Geometry of hypersurfaces and mapping theorems in Cn.

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Geometry of hypersurfaces and mapping theorems in $\mathbf{C}^n$

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§0. Introduction

The intrinsic pseudoconformal, or CR, geometry of a strictly pseudoconvex real hypersurface in $\mathbf{C}^n$ has recently attracted a great deal of attention because of its relation to the "equivalence problem" for such hypersurfaces and for the domains which they bound. Cf., for example, [5], [6], [15]. We present here a few applications of this geometric structure to some problems concerning mappings of pseudoconvex domains other than the equivalence problem itself, strictly speaking. The point is that auxiliary structures (measures, metrics, differential systems, etc.) on pseudoconvex boundaries can contribute interesting information about the behaviour of mappings of domains.

To state the main results, let D_i $(i = 1, 2)$ be relatively compact manifolds with $\mathscr{C}^\infty$ strictly pseudoconvex boundaries in complex manifolds M_i.

THEOREM I. *Let $f : D_1 \to D_2$ be a proper holomorphic map.*

(a) *If $D_1 = D_2$, then f extends smoothly up to the boundary ∂D_1.*

(b) *If ∂D_i is real analytic $(i = 1, 2)$ then f extends holomorphically past the boundary.*

For $D \subset M$ as above, let Aut (D) denote the biholomorphic automorphism group of D, and $\mathrm{Aut}^\circ(D)$ its identity component.

THEOREM II. (a) $\mathrm{Aut}^\circ(D)$ *is compact, unless D is biholomorphic to $\mathbf{B}^n$, the unit ball in $\mathbf{C}^n$ $(n = \dim_{\mathbf{C}} D)$.*

(b) *For $n \geq 3$,* $\mathrm{Aut}(D)$ *is compact unless $D \simeq \mathbf{B}^n$.*

Theorem I, for f biholomorphic, is Fefferman's theorem [7]. The case when $D_1 = D_2 = \mathbf{B}^n$ is due to H. Alexander [2].

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Since we first proved theorem II, more elementary proofs have been given by Wong [16] and Klembeck [11], simultaneously dropping the restriction $n \geq 3$ in II(b). Webster [15] proved a less precise analogue of II(a), for compact, integrable, strictly pseudoconvex CR-manifolds.

Along the way we collect some interesting structural facts about pseudoconvex hypersurfaces. The paper is organized as follows:

§1 contains the proof of I(a). This involves the construction and properties of an intrinsic measure on a strictly pseudoconvex hypersurface, and an analytic continuation procedure for extending the region of boundary regularity of maps such as our f. Some corollaries are given as well as an example of a (non-Stein) manifold D with proper self-map f which is not biholomorphic.

§2 proves I(b). Here we use a variant of the continuation argument in §1. The key point is to have some understanding of the domain of existence for solutions to the "chain"-equations of E. Cartan, and Chern–Moser. The lemmas providing the needed information here are of independent interest.

§3 contains the proof of II. Here several of the auxiliary structures are used at once. One first uses intrinsic pseudo-distances on the boundary of D to conclude that if Aut(D) were non-compact, then ∂D would be spherical in the sense of [4]. Theorem II(b) follows by a development map argument, as in [4], but a topological assumption intervenes causing the restriction $n \geq 3$. Theorem II(a) analyzes the situation directly from a fixed point on the boundary of a one-parameter group in Aut°(D).

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§1. Proper Self-Maps

We prove Theorem I(a) in this section. We first introduce a canonical measure on the boundary ∂D, and for this we need to recall some basic facts about pseudoconformal geometry (cf. [5], [6] and [15]).

Let $M \subset \mathbf{C}^n$ be a real hypersurface, $T(M)$ its real tangent bundle, $H(M) \subset T(M)$ the codimension 1 subbundle invariant by the almost complex structure J on $T(\mathbf{C}^n)$. The Levi form $\mathscr{L} : H(M) \otimes H(M) \to T(M)/H(M)$ is defined by $\mathscr{L}(X, Y) = [X, JY] \bmod H(M)$. The choice of $\theta \neq 0$ in $T^*(M)$, with $\theta \mid H(M) = 0$ defines $\mathscr{L}_\theta = \theta \cdot \mathscr{L}$. $\mathscr{L}_\theta$ is a hermitian form on $H(M)$, and our basic assumption is that for properly oriented θ, $\mathscr{L}_\theta$ is positive definite. Note that if we replace θ by $\lambda^2\theta$, λ a real-valued function on M, then $\mathscr{L}_{\lambda^2\theta} = \lambda^2 \mathscr{L}_\theta$.

The pseudoconformal geometry gives a structure bundle $T \to M$ and a Cartan connection ω on Y with values in $\mathcal{SU}(n, 1)$. If Ω denotes the curvature of this connection, then the component of lowest "weight" on Y, denoted $S^{\alpha}_{\beta\gamma\bar{\delta}}\omega^{\gamma}\wedge\omega^{\bar{\delta}}$ in [6], descends to a tensor on M, specifically, a section we'll denote by S of the bundle $\operatorname{End}(H(M))\otimes H(M)^*\otimes\overline{H(M)}^*$. Here, $H(M)^*$ denotes the complex linear dual of $H(M)$, $\overline{H(M)}^*$ the conjugate bundle, and $\operatorname{End}(H(M))$ the complex linear endomorphisms of $H(M)$.

Every choice of positively oriented θ as above gives $\mathcal{L}_\theta$, and hence a hermitian metric on $H(M)$, and by the natural extension, metrics on $H(M)^*$, $\overline{H(M)}^*$, and $\operatorname{End}(H(M))$. It was noted above that replacing θ by $\lambda^2\theta$ scales the hermitian form on $H(M)$ by a factor λ^2, and hence, scales the form on $\operatorname{End}(H(M))\otimes H(M)^*\otimes\overline{H(M)}^*$ by λ^{-4}. Letting $\|\cdot\|_\theta$ denote the norm function determined by the form $\mathcal{L}_\theta$, we conclude $\|S\|_{\lambda^2\theta}=\lambda^{-2}\|S\|_\theta$.

We denote by $\Sigma=\{x\in M \mid S(x)=0\}$ the "umbilic locus" of M. For $n>2$, it follows from the Bianchi identities that if S is $\equiv 0$ in an open set U, then $\Omega=0$ on U, and M is locally CR-equivalent to the unit sphere $S^{2n-1}\subset\mathbf{C}^n$. When $S\neq 0$, the 1-form $\theta_0=\|S\|_\theta\cdot\theta$ is $\mathcal{C}^\infty$, and independent of the choice of positively oriented θ. The non-degeneracy of the Levi-form implies that the $2n-1$ form

$$\theta_0\wedge\underbrace{d\theta_0\wedge\cdots\wedge d\theta_0}_{n-1\text{ times}}=\theta_0\wedge(d\theta_0)^{n-1}$$

is a strictly positive $2n-1$ form. More precisely, we have:

PROPOSITION 1.1. *There exists a* CR-*invariant measure* μ *on* M *with the following properties*:

(1) *If* dx *denotes a positive* $2n-1$ *form on* M, *then* μ *is absolutely continuous with respect to* dx
(2) $d\mu/dx=f$ *is a non-negative, continuous function*
(3) $\Sigma=\{x \mid f(x)=0\}$
(4) *If* M *is compact,* $0\le\int_M d\mu=\mu(M)<\infty$, *with equality holding if and only if* $M=\Sigma$.

