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Some rational computations of the Waldhausen algebraic K theory

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Introduction

The purpose of this paper is to compute the rational part of the algebraic K-theory defined by Waldhausen [W] of the following type of topological (semisimplicial) rings R.:

(a) R. is an associative topological (semisimplicial) ring with unit and $\Pi_0(\text{R.}) = \text{Z}$ (Z denotes the ring of integers).

(b) There exists a ring homomorphism $\iota : \Pi_0(\text{R.}) \to \text{R.}$ so that $\pi\iota = \text{id}$ where $\pi : \text{R.} \to \Pi_0\ (\text{R.})$[1] is the canonical projection.

(c) $\Pi_i(\text{R.}) \otimes_{\text{Z}} \text{Q} = 0$[1] for $i \neq r \geq 1$ and $\Pi_r(\text{R.}) \otimes_{\text{Z}} \text{Q} = \text{Q}$, where Q is the ring of rationals.

These computations provide in particular the computation of the rational part of the Waldhausen's algebraic K-theory of a space X, in the case X has the rational homotopy type of a $\text{K}(\text{Z}, 2r)$[2] and implicitly the computation of the rational homotopy type of ${}^{Pl}\text{Wh}(\text{X})$, ${}^{Pl}\text{Wh}\pm(\text{X})$, ${}^{Diff}\text{Wh}\ (\text{X})\pm$ (see [B.L] for notations), for $\text{K}(\text{Z}, 2r)$.

In this paper we give the results only for ${}^{Diff}\text{Wh}$, the problem of the computation of ${}^{Pl}\text{Wh}(\text{X})$, ${}^{Pl}\text{Wh}\pm(\text{X})$ will be contained in another paper on automorphisms of manifolds.

The methods of this paper allow the same computations for X of the rational homotopy type of a $\text{K}(\text{G}, 2r)$ but more "classical invariant theory" is necessary and the author has not yet worked it out.

The paper is organised as follows:

In section 1 we recall briefly the algebraic K-theory of Waldhausen for rings and for topological spaces and present the main results as a consequence of Theorem 3.1 of section 3. In section 2 we present the "invariant theory" necessary for the proof of Theorem 3.1 and in section 3 the proof of this theorem.

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[1] $\Pi_0(\text{R.})$ denotes the ring of connected components and $\Pi_i(\text{R.})$ the homotopy groups of R. with respect to the base point "0".

[2] $\text{K}(\text{G}, s)$ denotes the Eilenberg–MacLane space corresponding to G and s.

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§1

Let Ring^t be the category of topological (semisimplicial) rings which are always assumed to be associative and with unit, and continuous (semisimplicial) ring homomorphisms which are assumed to be unit preserving. Let Top_* be the category of based pointed topological spaces (semisimplicial complexes) and based point preserving maps, Gr^t be the category of topological (semisimplicial) groups and continuous (semisimplicial) homomorphisms and $\overset{\infty}{\Omega}$ the subcategory of Top_* consisting of ∞-loopspaces and ∞-loop space maps.

Following Waldhausen [W] one defines the algebraic K-theory as a functor $\mathbb{K}:\mathrm{Ring}^t \rightsquigarrow \overset{\infty}{\Omega}$ which is a homotopy functor in the sense that if f_1, f_2 are two homotopic morphisms (homotopic by "morphisms") then $\mathbb{K}(f_1)$ and $\mathbb{K}(f_2)$ are homotopic in $\overset{\infty}{\Omega}$ and if $f:\mathrm{R}. \to \mathrm{R}.$ is k-connected then $\mathbb{K}(f)$ is $(k+1)$-connected. Since we are not interested in the ∞-loop space structure of $\mathbb{K}$ we regard $\mathbb{K}$ as a functor with values in Top_* whose definition is the following.

