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On division by inner factors

by J. M. ANDERSON

Professor Albert Pfluger Zugeeignet

§1. Introduction

Let G denote a Banach space of function $f(z)$ analytic in the open disc $|z| < 1$, and suppose that G is contained within the Hardy space H^1 . If $h \in H^1$ we may write $h(z) = 0(z) \cdot I(z)$ where $0(z)$ is the outer factor of h , $0(z) \in H^1$ and $I(z)$ is the inner factor, which is a function in H^∞ for which $\lim_{r \rightarrow 1} f(re^{i\theta})$ exists and is of modulus one almost everywhere on $|z| = 1$. Such an inner factor may be written as

$$I(z) = B(z) \cdot S(z),$$

where $B(z)$ is the Blaschke product formed from the zeros of $h(z)$, and $S(z)$ is a singular inner factor of the form

$$S(z) = \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} d\mu(x) \right\}. \quad (1)$$

Here, $\mu(x)$ is a monotonic increasing function on $(0, 2\pi]$ which is singular with respect to Lebesgue measure on $(0, 2\pi]$. For the details of such a factorization and other properties of H^p spaces we refer to [3].

DEFINITION 1. A Banach space G , contained within H^1 is said to have the f -property if, given any $h \in G$ and any inner function I which divides h (i.e. $hI^{-1} \in H^1$) then $hI^{-1} \in G$.

This notation was introduced by Havin [6], reporting on earlier work by numerous authors [2], [9], [10], [11]; see also [15] and the lecture by Kahane [8]. In addition to the obvious spaces H^p ($p > 1$) many other spaces, such as various Lipschitz spaces, have the f -property.

If $\phi(z) \in H^\infty$ and $h(z) \in H^2$ we may write

$$\phi(e^{i\theta})h(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

and define the projection operator $P(\bar{\phi}h)$ by

$$P(\bar{\phi}h) \sim \sum_0^{\infty} a_n e^{in\theta}.$$

(i.e. P is the Toeplitz operator associated with $\bar{\phi} \in L^\infty$). Clearly

$$P(\bar{\phi}h)(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{\bar{\phi}(w)h(w)}{w-z} dw, \quad |z| < 1.$$

DEFINITION 2. A Banach space G contained in H^1 is said to have the K -property if given any $h \in G$ and $\phi \in H^\infty$ then $P(\bar{\phi}h) \in G$.

This property appears to have been considered first by Korenblum [9]. Its usefulness consists in the fact that if ϕ is an inner function dividing $h \in G$ then $P(\bar{\phi}h) = h\phi^{-1}$. Thus a space having the K -property also possesses the f -property.

The first example of a space not possessing the f -property and hence not possessing the K -property was given by Gurarii [5]. The space in question is l^1 (called W^\dagger by Gurarii), defined by

$$l^1 = \left\{ f : f(z) = \sum_0^{\infty} f_n z^n, \|f\| = \sum_0^{\infty} |f_n| < \infty \right\}.$$

The particular function considered is

$$h(z) = \frac{\pi}{2} \Gamma^2\left(\frac{3}{8}\right) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k + \frac{3}{8})}{(k + \frac{3}{8}) \Gamma(k + \frac{5}{8})} z^{(8k+3)^2}.$$

This function clearly belongs to l^1 but if we write

$$f(z) = \exp \left\{ \gamma \frac{1+z}{1-z} \right\} \cdot h(z),$$

where $\gamma = 2^{-9} \pi^2$ then $f \in H^1$ but $f \notin l^1$, ([5] p. 29).

The main difficulty in Gurarii's proof is in detecting the presence of the "jump" inner factor $\exp\{-\gamma \frac{1+z}{1-z}\}$ in $h(z)$. Gurarii does not assert that the resulting function $f(z)$ is outer—there might be other inner factors. It seems very difficult in general to detect the presence of singular inner factors of the form (1) when the representing function $\mu(t)$ is continuous.

§2. Results

In this paper we present another example of a subspace X of H^1 which does not have the f -property. We construct a function $F \in X$ such that

$$F(z) = \Phi(z) \cdot S(z),$$

where $\Phi(z)$ is an outer function not in X and $S(z)$ is an inner function of the form (1) with $\mu(t)$ a continuous, monotonic, increasing function of t , which is singular with respect to Lebesgue measure on $(0, 2\pi]$. This direct construction avoids the difficulty of showing the presence of an inner factor.

