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A Landesman–Lazer alternative theorem for a class of optimization problems

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In [1] we proved an alternative theorem for the existence of minima of functionals F_l defined by $F_l(u)=F(u)-(l, u)$ on, say, a reflexive Banach space B. Here $l\in B^*$ and F satisfies the following conditions

(1) $F:B\to\mathbf{R}\cup\{\infty\}$ *is lower semi-continuous in the weak topology of* B.
(2) F *is of polynomial type, i.e. if for some pair* $v, w\in B$

(i) $\sup t^{-1}F(w+tv)<\infty \qquad (t>t_0>0)$ *and*
(ii) $\inf F(w+tv)>-\infty \qquad (t\in\mathbf{R})$

then $F(w+tv)$ *is constant in* t.
(3) F *satisfies a surrogate convexity condition, i.e. for all* $u, w\in B, \alpha\in[0, 1]$, $F(w)<\infty$

$$F(1-\alpha)w+\alpha u)\le K_w+K_w\alpha F(u)$$

with some constant K_w.
(4) F *is semi-coercive, i.e. there exists a continuous projection* $Q:B\to V$ *onto a finite dimensional subspace* V *such that for all* $K>K_0$

$$\sup\{\|u\|/(1+\|Qu\|)\mid F(u)\le K\}<\infty$$

(5) F *is bounded from below on* B.

Introducing the set

$$D=\{v\in B\mid F(w+tv) \text{ is constant in } t\in\mathbf{R} \text{ for all } w\in B,\ F(w)<\infty\}$$

we obtained the following

THEOREM 0. *Under the conditions* (1)–(5) *the functional* F_l *has a minimum if and only if* $l\perp D$. *Furthermore,* $\dim D\le\dim V$.

A simple corollary (cf. [2], §1) yields that if in addition F has a Gateaux derivative $T: B \to B^*$ then the range of T is linear and has finite co-dimension.

In this paper we consider pertubations $F_l + G$ of the above functionals F_l where G has a so called *weak sub-asymptote* (cf. definition below). It then turns out that the set of $l \in B^*$ for which $F_l + G$ has a minimum becomes "thicker", i.e. is open and contains the closed set D. Under additional conditions we can characterize these elements l in the form of an alternative theorem. Our results are in the spirit of the "classical" Landesman–Lazer-alternative theorems, cf. Landesman–Lazer [8]. For a rather complete list of references to this subject, cf. [9] and also [4]. These theorems state that the range of perturbed semi-coercive linear differential operators like $-\Delta u - \lambda u + \operatorname{arctg} u$ subject to boundary conditions is open and can be characterized by the asymptotes of the perturbation.

If $F_l + G$ is Gateaux-differentiable our result yields a Landesman–Lazer-type theorem of the usual form but covers cases with strongly non-linear principal part of polynomial type. Our proof of this result is very simple. The theorem has a non-variational analogue which was presented in [3]. A Landesman–Lazer theorem for a class of equations with strongly non-linear principle part was presented by Hess in [5], [6], [7]. His approach and his results are rather different from the setting in the present paper.

DEFINITION. A mapping $a_0: D \to \mathbf{R}$ is called a weak sub-asymptote of the mapping $G: B \to \mathbf{R}$ if for every sequence $(u_i \in B,\ i = 1, \ldots,)$ with $\|u_i\| \to \infty$, $\|u_i\|^{-1} u_i \to v \in D$ weakly $(i \to \infty)$, $v \neq 0$, we have

$$a_0(v) \leq \liminf \|u_i\|^{-1} G(u_i) < \infty \qquad (i \to \infty) \tag{6}$$

We shall also assume for the pertubation G

$$\sup \|u\|^{-1} G(u) < \infty \qquad (u \in B, \quad u \neq 0) \tag{7}$$

and

$$F + G \text{ is lower semi-continuous in the weak topology of } B. \tag{8}$$

THEOREM 1. *Let B be a reflexive Banach space and $F: B \to \mathbf{R} \cup \infty$, $G: B \to \mathbf{R}$ mappings such that G has a weak sub-asymptote $a_0: D \to \mathbf{R}$ and F, G satisfy* (1)–(5), (7) *and* (8).

Then the functional $\Phi_l: B \to \mathbf{R}$ defined by

$$\Phi_l(u) = F(u) + G(u) - (l, u)$$

has a minimum on B for all $l \in B^*$ for which

$$(l, v) < a_0(v), \qquad v \in D, \qquad v \neq 0. \tag{9}$$

If in addition

$$\liminf t^{-1}(G(w+tv)-G(w)) < a_0(v) \qquad (t \to +0) \tag{10}$$

for all $w \in B$ and $v \in D, v \neq D$, then (9) is also necessary for the existence of a minimum of Φ_l.