For any n let $\widetilde{\mathrm{GL}}(\mathrm{R}., n)$ be the space[3] (semisimplicial complex) of $n \times n$ matrices $\{a_{ij}\}$, $a_{ij} \in \mathrm{R}.$, with $\{\pi(a_{ij})\}$ invertible. The composition of matrices endows $\widetilde{\mathrm{GL}}(\mathrm{R}., n)$ with a structure of associative H-space and the inclusion $\widetilde{\mathrm{GL}}(\mathrm{R}., n) \to \widetilde{\mathrm{GL}}(\mathrm{R}., n+1)$ defined by $\mathrm{A} \to \left(\begin{array}{c|c} \mathrm{A} & 0 \\ \hline 0 & 1 \end{array}\right)$ is a morphism of associative H-spaces. We take $\widetilde{\mathrm{GL}}(\mathrm{R}.) = \widetilde{\mathrm{GL}}(\mathrm{R}., \infty) = \lim_{\to} \widetilde{\mathrm{GL}}(\mathrm{R}., n)$ which is an associative H-space whose $\Pi_0(\widetilde{\mathrm{GL}}(\mathrm{R}.)) = \mathrm{GL}(\pi_0(\mathrm{R}.)) = \lim_{\to} \mathrm{GL}(\Pi_0(\mathrm{R}.); n)$. Applying the "classifying space" functor to $\widetilde{\mathrm{GL}}(\mathrm{R}.)$ one obtains $\mathrm{B}\widetilde{\mathrm{GL}}(\mathrm{R}.)$ whose $\Pi_1(\mathrm{B}\widetilde{\mathrm{GL}}(\mathrm{R}.)) = \mathrm{GL}(\Pi_0(\mathrm{R}))$ has the commutator a perfect group [L]. Consequently one can apply the Kervaire–Quillen's "+"-construction and the resulted space (semisimplicial complex) will be denoted by $\mathrm{B}\widetilde{\mathrm{GL}}(\mathrm{R}.)_+$. We define $\mathbb{K}(\mathrm{R}.) = \mathrm{Z} \times \mathrm{BGL}(\mathrm{R}.)_+$ where Z denotes the ring of integers. If $f:\mathrm{R}. \to \mathrm{R}'.$ is a morphism, it induces $\mathbb{K}(f):\mathbb{K}(\mathrm{R}.) \to \mathbb{K}(R'.)$ with the properties we have mentioned.

The "loop space" functor $\Omega:\mathrm{Top}_* \rightsquigarrow G_r^t$ in the semisimplicial case is the Kan's free group construction F and in topological case any "group type" construction

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[3] With the obvious topology.

* Added in proofs: Similar results have been independently obtained by Hsiang and Staffeldt; more recently, the author and Hsiang and Staffeldt have obtained upper bounds for $\dim \Pi_i(\mathscr{H}(\mathrm{X})) \otimes \mathrm{Q}$ for X 1-connected and with finite Betti numbers.

of the loop space for example the Milnor's construction $[M_2]$ or $X \rightsquigarrow |F(\operatorname{Sing} X)|$ where Sing denotes the singular complex and $|.\,.\,.|$ the "geometric realisation".

Let $\mathbf{Z}: G_r^t \rightsquigarrow \mathrm{Ring}^t$ be the functor which associates with any topological (semisimplicial) group G the ring $\mathbf{Z}(G)$ the topological (semisimplicial) analogous of the group ring; in the semisimplicial context it is actually the group ring, in topological context we can take $|\mathbf{Z}(\operatorname{Sing} G)|$ or any other functor from $G_r^t \rightsquigarrow \mathrm{Ring}^t$ which is essentially the infinite symmetric product. The composition $\mathcal{K} = \mathbb{K} \circ \mathbf{Z} \circ \Omega$ produces a functor defined on Top_* with values in $\overset{\infty}{\Omega}$ which is a homotopy functor and has the property that $f: X \to Y$ κ-connected implies $\mathcal{K}(f)$ κ-connected.

As we have mentioned, our purpose is to compute $\mathbb{K}(R.)$ for rings which satisfy the properties (a), (b), (c) mentioned in Introduction and in fact we prove the following theorem.