Proof. On $M-\Sigma$, we define μ as the smooth measure given by $\theta_0\wedge(d\theta_0)^{n-1}$. If θ is a local non-vanishing 1-form which annihilates $H(M)$, the formula

$$\theta_0\wedge(d\theta_0)^{n-1}=\|S\|_\theta^n\,\theta\wedge(d\theta)^{n-1}$$

valid on $M-\Sigma$, gives an extension of μ to all of M as a continuous, non-negative $(2n-1)$-form which vanishes exactly along Σ. The CR-invariance of μ follows from that of S and θ_0. The rest is by construction and the remarks above.

(For $n=2$, one must use E. Cartan's Q (as in [6]) instead of S as above.)

We note here that θ_0 may be used to construct a CR-invariant Riemannian metric on $M-\Sigma$ (see [15]). We'll specify this metric by specifying a particular kind of orthonormal basis for it. The basis will consist of independent vectors $\xi_1, \ldots, \xi_{2n-1}$ with $\xi_1, \ldots, \xi_{2n-2}$ in $H(M)$, $\mathcal{L}_{\theta_0}(\xi_i, \xi_j) = \delta_{ij}$, $1 \leq i, j \leq 2n-2$, and ξ_{2n-1} is determined by $\theta_0(\xi_{2n-1}) = 1$, and $d\theta_0(\xi, \cdot) \equiv 0$.

We recall here two results needed in this and the next §.

(1.2) (Pinčuk [13]). Let D_i $(i = 1, 2)$ be as in Theorem I and $f: D_1 \to D_2$ a proper holomorphic map. Then (a) f has an extension to $\bar{D}_1$ which is Hölder continuous of order 1/2, and (b) the jacobian of f is bounded on D_1.

(1.3) (Alexander [2]). D_i $(i = 1, 2)$, f as above. There exists an open subset $U \subset \partial D_2$ of full-measure in ∂D_2 such that the extension of f to ∂D_1 given in (1.2), when restricted to $f^{-1}(U) \subset \partial D_1$, is a $\mathscr{C}^\infty$, d-to-1 covering which is a local CR-equivalence (d = degree of $f: D_1 \to D_2$).

We remark that both theorems were originally stated for domains in $\mathbf{C}^n$, but both results extend to the present case by standard localization techniques.

Proof of Theorem 1(*a*):

We consider two possible cases:

(i) $\partial D \neq \Sigma$, i.e., $\Omega \not\equiv 0$ on ∂D.

(ii) $\partial D = \Sigma$, $\Omega \equiv 0$ on ∂D.

Case (*i*). We prove that in this case, $d = 1$, and hence f is biholomorphic. Thus, Fefferman's theorem [7] applies, and f is smooth at the boundary. To compute the degree, note

$$\begin{aligned} 0 < \mu(\partial D) &= \int_{\partial D} d\mu = \int_U d\mu \quad (U \text{ of full-measure}) \\ &= \frac{1}{d} \int_{f^{-1}(U)} d\mu \quad (\text{invariance of } d\mu;\ (1.3)) \\ &\leq \frac{1}{d} \int_{\partial D} d\mu \quad (\text{positivity of } d\mu) \end{aligned}$$

Since $0 < \mu(\partial D) < \infty$, and $d \geq 1$, this implies $d = 1$.

Case (*ii*). In this case we wish to argue more directly that f is smooth at the boundary. By (1.3), there is a point $p \in \partial D$ and an open neighborhood V of p such that the boundary values of f are a $\mathscr{C}^\infty$ CR-equivalence to $f(V)$ about $f(p)$. Let q be any other point in ∂D_1, and let $\gamma: [0, 1] \to \partial D$ be a continuous path with $\gamma(0) = p$, $\gamma(1) = q$. We claim that the boundary values of f are a smooth local

CR-equivalence at any point $\gamma(t)\in\partial D$, $0\le t\le 1$. Since q is arbitrary, and ∂D connected, the theorem would follow.

Suppose the boundary values of f are a smooth local CR-equivalence at $\gamma(t)$, for all $t<t_0$, with $t_0>0$. We have only to show that f gives a smooth local CR-equivalence in a neighborhood of $\gamma(t_0)$ in ∂D.

Since $\Omega\equiv 0$ on ∂D, we may choose neighborhoods $W_1\supset\gamma(t_0)$, $W_2\supset f(\gamma(t_0))$ in ∂D, and $\mathscr{C}^\infty$ CR-equivalences of the W_i with open subsets of $S^{2n-1}\subset\mathbf{C}^n$. By H. Lewy's extension theorem, these local CR-equivalences extend to local biholomorphic equivalences Φ_i, $i=1,2$, of D with $\mathbf{B}^n$, which are smooth at the boundary. Thus there exist open neighborhoods, $V_1\ni\Phi_1(\gamma(t_0))$ and $V_2\ni\Phi_2(f(\gamma(t_0)))$ in $\mathbf{C}^n$ such that $h=\Phi_2\circ f\circ\Phi_1^{-1}$ is continuous from $V_1\cap\bar{\mathbf{B}}^n$ to $V_2\cap\bar{\mathbf{B}}^n$, and is holomorphic from $V_1\cap\mathbf{B}^n$ to $V_2\cap\mathbf{B}^n$. By construction, h gives a smooth local CR-equivalence of $V_1\cap S^{2n-1}$ with $V_2\cap S^{2n-1}$ at points $\Phi_1(\gamma(t))$, $t<t_0$. Hence, by a result of H. Alexander [1], h must be given in a neighborhood of the points $\Phi(\gamma(t))$, $t<t_0$, by a linear fractional transformation T. Since $V_1\cap\mathbf{B}^n$ may be taken to be connected, we have $h=T$ on $V_1\cap\mathbf{B}^n$, by analytic continuation. Hence, $h=T$ on $V_1\cap\overline{\mathbf{B}^n}$, and, in particular, h is smooth to the boundary near $\Phi_1(\gamma(t_0))$, and $f=\Phi_2^{-1}\cdot h\circ\Phi_1$ is also smooth to the boundary near $\gamma(t_0)$, proving the theorem.

COROLLARY 1.4. *Assume D is as in Theorem* I(a), *and D is Stein. Then any proper map $f: D\to D$ is biholomorphic.*

Proof. This follows from I(a) and an argument of Pinčuk [13].

COROLLARY 1.5. *Assume D is as in Theorem* I(a), *and $\Omega\not\equiv 0$ on ∂D. Then any proper holomorphic map $f: D\to D$ is biholomorphic.*

Proof. Indeed, this is just the argument given to prove case (i) of I(a).

