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A homological characterization of foliations consisting of minimal surfaces

by DENNIS SULLIVAN

Say that p -dimensional foliation of a compact n -manifold is *geometrically taut* if there is a Riemann metric for which the leaves become minimal surfaces. Say that an oriented p -dimensional foliation is *homologically taut* if no foliation cycle $[S_1]$ constructed from an invariant transverse measure $[P]$ and $[RS]$ is approximately the boundary of a $(p+1)$ -chain tangent to the foliation.

THEOREM. For an orientable foliation to be geometrically taut it is necessary and sufficient that it be homologically taut.

Proof. The ingredients are Rummler's calculation $[R]$, an algebraic operation "purification" preserving a differential condition, Stokes theorem, and the Hahn Banach theorem in the general set-up of $[S_1]$.

Rummler's calculation. A p -dimensional foliation \mathcal{F} of a piece of Riemannian manifold has leaves which are minimal surfaces if and only if the characteristic p -form is relatively \mathcal{F} -closed. The characteristic p -form is obtained from the oriented volume form of \mathcal{F} by orthogonal projection of p -vectors onto the tangent planes of \mathcal{F} . A p -form is relatively \mathcal{F} -closed if its restriction to every $(p+1)$ -manifold tangent to \mathcal{F} is closed.

Purification. To each p -form ω on a vector space positive on an oriented p -dimensional subspace F associates the pair

$$(P_\omega, \omega/F) = (\text{projection onto } F, \text{volume form on } F).$$

Here “/” means restriction and P_ω is defined by the equation

$$P_\omega(v) \wedge (\omega/F) = (v \wedge \omega)/F$$

where “ \wedge ” means contraction. The pure form $\tilde{\omega} = P_\omega^*(\omega/F)$ is called the purification of ω .

Hahn–Banach. An orientation of a foliation \mathcal{F} allows one to regard a transversal invariant measure [RS] and [S₁] as a p -current and these form precisely the intersection of the closed p -currents with the “compact cone” of foliation currents (convex combinations of oriented tangent p -vectors) Theorem I.13 [S₁]. Homological tautness means the closed subspace S generated by boundaries of $(p+1)$ –chains tangent to \mathcal{F} strictly supports the intersection cone of foliation cycles. Hahn–Banach then applied as in Theorem I.7 [S₁] allows us to construct a p -form ω positive on \mathcal{F} and which annihilates this space S of boundaries. Such a form is relatively \mathcal{F} -closed by the obvious local argument.

Homological tautness complies *geometrical tautness*: take the form ω just constructed using Hahn–Banach and homological tautness. Now construct pointwise projections $\{P_\omega\}$ onto the tangent planes $\{F\}$ of \mathcal{F} using purification. Happily, purification is natural and equal to the identity on $(p+1)$ subspaces containing F . So the purified form $\tilde{\omega}$ is still relatively \mathcal{F} -closed. Now construct any metric on the family of subspaces $\{\text{kernel } P_\omega\}$ orthogonal direct sum any metric on the family of tangent planes $\{F\}$ giving the volume forms $\{\omega/F\}$. For this metric $\tilde{\omega}$ is the characteristic form, and by Rummler’s calculation the leaves of \mathcal{F} are minimal surfaces.

Stoke’s theorem implies the converse. If c_n is a sequence of $(p+1)$ chains tangent to \mathcal{F} so that ∂c_n approaches a foliation cycle z and ω is a p -form positive of \mathcal{F} which is relatively \mathcal{F} -closed, we arrive at the contradiction