THEOREM 1.1. *If R. satisfies the conditions (a), (b), (c) then* $\mathbb{K}(R.)$ *has the rational homotopy of* $\mathbb{K}(Z) \times T_{r+1}$ *where* $T_r = \prod_{i=1}^{\infty} K(Z, 2si)$ *if* $r = 2s$ *and* $T_r = \prod_{i=1}^{\infty} K(Z; (2s+1)(2i-1))$ *if* $r = 2s+1$.

As a consequence one obtains.

THEOREM 1.2. *If X has the rational homotopy type of* $K(Z, 2r)$, $r > 0$, *then* $\mathcal{K}(X)$ *has the rational homotopy type of* $\mathbb{K}(Z) \times \prod_{i=1}^{\infty} K(Z; 2ri)$.

COROLLARY 1.3. *If X has the rational homotopy type of* $K(Z, 2r)$, ${}^{\mathrm{Diff}}\mathrm{Wh}(X)$, ${}^{\mathrm{Diff}}\mathrm{Wh}\pm(X)$ *have the rational homotopy type of* ${}^{\mathrm{Diff}}\mathrm{Wh}(pt)$ *respectively* ${}^{\mathrm{Diff}}\mathrm{Wh}\pm(pt)$.

Recall from [W] that ${}^{\mathrm{Diff}}\mathrm{Wh}(pt)$[4] has the rational homotopy type of $\prod_{i=1}^{\infty} K(Z; 4i)$ and from [F, H] that ${}^{\mathrm{Diff}}\mathrm{Wh}_+$ respectively ${}^{\mathrm{Diff}}\mathrm{Wh}_-$ have the rational homotopy type of ${}^{\mathrm{Diff}}\mathrm{Wh}(pt)$ respectively *pt*.

Proof of Theorem 1.1. Let $M_n(\mathring{R}.)$ respectively $\mathcal{M}_n(\mathring{R}.)$ be the space (semisimplicial complex) of $n \times n$ matrices with entries in $\mathring{R}.$, the connected component of "0", endowed with the composition law "+" respectively "$*$" given by $M * N = M + N + MN$.

Consider the diagram

$$
(*) \quad \begin{array}{ccccc}
\mathcal{M}_n(\mathring{R}.) & \xrightarrow{\sigma_n} & \widetilde{GL}(R., n) & \xrightarrow{\omega_n} & GL(\Pi_0(R.), n) \\
\downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_n} \\
\mathcal{M}_{n+1}(\mathring{R}.) & \xrightarrow{\sigma_{n+1}} & \widetilde{GL}(R., n+1) & \xrightarrow{\omega_{n+1}} & GL(\Pi_0(R.), n+1)
\end{array}
$$

[4] Our ${}^{\mathrm{Diff}}\mathrm{Wh}$ is the loop space of the one defined by Waldhausen in [W].

where $\sigma_n(M)=(M+I)$, $\tilde{\iota}_n$ and $\bar{\iota}_n$ are defined by $A \rightsquigarrow \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$, i_n by $M \rightsquigarrow \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right)$ and ω_n by $\omega_n\{a_{ij}\}=\{\pi a_{ij}\}$.

Passing to the limit in diagram (*) one obtains the fibration

$$(^{**})\quad \mathcal{M}_\infty(R.) \xrightarrow{\sigma_\infty} \widetilde{GL}(R.) \xrightarrow{\omega_\infty} GL(\Pi_0(R.))$$

with all terms associative H-spaces and σ_∞ respectively ω_∞ homomorphisms. Applying the "classifying space" functor to (**) one obtains the fibration

$$(^{***})\quad B\mathcal{M}_\infty(\mathring{R}.) \xrightarrow{B\sigma_\infty} B\widetilde{GL}(R.) \xrightarrow{B\omega_\infty} BGL(\Pi_0(R.))$$