We close this § with an example to show that the above corollaries are "sharp" in some sense. Consider $\mathscr{U}=\{(z,w)\in\mathbf{C}^2\mid \operatorname{Im} w>|z|^2\}$; $\partial\mathscr{U}=\{(z,w)\in\mathbf{C}^2\mid \operatorname{Im} w=|z|^2\}$ is identifiable with the Heisenberg group, the multiplication given by $(z_1,w_1)\circ(z_2,w_2)=(z_1+z_2, w_1+w_2-2iz_1\bar z_2)$. $\Gamma=\{(\zeta,\eta)\in\partial\mathscr{U}\mid \zeta,\eta\in\mathbf{Z}[i]\}$ is a discrete cocompact subgroup of $\partial\mathscr{U}$ which acts holomorphically by right multiplication on $\partial\mathscr{U}$ and $\mathscr{U}$. It is well-known (*cf.*, e.g., [4]) that $\bar{\mathscr{U}}/\Gamma$ has a natural compactification $\bar D$ which is a disk bundle over $E=\mathbf{C}/\mathbf{Z}[i]$. Now the automorphism $(z,w)\to(2z,4w)$ of $\mathscr{U}$ descends to give a proper map $f: D\to D$, compatible with the projection of D to E. It is easy to compute that the degree of f is 16, and that f ramifies along the curve E embedded in D as the zero-section of the bundle. Note that $\Omega\equiv 0$ on ∂D, since ∂D is locally CR-equivalent to $\partial\mathscr{U}$.

2. Proper maps, real analytic case

Before proving Theorem I(b), we need to recall some more facts from pseudoconformal geometry. We use the $(n+1)\times(n+1)$ hermitian matrix

$$Q=\begin{bmatrix} & & i/2 \\ & I_{n-1} & \\ -i/2 & & \end{bmatrix}$$

to define

$$\mathfrak{su}\,(n, 1)=\{A\in M_n(\mathbf{C})\mid {}^t\bar{A}Q+QA=0\}.$$

On the structure bundle Y over the strictly pseudoconvex hypersurface M, there is a vector field $X=X_M$ determined by

$$\omega(X)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\in \mathfrak{su}\,(n, 1).$$

(ω is the Cartan connection on Y, with values in $\mathfrak{su}\,(n, 1)$.) The integral curves of X projected to M give the "chains" of E. Cartan, Chern and Moser, and the orbits of X in Y give a distinguished parameter (the integration parameter along X) and parallel transport along the corresponding chains.

A *parametrized chain* will denote a connected chain $\gamma=\gamma(t)$ with a choice of finite distinguished parametrization. Two points $p, q\in M$ are joined by a *broken chain* if there are points $q_1=p, q_2, \ldots, q_n$ in M, and parametrized chains γ_i, $i=1,\ldots,n-1$, so that $\gamma_i(a_i)=q_i$, $\gamma_i(b_i)=q_{i+1}$, for suitable a_i, b_i in the interior of the domain of definition of the parameter along γ_i.

LEMMA 2.1. *For M connected, any two $p, q\in M$ are joined by a broken chain.*

We will prove this later.

We note that by this lemma, and remarks as in the proof of case (ii) of I(a) above, it suffices for the proof of I(b) to show that if the f in I(b) is holomorphic past the boundary in a neighborhood of one point on a parametrized chain γ, then f extends holomorphically past the boundary in a neighborhood of any point on γ.

Let $\gamma=\gamma(t)$ be a parametrized chain in ∂D_1, with parameter $t\in(-\varepsilon, \varepsilon)$, and let $\tilde{\gamma}(t)=f(\gamma(t))$ be the (continuous) image of γ under f. (We are using (1.2) here.) Suppose f is holomorphic past the boundary in a neighborhood of $\gamma(t)$, for

$t \in (-\varepsilon, 0)$. Then $\tilde{\gamma}(t)$ is a parametrized chain in ∂D_2, for $t \in (-\varepsilon, 0)$. The main lemma in the proof of I(b) is the following:

LEMMA 2.2 *The chain $\tilde{\gamma}$ continues to a parametrized chain defined for $t \in (-\varepsilon, \delta)$, for some $\delta > 0$.*

This, too, will be proved later.

Given lemma 2.2, the proof of I(b) is concluded much as we argued for case (ii) of I(a) in §1. Let $\gamma = \gamma(t)$ be a parametrized chain in ∂D_1, and suppose that f is holomorphic past the boundary in a neighborhood of $\gamma(t)$, for $t < t_0$. By reparametrizing γ near $\gamma(t_o)$, we may assume $t_0 = 0$. Apply lemma 2.2 to conclude that $f \cdot \gamma(t) = \tilde{\gamma}(t)$ extends to be defined as a parametrized chain for $t \in (-\delta, \delta)$, for some $\delta > 0$. For δ small enough, we may take Moser normal coordinates $(z^1, \ldots, z^{n-1}, w)$ for ∂D_1 centered at $\gamma(-\delta)$, corresponding to the initial normalizations determined by the given parametrization t of the chain γ, and a choice of frame $\partial/\partial z^1, \ldots, \partial/\partial z^{n-1}$ for the space of complex tangents to ∂D_1 at $\gamma(-\delta)$. (The frame $\partial/\partial z^1, \ldots, \partial/\partial z^{n-1}$ is to be orthonormal for the Levi-form at $\gamma(-\delta)$. For more precision, *cf.* [6], §3.) This coordinate system is convergent in a neighborhood of $\gamma(t)$, $-2\delta < t < \delta$, for $\delta > 0$ sufficiently small.

We may also take Moser normal coordinates $(\tilde{z}^1, \ldots, \tilde{z}^{n-1}, \tilde{w})$ for ∂D_2, centred at $\tilde{\gamma}(-\delta)$, corresponding to the initial normalizations determined by the given parametrization of $\tilde{\gamma}(t)$, and the frame at $\tilde{\gamma}(-\delta)$ given by $f_*(\partial/\partial z^1), \ldots, f_*(\partial/\partial z^{n-1})$. These coordinates are also convergent in a neighborhood of $\tilde{\gamma}(t)$, $-2\delta < t < \delta$. Further, in these coordinates, the mapping f is given by

$$(\tilde{z}, \tilde{w}) = f(z, w) = (z, w), \tag{2.2}$$

by the uniqueness of Moser coordinates with given normalizations. (2.2) gives an analytic continuation of f to a neighborhood of $\gamma(t)$, $-2\delta < t < \delta$, proving Theorem I(b), modulo the lemmas.

We first prove lemma 2.2. The significance of the estimations performed in the course of the proof are clarified by Proposition 2.3 below.

In a neighborhood U of $\gamma(0)$ we pick a local frame of 1-forms for ∂D_1, θ, $\theta^1, \ldots, \theta^{n-1}$ where θ is a real 1-form annihilating $H(M)$; $\theta^1, \ldots, \theta^{n-1}$ are complex linear when restricted to $H(M)$, and

$$d\theta = \sqrt{-1} \sum_{i=1}^{n-1} \theta^i \wedge \bar{\theta}^i. \tag{2.3}$$

We take a similar frame $\tilde{\theta}, \tilde{\theta}^1, \ldots, \tilde{\theta}^{n-1}$ on ∂D_2 in a neighborhood of $\tilde{\gamma}(0) = f(\gamma(0))$.