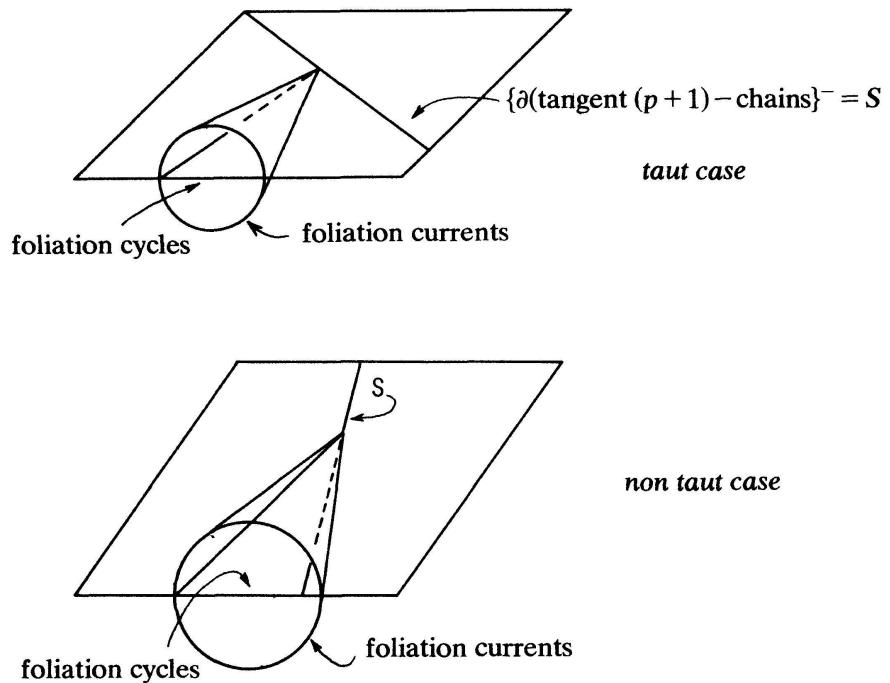
$$0 = \int_{c_n} d\omega = \int_{\partial c_n} \omega \rightarrow \int_z \omega > 0.$$

The proof of this theorem is now complete.

Remark. The proof shows that tautness of a foliation (geometrical or homological) is equivalent to either of the following conditions expressed by differential forms:

- (i) there is a p -form ω positive on the oriented leaves of the foliation so that $d\omega$ is zero on any $(p+1)$ manifold tangent to the foliation.
- (ii) there is a pure p -form ω satisfying the conditions of (i).

The equivalence of (ii) with geometrical tautness is Rummler’s calculation [R] while the algebraic operation of purification $\omega \rightarrow \tilde{\omega}$ shows (i) and (ii) are equivalent. Purification converts the *non-linear* problem (ii) into a *linear* problem (i) which may be treated as in [S₁] by Hahn–Banach. Thus we arrive at the necessary and sufficient homological condition of the theorem. The theorem is a generalization of the case $\eta = 1$ treated in [S₂] which was in turn motivated by an interesting open letter from Hermann Gluck.



EXAMPLES AND COROLLARIES

COROLLARY 1. *A foliation of a compact manifold has either a transversal invariant measure or for some Riemann metric all the leaves are minimal surfaces (of course, both can happen).*

COROLLARY 2. *A foliation is geometrically taut if no transversal invariant measure determines a trivial homology class.*

Proof. The boundaries form a closed subspace of currents.

EXAMPLE. Corollary 2 is illustrated by foliations which admit an immersed cross-section. (An immersed transversal submanifold cutting every leaf).

COROLLARY 3. *A codimension one oriented foliation is geometrically taut, if and only if every compact leaf is cut by a closed transversal curve.*

Proof. Transversal invariant measures carried on non-compact leaves intersect closed transversal curves and are thus determine foliation cycles essential in homology. By our assumption the same is true for the rest. (cf. Theorem II.20 [S₁] where this curve condition was shown to be equivalent to the existence of a transversal volume preserving flow.)

COROLLARY 4. *Foliations by geodesics are characterized by the condition that no flow cycle approximately bounds a tangent homology. (cf. [S₂]).*

Two general classes of foliations relevant to this discussion are “self-linking foliations” and “horospherical foliations.”

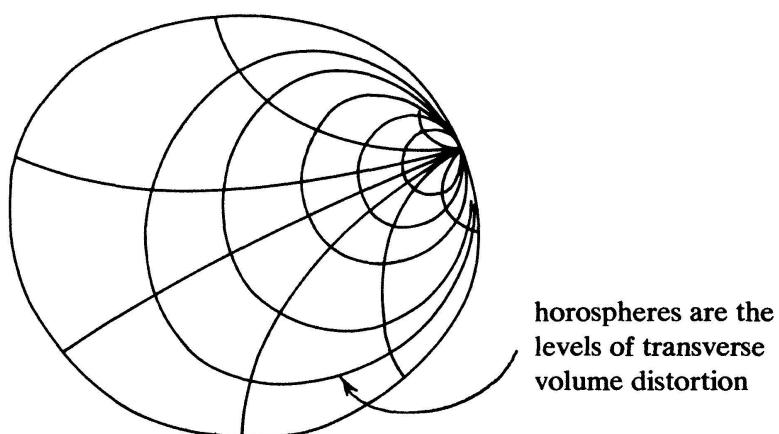
A “*self-linking foliation*” is by definition a p -dimensional foliation of a $2p+1$ manifold defined by an *exact* pure $(p+1)$ -form $d\omega$ so that $\omega \wedge d\omega$ is a *nowhere zero volume* form. Geometrically, such a foliation carries a diffuse foliation cycle (defined by $d\omega$) which is homologous to zero (ω is the homology) and the self-linking number $(\omega \wedge d\omega)$ is spread evenly over the entire manifold. When $p=1$ these self-linking foliations are just the contact flows.