Assume now that the ring R. satisfies our hypothesis (b), hence there exists a morphism $\iota:\Pi_0(R.)\to R.$ so that $\Pi.\iota=\mathrm{id}$; ι induces the group homomorphism $\bar{\iota}:GL(\Pi_0(R.)\to \widetilde{GL}(R.)$ and consequently we can define the representation ρ_∞ of $GL(\Pi_0(R.))$ on the H-space $M_\infty(\mathring{R}.)$ by $\rho_\infty(A;M)=\bar{\iota}(A)\cdot M\cdot \bar{\iota}(A)^{-1}$ for $A\in GL(\Pi_0(R.))$ $M\in\mathcal{M}_\infty(\mathring{R}.)$. Clearly

$$\rho_\infty(A;\ldots):\mathcal{M}_\infty(\mathring{R}.)\to\mathcal{M}_\infty(\mathring{R}.)$$

is an H-space isomorphism, consequently one can apply the classifying space functor to $\rho_\infty(A;\ldots)$ and obtain the action

$$B\rho_\infty: GL(\Pi_0(R.))\times B\mathcal{M}_\infty(\mathring{R}.)\to B\mathcal{M}_\infty(\mathring{R}.).$$

PROPOSITION 1.4. *The fibration* (***) *is the fibration over* $BGL(\Pi_0(R.))$ *associated with the action* $B\rho_\infty$.

Proof of Proposition 1.4. Let us recall the definition of the semidirect product of $\mathcal{M}_\infty(\mathring{R}.)\times_{\rho_\infty} GL(Z)$. This is the associative H-space structure defined on $\mathcal{M}_\infty(\mathring{R}.)\times GL(Z)$ by the following composition law

$$(M', A')\#(M, A)=(\{\rho_\infty(A^{-1};M)\} * M, A'\cdot A)$$

where $M, M'\in\mathcal{M}_\infty(\mathring{R}.)$ and $A, A'\in GL(Z)$. The natural projection $(M,A)\to A$ defines an homomorphism $p_2:\mathcal{M}_\infty(\mathring{R}.)\times_{\rho_\infty} GL(Z)\to GL(Z)$ whose kernel is exactly $\mathcal{M}_\infty(\mathring{R}.)$.

In order to prove Proposition 1.4 it is obviously enough to show that (**) is isomorphic to

$$\mathcal{M}_\infty(\mathring{R}.)\to\mathcal{M}_\infty(R.)\times_{\rho_\infty} GL(Z)\to GL(Z)$$

and this isomorphism is established by $\gamma:\widetilde{\mathrm{GL}}(\mathrm{R.})\to \mathcal{M}_\infty(\mathring{\mathrm{R}}.)\times_{\rho_\infty}\mathrm{GL}(\mathrm{Z})$ defined by

$\gamma(\mathrm{A})=(\sigma_\infty^{-1}\{\bar{\iota}(\omega(\mathrm{A})^{-1})\cdot \mathrm{A}-\mathrm{I}\},\ \omega(\mathrm{A}))$ which makes sense since

$\{\iota(\omega(\mathrm{A})^{-1})\cdot \mathrm{A}-\mathrm{I}\}\in\sigma_\infty(\mathrm{M}_\infty(\mathrm{R.}))$

q.e.d.

Remark. The same proof shows that

$\mathrm{B}\mathcal{M}_n(\mathring{\mathrm{R}}.)\to \mathrm{B}\widetilde{\mathrm{GL}}(\mathrm{R.},n)\to \mathrm{BGL}(\Pi_0(\mathrm{R.}),n))$ (with the hypothesis (b) on R.) is the fibration induced by the representation $\rho_n:\mathrm{GL}(\Pi_0(\mathrm{R.}),n)\times\mathcal{M}_n(\mathring{\mathrm{R}}.)\to\mathcal{M}_n(\mathring{\mathrm{R}}.)$ defined by the same formula.

Let us observe that if conditions (a), (b), (c) are satisfied then

$$\Pi_i(\mathrm{B}\mathcal{M}_\infty(\mathring{\mathrm{R}}.))\otimes_{\mathrm{Z}}\mathrm{Q}=\Pi_{i-1}(\mathcal{M}_\infty(\mathring{\mathrm{R}}.))\otimes_{\mathrm{Z}}\mathrm{Q}=\mathrm{M}_\infty(\Pi_{i-1}(\mathrm{R.})\otimes_{\mathrm{Z}}\mathrm{Q})=\begin{cases}0 & \text{if } i\neq r+1\\ \mathrm{M}_\infty(\mathrm{Q}) & \text{if } i=r+1\end{cases}$$