Along $\gamma(t)$, $-\varepsilon<t<0$, $f^*\tilde{\theta}=\lambda\theta$, where $\lambda=\lambda(t)$ is a positive factor of proportionality. We claim that there are positive constants c_1, c_2 such that

$$c_1\,|Jf(\gamma(t))|^{2/n+1}\le\lambda(t)\le c_2\,|Jf(\gamma(t))|^{2/n+1} \tag{2.4}$$

for $t\in(-\varepsilon,0)$, where Jf is the holomorphic jacobian determinant of f computed with respect to some holomorphic coordinates defined in neighborhoods of $\gamma(0)$ and $\tilde{\gamma}(0)$. Indeed, write

$$f^*\tilde{\theta}=\lambda\theta,\qquad f^*\tilde{\theta}^1=a^i_j\theta^j+b^i\theta, \tag{2.5}$$

along $\gamma(t)$, $t\in(-\varepsilon,0)$, and note that

$$c_1\,|\lambda\det(a^i_j)|\le|Jf(\gamma(t))|\le c_2\,|\lambda\det(a^i_j)|. \tag{2.6}$$

We compute from (2.3) and (2.5) that

$$f^*(\tilde{\theta}\wedge(d\tilde{\theta})^{n-1})=\lambda^n\theta\wedge(d\theta)^{n-1} \tag{2.7}$$

and that

$$f^*(\tilde{\theta}\wedge(d\tilde{\theta})^{n-1})=\lambda\theta\wedge\left(\sum_{i=1}^{n-1}\sqrt{-1}f^*\tilde{\theta}^i\wedge\overline{f^*\tilde{\theta}^1}\right)^{n-1}=\lambda\,|\det(a^i_j)|^2\,\theta\wedge(d\theta)^{n-1}$$

Hence, $|\det(a^i_j)|=\lambda^{(n-1)/2}$; substituting this into (2.6) proves the claim.

We must next examine explicitly the systems of ordinary differential equations defining $\gamma(t)$, $\tilde{\gamma}(t)$. This amounts to making the field X in (2.1) more explicit. We will denote the Cartan connection on ∂D_1 in the local frame $\theta,\theta^1,\ldots,\theta^{n-1}$ by ω', with

$$\omega'=\begin{bmatrix}\omega'^0_0 & \omega'^\beta_0 & \omega'^n_0\\ \omega'^0_\alpha & \omega'^\beta_\alpha & \omega'^n_\alpha\\ \omega'^0_n & \omega'^\beta_n & \omega'^n_n\end{bmatrix},\qquad 1\le\alpha,\beta\le n-1,$$

with

$$\omega'^n_0=2\theta,\qquad \omega'^\beta_0=\theta^\beta,\ \omega'^n_\alpha=2i\overline{\theta^\alpha}, \tag{2.9}$$

and where

$$\operatorname{Re}(\omega'^0_0)=0,\ \omega'^\beta_\alpha=-\omega'^\alpha_\beta,\ \omega'^0_\alpha=-2i\omega'^\alpha_n,\qquad \omega'^0_n=\overline{\omega'^0_n}$$

and Im $(\omega_0'^0)$, $\omega_\alpha'^\beta$, $\omega_\alpha'^0$, $\omega_n'^0$ are all of the form

$$A_\alpha\theta^\alpha + B_\alpha\overline{\theta^\alpha} + C\theta \tag{2.10}$$

where A_α, B_α, C are smooth functions defined in a neighborhood U of $\gamma(0)$. To express ω on Y, we have to introduce the group variables parametrizing the fibers of Y. These are given by the group $H \subset SU(n, 1)/(\text{center})$ of [6]; any element h is written $h = B \cdot A$, where:

$$A = \begin{bmatrix} \zeta^{-1}\zeta & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \rho\zeta \end{bmatrix}, \tag{2.11}$$

$$\det A = 1, \quad |\zeta| = 1, \quad \rho \in \mathbf{R}^+, \quad a \in U(n-1),$$

(A is understood to be taken modulo diagonal matrices λI_{n+1}, where $\lambda^{n+1} = 1$.)

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -2i{}^t\bar{b} & I_{n-1} & 0 \\ s - i\,|b|^2 & b & 1 \end{bmatrix}, \tag{2.12}$$

$$b = (b^1, \dots, b^{n-1}) \in \mathbf{C}^{n-1}, \quad |b|^2 = \sum_{i=1}^{n-1} |b^i|^2, \quad s \in \mathbf{R}.$$

The Cartan connection is given on all of Y by

$$\begin{aligned} \omega &= h^{-1}\omega' h + h^{-1}\,dh \\ &= A^{-1}B^{-1}\omega' BA + A^{-1}\,dA + A^{-1}(B^{-1}\,dB)A. \end{aligned} \tag{2.13}$$

We will denote by $\tilde{\omega}$, $\tilde{h} = \tilde{B} \cdot \tilde{A}$, *etc.*, the corresponding forms, variables, etc., on D_2, constructed in terms of the frame $\tilde{\theta}$, $\tilde{\theta}^1, \dots, \tilde{\theta}^{n-1}$ near $\tilde{\gamma}(0)$.

By a suitable choice of a frame for $H_{\gamma(0)}(\partial D_1)$, we may lift $\gamma(t)$ to an orbit $\Gamma(t) \in Y$ of the vector field X i.e.,

$$\omega(\Gamma(t)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.14}$$

The mapping f induces an H-equivariant mapping from Y over ∂D along $\gamma(t)$ to

$\tilde{Y}$ over ∂D_2 along $\tilde{\gamma}(t)$, $-\varepsilon<t<0$. Let $\tilde{\Gamma}(t)$ denote the image of $\Gamma(t)$; $\tilde{\Gamma}(t)$ is a lifting of $\tilde{\gamma}(t)$ to $\tilde{Y}$, and we have

$$\tilde{\omega}(\tilde{\Gamma}(t))=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \tag{2.15}$$

In order to show that $\tilde{\Gamma}(t)$ extends to an interval $(-\delta,\delta)$, it suffices to show that $\tilde{\Gamma}(t)$ remains bounded in $\tilde{Y}$ as $t\to 0$. Since $\tilde{\gamma}(t)\to f(\gamma(0))$ as $t\to 0$, it suffices to bound the fiber variables along $\tilde{\Gamma}(t)$, i.e., to bound:

$$\log\tilde{\rho}(t)=\log\tilde{\rho}(\tilde{\Gamma}(t)),\quad \tilde{b}(t)=\tilde{b}(\tilde{\Gamma}(t)),\quad \tilde{s}(t)=\tilde{s}(\tilde{\Gamma}(t))$$

as $t\to 0$. Writing out (2.15) explicitly, using (2.11), (2.12) and (2.13), one arrives, after some mildly tedious calculation, at the following system of equations (all repeated Greek indices are summed):

$$\tilde{\rho}^2\tilde{\omega}_0^n=1 \tag{2.16}$$

$$\tilde{\omega}_0^\beta+\tilde{b}^\beta\tilde{\omega}_0^n=0,\quad \beta=1,\dots,n-1 \tag{2.17}$$

$$\tilde{\omega}_0^0-2i\tilde{\omega}_0^\beta\ \overline{\tilde{b}^\beta}+\tilde{w}_0^n(\tilde{s}-i|\tilde{b}|^2)-\frac{d\tilde{p}}{\tilde{p}}-\frac{d\tilde{\zeta}}{\tilde{\zeta}}=0. \tag{2.18}$$

