivial normal subgroup of G is F. (Indeed: let N be a normal subgroup in G with $N \not\subset F$ and suppose there is $f \in F$ and $n \in N$ with $fn \neq nf$; then $nfn^{-1}f^{-1}$ is in $(F \cap N) - \{1\}$, so that $F \subset N$; as $N \not\subset F$ one has $\pi(N) = H$; it follows that $N = G$.)

Corollary 3 follows now from proposition 3 because the group of inner *-automorphisms of $\mathcal{R}$ is $PU(\mathcal{R})$ and because the quotient Out $(\mathcal{R})$ of the group of *-automorphisms of $\mathcal{R}$ by $PU(\mathcal{R})$ is simple by a theorem due to Connes [5, cor. 4]. (That the short exact sequence of concern here is non trivial is an easy fact, left to the reader.)

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