and $\Pi_1(\mathrm{B}\mathcal{M}_\infty(\mathring{\mathrm{R}}.))=\Pi_0(\mathcal{M}_\infty(\mathring{\mathrm{R}}.))=0$. Consequently the fibrewise "0-localisation" of the fibration (***) is the fibration

$$\mathrm{K}(\mathrm{M}_\infty(\mathrm{Q}),r+1)\to\mathrm{E}\to\mathrm{BGL}(\mathrm{Z})$$

associated with the action

$$\bar{\rho}_\infty:\mathrm{GL}(\mathrm{Z})\times\mathrm{K}(\mathrm{M}_\infty(\mathrm{Q}),r+1)\to\mathrm{K}(\mathrm{M}_\infty(\mathrm{Q}),r+1);$$

this action is determined by the adjoint representations $\rho_\infty:\mathrm{GL}(\mathrm{Z})\times\mathrm{M}_\infty(\mathrm{Q})\to\mathrm{M}_\infty(\mathrm{Q})$ given by $\rho_\infty(\mathrm{A}:\mathrm{M})=\mathrm{A.M.A}^{-1}$.

Warning. If (a) and (b) are satisfied and $\mathrm{B}\mathcal{M}_\infty(\mathring{\mathrm{R}}.)$ has trivial rational Postnicov invariants, we might be tempted to believe that the fibrewise "0-localisation" of the fibration (***) is the fibration with fibre $\prod_{s=2}^{\infty}(\mathrm{K}(\mathrm{G}_s\otimes\mathrm{Q}),s))$ associated with the action $\prod_{s=2}^{\infty}s_{\rho_\infty}$ where s_{ρ_∞} is the action induced by the representation $s_{\rho_\infty}:\mathrm{GL}(\mathrm{Z})\times\mathrm{M}_\infty(\mathrm{G}_s\otimes\mathrm{Q})\to\mathrm{M}_\infty(\mathrm{G}_s\otimes\mathrm{Q})$ defined by $s_{\rho_\infty}(\mathrm{A},\mathrm{M})=\mathrm{A}\cdot\mathrm{M}\cdot\mathrm{A}^{-1}$ this is not always the case.

The proof of Theorem 1.1 follows now immediately from Theorem 3.1.

Proof of Theorem 1.2. If $\mathrm{X}=\mathrm{K}(\mathrm{Z},2r)$ then $\Omega\mathrm{X}=\mathrm{K}(\mathrm{Z},2r-1)$ and consequently $\mathbf{Z}\Omega(\mathrm{X})$ has as homotopy groups the homology groups of $\Omega(\mathrm{X})$ since

$\mathbf{Z}\Omega(X)$ is essentially the infinite symmetric product of ΩX. This makes clear that $\mathbf{Z}\Omega(X)$ satisfies (c) since X is 1-connected (a) is also satisfied and (b) is trivially satisfied since $Zpt = Z$. Consequently the theorem is true by Theorem 1.1 for $K(Z, 2r)$. The construction of the functor $\mathcal{H}$ implies immediately that if X and Y are rationally homotopy equivalent then $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are.

q.e.d

Proof of Corollary 1.3. In [W] Waldhausen defines two natural transformation $\mathcal{H}(\ldots) \to \mathcal{H}^s(\ldots)$ where $\mathcal{H}^s(\ldots)$ is the stabilized functor associated with $\mathcal{H}$, which is an unreduced homology theory, and $h(\ldots; \mathcal{H}(pt)) \to \mathcal{H}(\ldots)$ where $h(\ldots; \mathcal{H}(pt))$ is the reduced homology theory produced by the ∞-loop space $\mathcal{H}(pt)$.