We denote by B the Banach space of Bloch functions, i.e. functions $f(z)$ analytic in $|z| < 1$ for which the norm

$$\|f\| = |f(0)| + \sup_{|z| < 1} (1 - |z|^2) |f'(z)|$$

is finite. All polynomials belong to B and the closure of the polynomials in the Bloch norm is denoted by B_0 . Alternatively, B_0 consists of those $f \in B$ for which

$$(1 - |z|^2)f'(z) \rightarrow 0 \quad \text{as} \quad |z| \rightarrow 1-.$$

It is evident that $H^\infty \subset B$, but $H^\infty \not\subset B_0$ —for these and other properties of Bloch functions we refer to [1]. The space X we wish to consider is

$$X = H^\infty \cap B_0.$$

It is easy to verify that if f and g belong to X so does $f \cdot g$ so that, with a suitable choice of norm, X is a subalgebra of H^∞ .

THEOREM 1. *X does not have the f -property (and so does not have the K -property).*

There is an interesting characterization of X which appears to be due, in the first instance, to M. Behrens. Let M denote the maximal ideal space of H^∞ and let \hat{f} denote the transform of f in the standard Gelfand representation. Then $H^\infty \cap B_0$ consists precisely of all those $f \in H^\infty$ such that f is constant on all the non-trivial Gleason parts of M other than the image of the disc $|z| < 1$. The parts are, of course, either single points or analytic discs, and the proof is a simple consequence of work of Hoffman [7]. The space X is thus a closed subalgebra of H^∞ , a fact which can also be verified directly.

§3. Interference

To construct the function in X whose outer part (which belongs to H^∞ and so to B) does not belong to B_0 , we require the following theorem of Shapiro [14 Theorem 4] regarding interference.

THEOREM A. *Let $\phi(x)$ be any increasing function on $[0, 2\pi]$ having continuous second derivative on the open interval $(0, 2\pi)$. Then there exists a function $f(x)$ continuous on $[0, 2\pi]$ such that*

$$\mu(x) = \phi(x) + f(x) \tag{2}$$

is increasing and singular,

$$|f(x+t) + f(x-t) - 2f(x)| < Kt|\log t|^{-\frac{1}{2}}, \tag{3}$$

for some absolute constant K and all x, t satisfying $0 < t < \frac{1}{2}$, $0 \leq x-t < x+t \leq 2\pi$.

Clearly $\mu(x)$ is continuous. As regards the function f , it is sufficient for our purposes to note that (3) implies that f belongs to the Zygmund class λ_* . Thus, if we define the function $F(z)$ by

$$F(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} df(x) \right\}, \tag{4}$$

then

$$F'(z) = 0(1 - |z|^2)^{-1}, \quad |z| \rightarrow 1-,$$

and so $F \in B_0$. (See [4] Theorem 1 for details.)

It is convenient to base the proof of Theorem 1 on another theorem, which seems of independent interest.

THEOREM 2. *There exists a monotonic increasing, infinitely differentiable function $\phi(x)$ defined for $0 < x < 2\pi$ such that the function*

$$\Phi(z) = \exp \left\{ + \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} d\phi(x) \right\} \quad (5)$$

belongs to H^∞ , but not to B_0 .

Proof of Theorem 1. Let $\phi(x)$ be as in Theorem 2 and $f(x)$ as in Theorem A. Then $\mu(x) = \phi(x) + f(x)$, or $f(x) = \mu(x) + (-\phi(x))$ with $\mu(x)$ singular and $-\phi(x)$ absolutely continuous with respect to Lebesgue measure. Thus

$$\begin{aligned} F(z) &= \exp \left\{ - \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} df(x) \right\} \\ &= \exp \left\{ + \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} d\phi(x) \right\} \cdot \exp \left\{ - \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} d\mu(x) \right\} \\ &= \Phi(z) \cdot S(z). \end{aligned}$$

Since $\Phi(z)$ and $S(z)$ belong to H^∞ so does $F(z)$, and $\Phi(z)$ is the outer part of $F(z)$. Thus $F(z) \in H^\infty \cap B_0 = X$ by (3) but, by construction, $\Phi(z)$ does not belong to B_0 . Thus X does not have the f -property.