COROLLARY 5. *Self-linking foliations are geometrically taut.*

Proof. Since $\omega \wedge d\omega$ is a volume form, ω is never zero on the leaves $\{\text{kernel } d\omega\}$. Clearly $d\omega$ is zero on any manifold tangent to the foliation of dimension $\leq 2p$ (in particular for $p+1$). Thus ω fulfills the condition of the Remark following the theorem. In fact, for any metric whose leaf volumes are $\{\omega \mid \text{leaf}\}$ and whose orthogonal projection agree with $\{P\omega\}$ the leaves are minimal surfaces.

Finally, non-taut examples can be constructed using the classical horospherical foliation as models. Define a “*horospherical foliation*” as one arising from another foliation of dimension one higher as the levels of distortion for a given transverse volume. More precisely, if ω defines one foliation then $d\omega = \eta \wedge \omega$ is pure and if it is nowhere zero defines a second foliation. This is the horospherical foliation associated to the first foliation defined by ω . In the tangent bundles of negatively curved manifolds the first foliation is made of all geodesics asymptotic at ∞ and the second is the foliation by horospheres.

COROLLARY 6. *The (generalized) horospherical foliations are never taut (geometrically or homologically).*



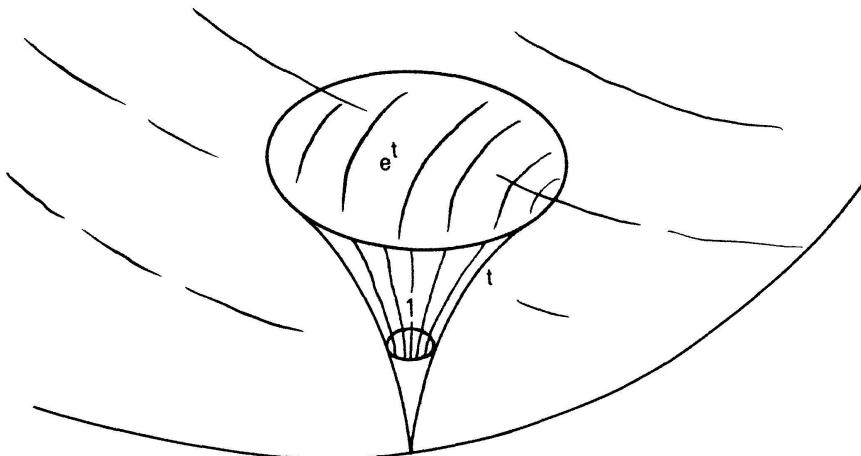
Proof. The diffuse foliation cycle defined by $d\omega$ actually bounds an (infinite) tangent homology defined by ω . This infinite homology may be approximated easily by a finite tangent homology $[S_1]$ or one may simply use Stokes again. For example let α be a form as in the Remark following the theorem. Then

$$0 < \int_M \alpha \wedge d\omega = \int_M d\alpha \wedge \omega = 0$$

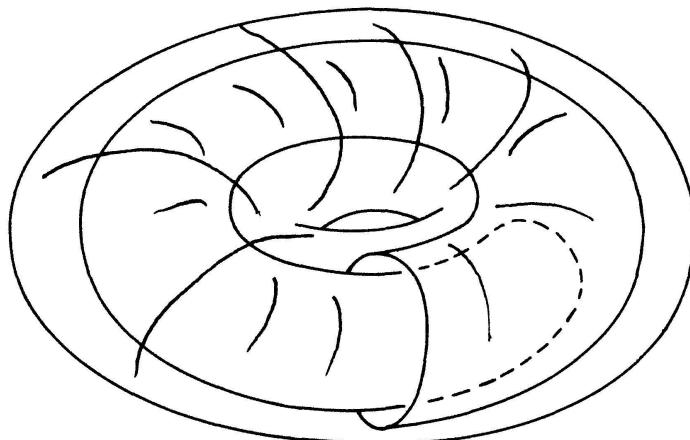
a contradiction.

EXAMPLES OF TANGENT HOMOLOGIES

In the classical horospherical tangent examples there is a picture* of the approximating homology defined by the region bounded by pieces of geodesics and horospheres of the indicated diameters.



Finite tangent homologies are easy to imagine. For example, Reeb components.



* Contained in Plante's thesis for the upper half plane. (cf. [P]).

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