The composition $h(\ldots; \mathcal{H}(pt)) \to \mathcal{H}^s(\ldots)$ is a natural transformation of homology theory and because $\mathcal{H}(pt) \to \mathcal{H}^s(pt)$, is rationally homotopy surjective, $h(X : \mathcal{H}(pt)) \to \mathcal{H}^s(X)$ is rationally homotopy surjective. On the other side $B^{Diff}Wh(X)$ is the fibre of $\mathcal{H}(X) \to \mathcal{H}^s(X)$. Consequently $\mathcal{H}(X)$ and $B^{Diff}Wh(X) \times \mathcal{H}^s(X)$ are rationally homotopy equivalent. (Waldhausen claims a much stronger fact namely $A(X)$ and $B^{Diff}Wh(X) \times A^s(X)$ are homotopy equivalent which will imply the mentioned rational homotopy equivalence).

§2

Let k be one of the fields Q, R, C of rational, real or complex numbers, and $M_n(k)$ be the Lie algebra of $GL(k, n)$, i.e. the Lie algebra of $n \times n$ matrices over k.

We denote by ${}^k\rho_n$ or ρ_n, using ${}^k\rho_n$ only when we are interested to explicit the field, the adjoint representation of $GL(k, n)$ on $M_n(k)$ defined by $\rho_n(A, M) = A \cdot M \cdot A^{-1}$ for $A = GL(k, n)$ and $M \in M_n(k)$, by ρ_n^* its dual representation and by $\Lambda^k \rho_n$, $S^k \rho_n$, $\Lambda^k \rho_n^*$, $S^k \rho_n^*$ the k-times exterior respectively symmetric power of ρ_n respectively ρ_n^*. Denote by $\operatorname{Inv} \xi$ the fixed point subspace of the representation ξ. The following theorem contains well known facts; since the present formulation is not easy to be found in literature we enclose the proof.

THEOREM 2.1. (1) *There exists an injective linear map* $s_n^k : H^k(U(n) : k) \to \Lambda^k M_n^*(k)^5$ *with* $s_n^k(H^k(U(n) : k)) = \operatorname{Inv} \Lambda^k({}^k\rho_n^*)$ *such that the following diagram is commutative.*

$$\begin{array}{ccc} H^k(U(n) : k) & \xrightarrow{s_n^k} & \overset{k}{\Lambda} M_n^*(k) \\ \uparrow{\scriptstyle j_n^*} & & \uparrow{\scriptstyle \Lambda^k i_n^*} \\ H^k(U(n+1) : k) & \xrightarrow{s_{n+1}} & \overset{k}{\Lambda} M_{n+1}^*(k) \end{array}$$

where j_n^* *is induced by the canonical inclusion* $\mathrm{U}(n) \xrightarrow{j_n} \mathrm{U}(n+1)$ *and* i_n *is the canonical Lie algebra inclusion* $i_n(\mathrm{M}) = \left|\begin{array}{c|c} \mathrm{M} & 0 \\ \hline 0 & 0 \end{array}\right|$.

(2) *There exists an injective linear map*

$$q_n^k : \mathrm{H}^{2k}(\mathrm{BU}(n) : \mathscr{k}) \to \overset{k}{\mathrm{S}}\, \mathrm{M}_n^*(\mathscr{k})$$

with

$$q_n^k(\mathrm{H}^{2k}(\mathrm{BU}(n) : \mathscr{k})) = \mathrm{Inv}\, \overset{k}{\mathrm{S}}\, {}^{\mathscr{k}}\rho_n^*$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{H}^{2k}(\mathrm{BU}(n) : \mathscr{k}) & \xrightarrow{q_n^k} & \overset{k}{\mathrm{S}}\, \mathrm{M}_n^*(\mathscr{k}) \\ \uparrow{\scriptstyle 'j_n^*} & & \uparrow{\scriptstyle \mathrm{S}^k i_n^*} \\ \mathrm{H}^{2k}(\mathrm{BU}(n+1) : \mathscr{k}) & \xrightarrow{q_{n+1}^k} & \overset{k}{\mathrm{S}}\, \mathrm{M}_{n+1}^*(\mathscr{k}) \end{array}$$

where $'j_n^*$ *is induced by the canonical inclusion* $\mathrm{BU}(n) \xrightarrow{j_n} \mathrm{BU}(n+1)$.