§4. Proof of Theorem 2

We construct the required function $\phi(x)$ satisfying $\phi'(x) \leq 1$ for all x . This will imply that $|\Phi(z)| \leq e^{2\pi}$ for $|z| \leq 1$ and so $\Phi(z) \in H^\infty$. The standard example of a function in H^∞ but not in B_0 is the inner function $\exp \{-1 + z/(1-z)\}$ which is of the form (1) with $\mu(x)$ discontinuous. Our construction is based on the modified function

$$\Phi(\delta, z) = \exp \left\{ \int_{-\delta}^{\delta} \frac{e^{ix} + z}{e^{ix} - z} d(x + \delta) \right\}.$$

An integration yields

$$\Phi(\delta, z) = \exp \left\{ -2\delta - 2i \log \frac{e^{i\delta} - z}{e^{-i\delta} - z} \right\}.$$

We now consider z_0 real and equal to $1 - \delta$. Then

$$(1 - |z_0|^2) \Phi'(\delta, z_0) = \delta(2 - \delta) e^{-2\delta} \exp \left[-2i \log \frac{e^{i\delta} - z_0}{e^{-i\delta} - z_0} \right] 2i \left(\frac{1}{e^{i\delta} - z_0} - \frac{1}{e^{-i\delta} - z_0} \right)$$

An easy calculation now shows that

$$\begin{aligned} \lim_{\delta \rightarrow 0} (1 - |z_0|^2) \Phi'(\delta, z_0) &= -4 \exp \left\{ -2i \log \frac{1+i}{1-i} \right\} \\ &= -4 e^{\pi}. \end{aligned}$$

We now define sequences $\{\delta_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ so that the function

$$\Phi(z) = \prod_{n=1}^{\infty} \Phi(\delta_n, ze^{i\theta_n})$$

has the required properties. We choose $\theta_1 = 0$ and $\theta_n = \pi(1 - 1/n)$, $n > 1$. We then choose each δ_n so small so that, with $z_n = (1 - \delta_n) e^{-i\theta_n}$,

$$(1 - |z_n|^2) \Phi^1(\delta_n, z_n) \approx -4 e^{\pi}$$

but

$$|\Phi(\delta_n, z_m)| \approx 1 \quad (m \neq n).$$

It is clear that this can be achieved simply by choosing the sequence $\{\delta_n\}$ sufficiently small. Thus for our function $\Phi(z)$ we have

$$\lim_{n \rightarrow \infty} (1 - |z_n|^2) \Phi'(z_n) > 0,$$

whereas $|z_n| \rightarrow 1$ as $n \rightarrow \infty$. Thus $\Phi(z) \notin B_0$.

If $\Phi(z)$ as constructed above has the representation (5), the representing function $\phi(x)$ will not be differentiable at the points $x = \theta_n + \delta_n$, $n = 1, 2, 3, \dots$. Nevertheless by adjusting the function $\phi(x)$ in intervals $[\theta_n - \delta_n - \epsilon_n, \theta_n - \delta_n + \epsilon_n]$ and $[\theta_n + \delta_n - \epsilon_n, \theta_n + \delta_n + \epsilon_n]$ where ϵ_n is now chosen very small compared with δ_n we may make the function $\Phi(x)$ as smooth as we please. Thus Theorem 2 is proved.

§5. Analytic VMO

We denote by BMOA the Banach space of functions $f(z)$ analytic for $|z| < 1$ for which the norm

$$\|f\| = |f(0)| + \sup_{|\xi| < 1} \|f_3\|_2 < \infty.$$

Here

$$(\|f_3\|_2)^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\frac{z+\xi}{1+\xi z}\right) - f(\xi) \right|^2 |dz|, \quad z = e^{i\theta}.$$

The subspace of BMOA for which

$$\|f_\xi\|_2 \rightarrow 0 \quad \text{as} \quad |\xi| \rightarrow 1- \tag{6}$$

is denoted by VMOA. A function belongs to VMOA if and only if its boundary values on $|z| = 1$ belong to the space of functions of vanishing mean oscillation as defined, for example in [13]. This characterization of VMOA was introduced in [12] where it was shown that $f \in \text{VMOA} \Rightarrow f \in B_0$.

From the above characterization two interesting facts follow immediately:

(i) No inner function in H^∞ can belong to VMOA. To see this we note that if f is inner then there is a sequence $\{\xi_n\}$ with $|\xi_n| \rightarrow 1$ as $n \rightarrow \infty$ such that $f(\xi_n) \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\frac{z+\xi}{1+\xi z}\right) \right|^2 d\theta = 1$$

for all ξ with $|\xi| < 1$ it follows that (6) does not hold.

(ii) If $Y = H^\infty \cap \text{VMOA}$ then $f, g \in Y$ implies $f \cdot g \in Y$. Thus Y , with suitable choice of norm can be made into a Banach algebra. The space Y is the “analytic” part of the space QC of quasicontinuous functions introduced by Sarason [13]. Since it was observed by Sarason that $QC = (H^\infty + C) \cap (H^\infty + C)$, we see that $Y = H^\infty \cap (H^\infty + C)$. In particular we see that Y is again a closed subalgebra of H^∞ , a fact which can also be verified directly.