$$-d\tilde{b}^\beta+\tilde{b}^\beta\overline{\tilde{\omega}_0^0}-\tilde{\omega}_n^\beta-\overline{\tilde{\omega}_\alpha^\mu\tilde{b}^\mu}+2i\tilde{b}^\beta\overline{\tilde{\omega}_0^\alpha\tilde{b}^\alpha}=0 \tag{2.19}$$

and

$$\begin{aligned}d\tilde{s}&+2\operatorname{Im}(\overline{\tilde{b}^\alpha}\,d\tilde{b}^\alpha)+\tilde{\omega}_n^0-2\operatorname{Im}(\tilde{b}^\beta\tilde{\omega}_\beta^\alpha\overline{\tilde{b}^\beta})-2\operatorname{Re}((\tilde{s}+i|\tilde{b}|^2)\tilde{\omega}_0^0)\\&+2\operatorname{Re}(2i\tilde{b}^\beta\overline{\tilde{\omega}_n^\beta})+2\operatorname{Re}(2i(\tilde{s}+i|\tilde{b}|^2)\tilde{\omega}_0^\alpha\overline{\tilde{b}^\alpha})-(\tilde{s}^2+|\tilde{b}|^4)\tilde{\omega}_0^n=0.\end{aligned} \tag{2.20}$$

Note that here we are abbreviating: all one forms are to be evaluated on $\tilde{\Gamma}(t)$. Thus, for example, $d\tilde{s}$ stands for $(d/dt)\tilde{s}(t)$, and $\tilde{\omega}_0^n$ stands for $\tilde{\omega}_0^n(\tilde{\Gamma}(t))$. We also have omitted the equations involving $d\tilde{a}$, since $\tilde{a}$ is automatically bounded. Similarly, since $|\tilde{\zeta}|=1$, we are only interested in the real part of (2.18). We substitute (2.16) and (2.17) into (2.19), (2.20) and the real part of (2.18) to obtain equations

$$\frac{d}{dt}\log(\tilde{\rho})-\tilde{s}\tilde{\rho}^{-2}=0, \tag{2.21}$$

$$\frac{d\tilde{b}^{\beta}}{dt}-\tilde{b}^{\beta}\overline{\tilde{\omega}_0^0}+\tilde{\omega}_n^{\beta}-\tilde{\omega}_{\alpha}^{\beta}\tilde{b}^{\alpha}+2i\,|\tilde{b}|^2\,\tilde{b}^{\beta}\tilde{\rho}^{-2}=0, \tag{2.22}$$

$$\frac{d\tilde{s}}{dt}+\tilde{\omega}_n^0+2\,\mathrm{Im}\,(\overline{\tilde{b}^{\beta}}\tilde{\omega}_n^{\beta})-(\tilde{s}^2+|\tilde{b}|^4)\tilde{\rho}^{-2}=0. \tag{2.23}$$

From (2.22) we derive

$$0=\mathrm{Re}\left(\overline{\tilde{b}^{\beta}}\frac{d\tilde{b}^{\beta}}{dt}\right)+\mathrm{Re}\,(\overline{\tilde{b}^{\beta}}\tilde{\omega}_n^{\beta}),$$

or

$$\frac{d\,|\tilde{b}|^2}{dt}=-2\,\mathrm{Re}\,(\overline{\tilde{b}^{\beta}}\tilde{\omega}_n^{\beta}). \tag{2.24}$$

Now examples show that equations (2.21), (2.23), (2.24) are not, in themselves, sufficient to bound by $\tilde{\rho}$, $|\tilde{b}|^2$ and $\tilde{s}$ along $\tilde{\Gamma}(t)$, cf. [5] and [8]. We therefore use the extra information provided by (2.4) above and Pinčuk's theorem (1.2)(b). Note that, by construction, if $f:\Gamma(t)\to\tilde{\Gamma}(t)$, then

$$f^*(\tilde{\rho}(\tilde{\Gamma}(t))^2\tilde{\theta}_{\tilde{\gamma}(t)})=\rho(\Gamma(t))^2\theta_{\gamma(t)},\qquad -\delta<t<0.$$

Hence,

$$\tilde{\rho}(\tilde{\Gamma}(t))^2\lambda(\gamma(t))=\rho(\Gamma(t))^2,$$

λ as in (2.4) above, and so

$$\tilde{\rho}(\tilde{\Gamma}(t))^{-1}=\lambda(\gamma(t))^{1/2}\rho(\Gamma(t))^{-1}.$$

Since the curve $\Gamma(t)$ in Y extends past $\Gamma(0)$, $\rho(\Gamma(t))^{-1}$ is bounded for $t\in(-\varepsilon,0)$. $\lambda(\gamma(t))^{1/2}$ is bounded there by (2.4) and Pincuk's theorem. Hence, $\tilde{\rho}(\tilde{\Gamma}(t))^{-1}$ is *bounded* for $t\in(-\varepsilon,0)$. This bound and the equations will bound $|\tilde{b}|^2$, $\tilde{s}$ and $\tilde{\rho}$.

Inserting (2.10), (2.16) and (2.17) into (2.24) gives

$$\frac{d\,|\tilde{b}|^2}{dt} = -2\,\mathrm{Re}\,(\overline{\tilde{b}^\beta}(A^\beta_\alpha\theta^\alpha(\tilde{\gamma}(t))+B^\beta_\alpha\overline{\theta^\alpha}(\tilde{\gamma}(t))+C^\beta\theta(\tilde{\gamma}(t)))$$
$$= -2\,\mathrm{Re}\,(-\tfrac{1}{2}\overline{\tilde{b}^\beta}\tilde{\rho}^{-2}\tilde{b}^\alpha A^\beta_\alpha-\tfrac{1}{2}\overline{\tilde{b}^\beta\tilde{b}^\alpha}B^\beta_\alpha\tilde{\rho}^{-2}+\tfrac{1}{2}\tilde{\rho}^{-2}C^\beta\overline{\tilde{b}^\beta}).$$

Here A^β_α, B^β_α, C^β are suitable smooth functions on ∂D_2 near $\tilde{\gamma}(0)$. Hence, using that $\tilde{\rho}^{-1}(\tilde{\Gamma}(t))$ is bounded, and Schwartz's inequality, one obtains

$$\frac{d}{dt}|\tilde{b}|^2 \le C_1\,|\tilde{b}|^2 + C_2, \qquad t\in(-\varepsilon, 0)$$

where C_1, C_2 are positive constants. Hence, $|\tilde{b}|^2$ remains bounded as $t\to 0$.
Next, taking (2.21) and (2.23) together, one derives

$$\frac{d\tilde{s}}{dt}-\frac{\tilde{s}}{\tilde{\rho}}\frac{d\tilde{\rho}}{dt} = -\tilde{\omega}^0_n - 2\,\mathrm{Im}\,(\tilde{b}^\beta\tilde{\omega}^\beta_n) + |\tilde{b}|^4\,\tilde{\rho}^{-2},$$

i.e.,

$$\frac{d}{dt}\left(\frac{\tilde{s}}{\tilde{\rho}}\right) = -\tilde{\rho}^{-1}\tilde{\omega}^0_n - 2\tilde{\rho}^{-1}\,\mathrm{Im}\,(\tilde{b}^\beta\tilde{\omega}^\beta_n) + \tilde{\rho}^{-3}\,|\tilde{b}|^4 \le C_3,$$

the last inequality by (2.10), (2.16), (2.17) and the bounds on $\tilde{\rho}^{-1}$ and $|\tilde{b}|^2$. Hence, $\tilde{s}$ is bounded as $t\to 0$. Finally, (2.21) and the bounds already achieved imply that $\log\tilde{\rho}$ is bounded, completing the proof of lemma 2.2.