Proof. Since ${}^{\mathrm{R}}\rho_n^*$ is a real form of ${}^{\mathrm{C}}\rho_n^*$ it is clear that the proof for $\mathscr{k} = \mathrm{R}$ implies the result for $\mathscr{k} = \mathrm{C}$.

Proof of (1). ($\mathscr{k} = \mathrm{R}, \mathrm{Q}$). Let η_n be the adjoint representation of $\mathrm{U}(n)$ on its Lie algebra; η_n^* is a real form of ${}^{\mathrm{R}}\rho_n^*$. Analogously let ${}^{\mathrm{Q}}\eta_n$ be the adjoint representation of the group

$${}^{\mathrm{Q}}\mathrm{U}(n) = \{\mathrm{A} = \{a_{tj} = \alpha_{tj} + i\beta_{tj}\} \mid \mathrm{A} \in \mathrm{U}(n),\ \alpha_{tj}, \beta_{tj} \in \mathrm{Q}\}$$

on the Q-Lie algebra

$$\mathrm{m} = \{\mathrm{M} = \{m_{tj} = \alpha_{tj} + i\beta_{tj}\} \mid m_{jt} + \bar{m}_{tj} = 0,\ \alpha_{tj}, \beta_{tj} \in \mathrm{Q}\}$$

given by ${}^{\mathrm{Q}}\eta_n(\mathrm{A}, \mathrm{M}) = \mathrm{A} \cdot \mathrm{M} \cdot \mathrm{A}^{-1}$.

[5] For a $\mathscr{k}$ vector we denote by V^* its dual.

Clearly it is enough to prove (1) for R η_n^* respectively ${}^{Q}\eta_n^*$ in order to have it proved for ${}^{R}\rho_n^*$ respectively ${}^{Q}\rho_n^*$.

Let us recall that de Rham theory permits to associate with any closed differential form on a differentiable manifold a precise singular cohomology class with coefficients in R.

Therefore we have the linear map $t_n : \mathrm{Inv}\,(\Lambda^k\, \eta_n^*) \to \mathrm{H}^k(\mathrm{U}(n) : \mathrm{R})$ constructed as follows; an element of $\mathrm{Inv}\,(\Lambda^k\, \eta_n^*)$ is regarded as a k-form on the Lie algebra of $\mathrm{U}(n)$ which by translation is extended to a k-differential form on the compact Lie group $\mathrm{U}(n)$; since the element we started with is in $\mathrm{Inv}\,(\Lambda^k\, \eta_n^*)$ the obtained differential form is biinvariant therefore closed.

It is well known (for any compact Lie group) that t_n is an isomorphism. Moreover $t_n/\Lambda^k\, {}^{Q}\eta_n^*$ factors through $\mathrm{H}^k(\mathrm{U}(n) : \mathrm{Q})$ since a form in $\mathrm{Inv}\,(\Lambda^k\, \eta_n^*)$ with rational coefficients (with respect to the canonical base) produces a cohomology class with rational periods on all integral cycles. Consequently we have the cummutative diagram

$$\begin{array}{ccc} \mathrm{Inv}\,\overset{k}{\Lambda}\mathrm{R}\eta_n^* & \xrightarrow{t_n{}^k} & \mathrm{H}^k(\mathrm{U}(n) : \mathrm{R}) \\ \uparrow{\scriptstyle \bar{U}} & & \uparrow{\scriptstyle \bar{U}} \\ \mathrm{Inv}\,\overset{k}{\Lambda}\, {}^{Q}\eta_n^* & \xrightarrow{{}^{Q}t_n{}^k} & \mathrm{H}^k(\mathrm{U}(n) : \mathrm{Q}) \end{array}$$

which implies ${}^{Q}t_n^k$ is an isomorphism.