Proof. We may assume that $F \neq \text{const} = \infty$. Let (u_i) be a minimizing sequence for Φ_l. Suppose that (u_i) were unbounded. Then we may assume that $\|u_i\| \to \infty$ and that $\|u_i\|^{-1}u_i := v_i \to v$ weakly in $B (i \to \infty)$. By (4) there is a constant $C > 0$ such that $\|u_i\| \leq C + C\|Qu_i\|$ and hence $1 \leq C \liminf \|Qv_i\|$ $(i \to \infty)$. Since $\dim QB < \infty$ we have $Qv_i \to Qv$ strongly and, therefore, $1 \leq C\,\|Qv\|$, and

$$v \neq 0.$$

We intend to show that $v \in D$. By the convexity condition (3)

$$F(1-\alpha)w + \alpha u_i) \leq K_w + \alpha K_w F(u_i) = K_w + \alpha K_w[\Phi_l(u_i) - G(u_i) + (l, u_i)]$$

for all w such that $F(w) < \infty$.

Since $\|u_i\| \to \infty$ we may set $\alpha = t\,\|u_i\|^{-1}$ for $t > 0$, $i > i(t)$. Passing to the limit $i \to \infty$ we obtain in view of the lower semi-continuity of F.

$$F(w+tv) \leq K_w - tK_w \liminf \|u_i\|^{-1} G(u_i) + tK_w(l, v) \tag{11}$$

and by (7)

$$t^{-1}F(w+tv) \leq K_{wl}, \qquad t > 1.$$

From (5) and (2) we then conclude that $F(w+tv)$ is constant in $t \in \mathbf{R}$ for all w with $F(w) < \infty$ and hence

$$v \in D.$$

Since G has a weak sub-asymptote $a_0 : D \to \mathbf{R}$, we obtain from (11)

$$(F(w) =) F(w+tv) \leq K_w - tK_w a_0(v) + tK_w(l, v)$$

and passing to the limit $t \to \infty$ we arrive at the inequality

$$a_0(v) \leq (l, v).$$

This contradicts (9) and hence the assumption of (u_i) being unbounded leads to a contradiction. The first statement follows in view of (8) and the reflexivity of B. The necessity of (9) can be seen from the following simple argument: If u is a minimum of Φ_l on B and if $v \in D$, then

$$F(u) + G(u) - (l, u) \leq F(u + tv) + G(u + tv) - (l, u + tv)$$

$$G(u) \leq G(u + tv) - t(l, v)$$

since $F(u + tv)$ is constant in t. Hence, by (10),

$$(l, v) \leq \liminf t^{-1}(G(u + tv) - G(u)) < a_0(v) \qquad (t \to +0)$$

as claimed. The theorem is proved.

EXAMPLES. In the following let Ω be a bounded connected open set in $\mathbf{R}^n$ and $H^{1,p}$ the usual Sobolev space over Ω. The corresponding Sobolev space of r-vector functions is denoted by $[H^{1,p}]^r$.

(i) Let $B = H^{1,p}(\Omega)$, $l \in B^*$, and

$$F(u) = \frac{1}{p} \int |\nabla u|^p \, dx, \qquad G(u) = \int [u \operatorname{arctg} u - \tfrac{1}{2} \ln (1 + |u|^2)] \, dx$$

Here and in the following $\int$ denotes integration over Ω. Then $D = \{z \in H^{1,p} \mid z = \text{const.}\}$ and $a_0(v) = \pi/2 \int |v| \, dx$. Since F and G satisfy the hypotheses of Theorem 1, cf [1], §3, the functional F_l defined by

$$F_l(u) = F(u) + G(u) - (l, u)$$

has a minimum when

$$|(l, 1)| < \frac{\pi}{2} |\Omega|. \tag{12}$$

The minimum u of F_l is a weak solution of the differential equation.

$$-\sum_{i=1}^{n} \partial_i(\partial_i u\,|\nabla u|^{p-2})+\operatorname{arctg} u = l. \tag{13}$$

It is a simple exercise to prove that the above functional G satisfies condition (10); Hence (12) is also necessary for the existence of a minimum of F_l. The characterization of the range of the operator on the left hand side of (13) can be obtained using the methods of Peter Hess, cf. references.

A non-differentiable variant of this example is obtained when $F(u)$ is replaced by

$$F(u)=\int\left[\frac{1}{p}|\nabla u|^p+|\nabla u|\right]dx$$

and condition (12) remains as the necessary and sufficient condition for the existence of minima of the functional F_l.

(ii) Let $B=[H^{1,p}(\Omega)]^2$, $l\in B^*$ and

$$F(u)=\int[|\nabla u_1|^p+u_1^p+\lambda \sin u_1+u_1\,\partial_1 u_2+|\nabla u_2|^p]\,dx, \qquad u=(u_1, u_2),$$

$$G(u)=\int[u_2 \operatorname{arctg} u_2-\tfrac{1}{2}\ln(1+|u_2|^2)]\,dx.$$

Again, F and G satisfy the hypotheses of the theorem. The surrogate convexity of F follows by splitting the integrand into a sum of convex and bounded functions. The set D has the form

$$D=\{(0, c)\in[H^{1,p}(\Omega)]^2 \mid c\in\mathbf{R}\}.$$

The functional F_l, $l=(l_1, l_2)\in B^*$ has a minimum if and only if

$$|\langle l_2, 1\rangle| < \frac{\pi}{2}|\Omega|.$$

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