If A is defined by

$$A = \{f(z) : f(z) \text{ analytic for } |z| < 1, \text{ continuous for } |z| \leq 1\}.$$

then we have that $A \subset Y \subset X \subset H^\infty$. It is easy to see that all of the above inclusions are strict. An example of a function in X but not in Y is a function of the form (4) where $f(x)$ is a monotonic singular function satisfying (3). That such functions exist is known from [14, Theorem 2]. An example of a function in Y but not in A is a function $f(z)$ mapping $|z| < 1$ $1-1$ conformally onto a bounded domain which has a bad prime end. Clearly $f \notin A$ but since f is univalent and in B_0 it is in VMOA ([12] Satz 1).

It would be nice to find a characterization of $Y = H^\infty$ VMOA similar to that of X mentioned in §2. Even though no inner function can belong to Y there are functions in Y which have an inner factor—the function constructed by Gurarii is in A and has a jump inner factor.

In conclusion it seems worthwhile to point out the following theorem

THEOREM 3. *Let $F(z)$ be given by (4) where $f(x)$ satisfies (3). Then there is a constant K_1 so that*

$$|F'(z)| < \frac{K_1}{1-|z|^2} \left[\log \left(\frac{1}{1-|z|^2} \right) \right]^{-\frac{1}{2}}$$

for $|z| < 1$.

For our purposes we needed only that $F'(z) = O(1-|z|^2)^{-1}$, $|z| \rightarrow 1-$. The proof of this theorem is a straightforward adaptation of the proofs of ([4] Theorem 1) and ([16] Theorem 7), and so is omitted.

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REFERENCES

1. ANDERSON, J. M. CLUNIE J. and Ch. POMMERENKE, *On Bloch functions and normal functions* J. Reine Angew. Math. 270 (1974) 12–37.
2. CARLESON, L., *A representation formula for the Dirichlet Integral* Math. Zeit. 73 (1960) 190–196.
3. DUREN, P. L., *Theory of H^p Spaces*, New York, 1970.
4. DUREN, P. L., SHAPIRO, H. S. and SHIELDS, A. L., *Singular measures and domains not of Smirnov type*, Duke Math. J. 33, (1966) 247–254.
5. GURARII, V. P., *On the factorization of absolutely convergent Taylor series and Fourier Integrals* (in Russian). Zap. Nauch. Sem. LOMI. 30 (1972), 15–32.
6. HAVIN, V. P., *On the factorization of analytic functions, smooth up to the boundary* (in Russian), Zap. Nauch. Sem. LOMI. 22 (1971) 202–205.
7. HOFFMAN, K., *Bounded analytic functions and Gleason parts*, Ann. of Math, 86, (1967) 74–111.
8. KAHANE, J. P., *Two problems on H^p classes* Symposium on Complex Analysis, Canterbury 1973, London Math. Soc. Lecture Note Series, 12, Cambridge, 1975, 93–95.

9. KORENBLYUM, B. I., *On an extremal property of outer functions* (in Russian) *Mat. Zamet.* 10 (1971), 53–56.
10. KORENBLYUM B. I. and KOROLEVICH, V. S., *On analytic functions, regular in the disc and smooth on its boundary* (in Russian) *Mat. Zamet.* 7 (1970) 165–172.
11. KORENBLYUM B. I. and FAIVYSHEVSKII, V. M., *On a class of projection operators connected with the division of analytic functions*, (in Russian), *Ukrain. Mat. Zhurnal*, 24 (1972) 692–695.
12. POMMERENKE, Ch., *Schlichte Funktionen und analytische Funktionen beschränkter mittlerer Oszillation* *Comment. Math. Helv.* 52 (1977) 591–602.
13. SARASON, D., *Functions of vanishing mean oscillation* *Trans. Amer. Math. Soc.* 207 (1975) 391–405.
14. SHAPIRO, H. S., *Monotonic singular functions of high smoothness*, *Michigan Math. J.* 15 (1968) 265–275.
15. SHIROKOV, N. A., “Ideal and factorisation in algebras of functions smooth up to the boundary (in Russian)” *Trudy Math. Inst. imeni V. A. Steklova* 130 (1978) 196–220.
16. ZYGMUND, A., *Smooth functions* *Duke Math. J.* 12 (1945) 47–76.

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