Proof of lemma 2.1. Note that it suffices to show p may be joined to all q in some neighborhood of p. We use the computations made in the proof of lemma 2.2. Let X denote the vector field (2.1) on Y. Let $z^1,\dots,z^{n-1}$, $w=u+\sqrt{-1}\,v$ be holomorphic coordinates centered at p, and $\theta_p = du$, $\theta^i_p = dz^i$. Introducing group variables in Y as above, we write

$$X = g\cdot\left(\frac{\partial}{\partial u}+\alpha^i\frac{\partial}{\partial x^i}+\beta^i\frac{\partial}{\partial y^i}+(\text{vertical components})\right)$$

Here $x^i=\mathrm{Re}\,z^i$, $y^i=\mathrm{Im}\,z^i$, $i=1,\dots,n-1$, and u are taken as coordinates on M, and g is a non-vanishing function in a neighborhood $U\times H\subset Y$, U a neighborhood of p in M. We evaluate g, α^i, β^i along $\{p\}\times H\subset Y$ in terms of the group variables, using (2.16) and (2.17).

$$g=\tfrac{1}{2}\rho^{-2} \tag{2.23}$$

$$\alpha^i+\sqrt{-1}\,\beta^i=-b^i \tag{2.24}$$

Let π denote the projection $\pi: Y\to M$, and let us consider the map sending (t, b) near $0\in\mathbf{R}^{2n-1}$ to $\pi(\Phi_t(p_0\cdot B(b)))$, where Φ_t is the exponential map for the flow determined by X at time t, p_0 is point of $\{p\}\times H\subset Y$ determined by setting $\rho=1$, $a=I$, $\zeta=1$, $b=0$, $s=0$, and

$$B(b)=\begin{bmatrix} 1 & 0 & 0\\ -2i{}^t\bar{b} & I & 0\\ -\dfrac{i|b|^2}{4} & b & 1\end{bmatrix}$$

Note that if b is fixed and t varies, $\pi(\Phi_t(p_0\cdot B(b)))$ describes a parametrized chain in M.

Write

$$u(t, b)=u(\pi(\Phi_t(p_0\cdot B(b))))$$
$$x^i(t, b)=x^i(\pi(\Phi_t(p_0\cdot B(b))))$$
$$y^i(t, b)=y^i(\pi(\Phi_t(p_0\cdot B(b))))$$

and define

$$J(t)=\det\begin{bmatrix} \dfrac{\partial u}{\partial t}(t,0) & \dfrac{\partial u}{\partial \alpha^i}(t,0) & \dfrac{\partial u}{\partial \beta^i}(t,0)\\ \dfrac{\partial x^j}{\partial t}(t,0) & \dfrac{\partial x^j}{\partial \alpha^i}(t,0) & \dfrac{\partial x^j}{\partial \beta^i}(t,0)\\ \dfrac{\partial y^j}{\partial t}(t,0) & \dfrac{\partial y^j}{\partial \alpha^i}(t,0) & \dfrac{\partial y^j}{\partial \beta^i}(t,0)\end{bmatrix}$$

Note that we are using (2.24) here. Note also that $u(0, b)=x^i(0, b)=y^i(0, b)=0$. Hence, we obtain, using (2.24), that:

$$\frac{d^{2n-2}J}{dt^{2n-2}}(0)=\det\begin{bmatrix} 1/2 & 0 & 0\\ 0 & 1/2I & 0\\ 0 & 0 & 1/2I\end{bmatrix}=2^{-(2n-1)}\neq 0.$$

Hence, for t small enough, there is a neighborhood of $\gamma(t)=\pi(\Phi_t(p_0))$ each point of which is joined to p by a chain. Now the computation above is clearly uniform in a neighborhood of p_0 in Y. Hence, for t small enough, we may apply the above argument with p replaced by $\gamma(t)$, and conclude there is a neighborhood of p every point of which can be joined to $\gamma(t)$ by a parametrized chain. Hence, using two segments of parametrized chain, one may join p to any point in a neighborhood of p.

We remark that the proof of lemma 2.1 just given requires high differentiability of the hypersurface. One may avoid this by using the Lorentz structure over the hypersurface ([3], [8]). A more or less standard argument there will prove the same result, using only that M is C^4. Since this is not a central point, we'll avoid the extra formalism involved.

Finally, we'd like to close this section with an interpretive proposition which gives the significance of the principal estimate in our argument above, viz., the estimate of $|\tilde{b}|^2$. We thank J. Moser for his insistence that such a simple interpretation should exist. The main problem that lemma 2.2 is supposed to deal with is the existence of spiralling chains, as exhibited in [8]. These are chains which may be extended continuously through a limit point, but not smoothly as a solution of the chain equations. If a chain continues smoothly through a limit point p, then clearly the angle that its tangent vector $\dot{\gamma}$ makes with $H_{\gamma(t)}(M)$ remains bounded from below, since γ is transverse to $H_{\gamma(t)}(M)$. Conversely, we have

PROPOSITION 2.3. *Suppose $\gamma(t)$ is a chain (the parameter t need not be a distinguished parameter), such that $\lim_{t\to t_\infty}\gamma(t)=p$ and such that the angle between $\gamma(t)$ and $H_{\gamma(t)}(M)$ remains bounded from below. Then, after suitable smooth reparametrization, γ extends past p as a smooth solution of the chain equations.*

Proof. Introduce a local frame $\theta, \theta^1, \ldots, \theta^{n-1}$ near p as in (2.3) above. We reparametrize γ (smoothly) to normalize

$$\theta(\dot{\gamma})=1. \tag{2.25}$$

We recall from [4] that with this normalization, the chain equations are given by

$$\frac{d}{dt}\theta^i(\dot{\gamma})=-\sqrt{-1}\left\{\sum_{j=1}^{n-1}|\theta^j(\dot{\gamma})|^2\right\}\theta^i(\dot{\gamma})+\sum_{j=1}^{n-1}\theta^j(\dot{\gamma})\omega_j'^i(\dot{\gamma})+\omega_n'^i(\dot{\gamma})$$

where the notation $\omega_j'^i$, $\omega_n'^i$ is as in (2.9) above.

Note that if α is the angle between $\dot{\gamma}$ and $H_\gamma(M)$, then $\sin^2(\alpha)$ is of the exact order of magnitude

$$\frac{\theta(\dot{\gamma})^2}{\theta(\dot{\gamma})^2+\sum_{i=1}^{n-1}|\theta^i(\dot{\gamma})|^2}$$

Hence, to say $\sin^2(\alpha)\geq\varepsilon\geq 0$, with our normalization (2.25), is to say $\sum_{i=1}^{n-1}|\theta^i(\dot{\gamma})|^2$ remains bounded along $\gamma(t)$. Hence, the standard theorem on systems such as (2.26) implies that the given solution extends smoothly through p, proving the proposition.

We remark that showing $|\tilde{b}|^2$ is bounded in the proof of lemma 2.2 shows that $\sin^2(\alpha)$ is bounded from below on $\tilde{\gamma}(t)$, as $t\to 0$. We argue further there to insure that our given parametrization continues through $t=0$.

3. Compactness of automorphism groups

In this section we prove Theorem II. For this purpose, we introduce an intrinsic, CR-invariant pseudo-distance on the non-umbilic points of a strictly pseudoconvex hypersurface. It is closely related to the arc-length of the distinguished metric mentioned in §1, but its degeneration at the umbilic points is easier to manage.

Let M be a connected, strictly pseudoconvex hypersurface. Let $\Sigma\subset M$ denote, as in §1, the umbilic points of M. We define, for p, $q\in M$, the restricted pseudo-distance $d_H(p,q)$ as

$$d_H(p,q)=\inf_{\gamma\in F(p,q)}\left\{\int_a^b\|\dot{\gamma}\|_\theta\,\|S\|_\theta^{1/2}\,dt\right\}. \tag{3.1}$$

Here θ, $\|\cdot\|_\theta$, S are as in §1, and $F(p,q)$ is the set of all piece-wise $\mathscr{C}^1$ paths $\gamma:[a,b]\to M$, with $\gamma(a)=p$, $\gamma(b)=q$, and $\dot{\gamma}(t)\in H_{\gamma(t)}M$, for all $t\in[a,b]$. Let $d(p,q)$ denote the distance between p and q computed with respect to some fixed Riemannian metric on M. The basic facts about d_H are collected in the following lemma.

LEMMA 3.1. (a) *For any $p,q\in M$, $F(p,q)$ is non-empty.*

(b) $d_H(p,q)=d_H(q,p)$; $d_H(p,q)\leq d_H(p,n)+d_H(n,q)$, *for any $n\in M$.*

(c) *On the compact sets, $d_H(p,q)\leq$* const. $d(p,q)^{1/2}$, *where const. might depend on the compact set.*

(d) *For $p\in M-\Sigma$, $d_H(p,q)=0$ if and only if $p=q$.*

Proof. (a) Since M is connected, it suffices to show that for each $p \in M$, and all q sufficiently close to p, $F(p, q)$ is non-empty. To prove this, let $X_1, \ldots, X_{2n-2}$ be a local frame for $H(M)$ near p, with $X_{2n-1} = [X_1, X_2]$ independent of $X_1, \ldots, X_{2n-2}$ at all points in a neighborhood of p. This is possible by non-degeneracy of the Levi-form. Let Φ^i_τ denote the flow of X_i at time τ. Define a mapping Φ from a neighborhood of $0 \in \mathbf{R}^{2n-1}$ to a neighborhood of $p \in M$ by

$$\Phi(t_1, \ldots, t_{2n-1}) = \begin{cases} \Phi^2_{-\sqrt{t_{2n-1}}} \circ \Phi^1_{-\sqrt{t_{2n-1}}} \cdot \phi^2_{\sqrt{t_{2n-1}}} \cdot \phi^1_{\sqrt{t_{2n-1}}} \cdot \Phi^{2n-2}_{t_{2n-2}} \circ \cdots \circ \Phi^2_{t_2} \circ \Phi^1_{t_1}(p), & t_{2n-1} \geq 0. \\ \Phi^2_{\sqrt{|t_{2n-1}|}} \circ \Phi^1_{\sqrt{|t_{2n-1}|}} \cdot \Phi^2_{-\sqrt{|t_{2n-1}|}} \circ \Phi^1_{-\sqrt{|t_{2n-1}|}} \circ \Phi^{2n-2}_{t_{2n-2}} \circ \cdots \circ \Phi^1_{t_1}(p), & t_{2n-1} \leq 0. \end{cases}$$

This function is well-defined for $t = (t_1, \ldots, t_{2n-1})$ small enough, and is clearly smooth for $t_{2n-1} \neq 0$. That Φ is actually $\mathscr{C}^1$ along $t_{2n-1} = 0$ comes from the standard fact that locally, for $\tau \geq 0$,

$$\Phi^2_{-\sqrt{\tau}} \circ \Phi^1_{-\sqrt{\tau}} \circ \Phi^2_{-\sqrt{\tau}} \circ \Phi^1_{-\sqrt{\tau}}(q) = \Phi^{2n-1}_\tau(q) + 0(\tau^{3/2}),$$

and similarly, for $\tau \leq 0$,

$$\Phi^2_{\sqrt{|\tau|}} \circ \Phi^1_{\sqrt{|\tau|}} \cdot \Phi^2_{-\sqrt{|\tau|}} \circ \Phi^1_{-\sqrt{|\tau|}}(q) = \Phi^{2n-1}_\tau(q) + 0(|\tau|^{3/2}).$$

The derivative of Φ at $t = 0$ sends $\partial/\partial t_i$ to X_i, $1 \leq i \leq 2n-1$, hence Φ is a local $\mathscr{C}^1$ diffeomorphism. The image in M of a small cube about $0 \in \mathbf{R}^{2n-1}$ is the desired neighborhood of $p \in M$. Indeed, if $q = \Phi(t)$, $t = (t_1, \ldots, t_{2n-1})$, join p to q by first taking the path $\Phi^1_s(p)$ to $p_1 = \Phi^1_{t_1}(p)$, then $\Phi^2_s(p_1)$ to $p_2 = \Phi^2_{t_q}(p_1)$, etc., as prescribed in the definition of Φ.

(b) is immediate from the definitions. It suffices to show (c) for $p, q \in M$ such that $d(p, q) \leq \varepsilon$. We first fix p. Using the map Φ of part (a), we have $d(p, q) \geq c_1(p)\,|t|$ for $d(p, q) \leq \varepsilon = \varepsilon(p)$, and $q = \Phi(t)$. The path from p to q exhibited in (a) shows $d_H(p, q) \leq c_2(p)(|t_1| + \cdots |t_{2n-2}| + |t_{2n-1}|^{1/2})$ where the constant $c_2(p)$ depends only on the size of the X_i near p, and $\|S\|_\theta$ near p. Hence, $d_H(p, q) \leq c_3(p)\, d(p, q)^{1/2}$, where $c_3(p)$ depends only on $c_1(p)$, $c_2(p)$ and n. Finally, we note that for p' close to p, we may also use $X_1, \ldots, X_{2n-1}$ for the construction of the map Φ at p', and $c_1(p')$, $c_2(p')$, $c_3(p')$, $\varepsilon(p)$ depend continuously on p', proving (c).

For (d), if $p \neq q$, pick a positive $\delta < d(p, q)$ so that $\bar{B}(\delta, p) = \{s \in M \mid d(p, s) \leq \delta\} \subset M - \Sigma$. There is a $c > 0$ so that $c\,\|X\|_{\theta_0} \geq \|X\|$, for $X \in H_s(M)$,

$s \in \bar{B}(\delta, p)$, where $\|X\|$ is the length of X with respect to the Riemannian metric used to calculate $d(p, q)$, and $\theta_0 = \|S\|_\theta \cdot \theta$, as in §1. From this it follows that $d_H(p, q) \geq \delta/c$, completing the proof of the lemma.

From (a) and the definition, we know that $d_H(p, q)$ is well-defined and finite. From (b) and (c) we know it is continuous, and from (d) we know that, if $p \in M - \Sigma$, and M is compact, then $d_H(p, \Sigma) = \inf_{q \in \Sigma} d_H(p, q)$ is finite and positive. Set $U_\varepsilon = \{p \in M - \Sigma \mid d_H(p, \Sigma) \geq \varepsilon\}$. For ε sufficiently small, we know that U_ε is a compact set in $M - \Sigma$, with non-empty interior, and that U_ε is *invariant* under all smooth CR-automorphisms of M (since d_H and Σ are).

LEMMA 3.2. *Let D be as in Theorem* II. *If* Aut D *is non-compact, then ∂D is spherical, i.e., $\Sigma = \partial D$.*

Proof. The proof is by contradiction. Let $\{f_i\}$ be a sequence in Aut(D) with no convergent subsequence. By Montel's theorem, a subsequence of the f_i converges uniformly on compact sets to a map f_0. For $q \in D$, let $p = f_0(q)$. If p were in D, then the f_i would converge uniformly on compact sets, with all derivatives, to a map $f \in \text{Aut}(D)$: this is because the f_i's are isometries of the Bergmann metric of D(cf. [9], IV 2.2). Hence, $p \in \partial D$. By the strict pseudoconvexity of ∂D, $f(D) = \{p\}$.

Suppose $\partial D - \Sigma$ is non-empty. By C. Fefferman's extension theorem, every $f \in \text{Aut}(D)$ induces a $\mathscr{C}^\infty$ CR-automorphism of ∂D (still denoted f). Thus, on ∂D, each f_i preserves the sets U_ε constructed above, and each f_i induces an isometry of the distinguished metric on $\partial D - \Sigma$. Pick $s \in U_\varepsilon$, and pass again to a subsequence of the f_i so that $\{f_i(s)\}$ is a convergent sequence. This is possible because U_ε is compact. Hence, near s, the f_i converge uniformly, together with all derivatives to a local isometry f of $\partial D - \Sigma$ in the distinguished metric ([9], loc. cit.), and f must also be a local CR equivalence. By Lewy's extension theorem, f is the boundary value of a biholomorphic equivalence (still denoted f) of a neighborhood of s in $\bar{D}$ with a neighborhood of $f(s)$ in $\bar{D}$. However, by the strict pseudoconvexity of ∂D, and the maximum principle, there is a compact set $K \subset D$ such that the f_i converge uniformly to f on K, and $f(K) \subset D$. This contradicts the conclusion previously reached, that $f_i(K) \to p$, proving the lemma.

Suppose that Aut(D) is non-compact. The first part of the proof just given shows that the homotopy groups $\pi_k(D) = \pi_k(\bar{D})$ are trivial, for all k. Indeed, if $\gamma : S^k \to D$ represents an element of $\pi_k(D)$, then $f_i \circ \gamma$ converges uniformly to a point map, hence is null-homotopic for $i \gg 0$, but $f_i \cdot \gamma$ represents $f_{i\#}(\gamma) \in \pi_k(D)$, and $f_{i\#}$ is an isomorphism. Since γ was arbitrary, $\pi_k(D) = 0$.

Now consider a segment of the homotopy sequence for $(\bar{D}, \partial D)$:

$$\pi_2(\bar{D}, \partial D) \to \pi_1(\partial D) \to \pi_1(D).$$

By the above, $\pi_1(D)=0$. On the other hand, by the Lefschetz theorem, if $n=\dim_{\mathbf{C}} D\geq 3$, $\pi_2(\bar{D}, \partial D)=0$. Thus, if $n\geq 3$, $\pi_1(\partial D)=0$. Since we know by lemma 3.2 that ∂D is spherical, we conclude ∂D is CR-equivalent to $S^{2n-1}\subset \mathbf{C}^n$, and D is biholomorphic to $\mathbf{B}^n$ ([4], §1). This completes the proof of Theorem II(b).

We sketch the proof of the weaker statement II(a). We note first that by lemma 2.2, ∂D is spherical, if $\mathrm{Aut}^\circ(D)$ is not compact. Let f_t be a one-parameter subgroup of $\mathrm{Aut}^\circ(D)$, not contained in any compact set. As above we conclude that there exists a sequence $\{t_i\}$ so that $f_{t_i}(p)\to q\in\partial D$ uniformly on compact sets as $t_i\to+\infty$. Note, then, that each f_t is smooth to the boundary, and

$$f_t(q)=f_t(\lim_{i\to\infty} f_{t_i}(p))=\lim_{i\to\infty} f_t(f_{t_i}(p))=\lim_{i\to\infty} f_{t_i}(f_t(p))=q.$$

Now ∂D is spherical near q, so we may take, as in §1, a local biholomorphic equivalence Φ of $\bar{D}$ near q with the quadric $\partial\mathcal{U}=\{(z,w)\mid \mathrm{Im}\,(w)=|z|^2,\ w\in\mathbf{C},\ z\in\mathbf{C}^{n-1}\}$ near $(0,0)$. The germ of one-parameter group f_t at $q\in\bar{D}$ is conjugated by Φ to a 1-parameter group of linear fractional transformations of $\mathcal{U}=\{(z,w)\mid \mathrm{Im}\,(w)>|z|^2\}$ which fix 0. If ξ denotes the infinitesimal generator of this 1-parameter group on $\mathcal{U}$, one may check explicitly that there are arbitrarily small connected open sets $U_i\subset\mathcal{U}$, with $(0,0)\subset\bar{U}_i$, such that the $\exp(t\xi): U_i\to U_i$, for all $t>0$, and that $\exp(t\xi)(p)\to(0,0)$, uniformly on compact sets in $\mathcal{U}$, for $t>0$, $t\to+\infty$. Conjugating this back to D via Φ, one concludes: (a) the existence of a connected open set V in D, with $q\in\bar{V}$ and $\bar{V}$ in the domain of the local equivalence Φ; (b) that $f_t: V\to V$, for all $t>0$, and $f_t(p)\to q$ as $t\to+\infty$, uniformly on compact sets in D. Using this, we can make the local equivalence Φ between D and $\mathbf{B}^n$ a global one as follows.

For arbitrary $p\in D$, define

$$\Phi(p)=\exp(-t\xi)(\Phi(f_t(p))$$

for t sufficiently large. First of all note that for $p\in V$, this is independent of t, and agrees with Φ as already defined there, by the definition of ξ. On any compact set $K\subset D$, for $t\gg 0$, $f_t(K)\subset$ the domain of our original Φ, and the definition above makes sense on K for such t. Finally, these definitions for different t will all agree, by analytic continuation, since they all agree on V. The procedure may be reversed, setting

$$\Phi^{-1}(p)=f_t^{-1}((\Phi^{-1}(\exp(t\xi)(p))).$$

Hence Φ gives a biholomorphism of D with $\mathbf{B}^n$.

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