We also observe that $\Lambda^k i_n^* : \Lambda^k \eta_{n+1}^* \to \Lambda^k \eta_n^*$ sends $\mathrm{Inv}\,\Lambda^k \eta_{n+1}^*$ into $\mathrm{Inv}\,\Lambda^k \eta_n^*$ where i_n is the canonical inclusion of the Lie algebra of $\mathrm{U}(n)$ into the Lie algebra of $\mathrm{U}(n+1)$, (analogously $\Lambda^{kQ} i_n^*$ sends $\mathrm{Inv}\,\Lambda^{kQ}\eta_{n+1}^*$ into $\mathrm{Inv}\,\Lambda^{kQ}\eta_n^*$) and the following diagram is commutative

$$\begin{tikzcd} \Lambda^k \eta_{n+1}^* \arrow[r] \arrow[d, "t_{n+1}^k"] & \Lambda^k \eta_n^* \arrow[d, "t_n^k"] \\ \mathrm{H}^k(\mathrm{U}(n+1); R) \arrow[r] & \mathrm{H}^k(\mathrm{U}(n); \mathrm{R}) \end{tikzcd}$$

since the correspondence "biinvariant forms" $\rightsquigarrow$ "cohomology" is a functorial isomorphism for the category of compact Lie groups. If we take $s_n^k = (t_n^k)^{-1}$ and ${}^{Q}s_n^k = ({}^{Q}t_n^k)^{-1}$, (1) is proved.

Proof of 2. ($\mathscr{k} = \mathrm{Q}, \mathrm{R}$). Let us consider $c_n(\mathscr{k})$ the Lie subalgebra of $\mathrm{M}_n(\mathscr{k})$ consisting of the diagonal matrices and ${}^{\mathscr{k}}\Theta_n$ the representation of the symmetric group on $c_n(\mathscr{k})$.

$S^k M_n^*(\mathscr{k})$ respectively $S^k c_n^*(\mathscr{k})$ can be identified to the vector space of the degree k homogeneous polynomials on $M_n(\mathscr{k})$ respectively on $c_n(\mathscr{k})$; let ${}_n\pi^\kappa : S^k M_n^*(\mathscr{k}) \to S^k c_n^*(\mathscr{k})$ be the linear map defined by "restriction to $c_n(\mathscr{k})$." Clearly we have the commutative diagram

$$\begin{array}{ccc} \overset{k}{S}\, M_n^*(\mathscr{k}) & \xleftarrow{S\, i_n^*} & \overset{k}{S}\, M_{n+1}^*(\mathscr{k}) \\ \downarrow {\scriptstyle {}_n\pi^k} & & \downarrow {\scriptstyle {}_{n+1}\pi^k} \\ \overset{k}{S}\, c_n^*(\mathscr{k}) & \xleftarrow{S\, i_n^*} & \overset{k}{S}\, c_{n+1}^*(\mathscr{k}) \end{array}$$

where $S^k \; i_n^*(\operatorname{Inv} S^k \rho_{n+1}^*) \subset \operatorname{Inv} S^k \rho_n^*$ and $S^k i_n^*(\operatorname{Inv} S^k \Theta_{n+1}^*) = \operatorname{Inv} S^k (\Theta_n^*)$, ${}_r\pi^k(\operatorname{Inv} S^k \rho_r^*) \subset \operatorname{Inv} S^k \Theta_r^*$; $\operatorname{Inv} S^k \Theta_n^*$ is the fixed point subspace of $S^k \Theta_n^*$. Let ${}_r\bar{\pi}^k$: $\operatorname{Inv}(S^k \rho_r^*) \to \operatorname{Inv} S^k \Theta_r^*$ be the same map as ${}_r\pi^K$ with the target restricted to $\operatorname{Inv} S^k \Theta_r^*$. We will prove that ${}_r\bar{\pi}^k$ is surjective checking that ${}_r\bar{\pi}^* = \oplus_{k=0}^{\infty} {}_r\bar{\pi}^k$, ${}_r\bar{\pi}^*$:$\operatorname{Inv} S\rho_r^* \to \operatorname{Inv} S\Theta_r^*$ where $S \cdots = \oplus_k S^k \cdots$ is. For this purpose we define

$$\mu_r : M_r(\mathscr{k}) \to \mathscr{k}^r \qquad \mu_r = (\mu_r^1, \ldots, \mu_r^r) \quad \text{with} \quad \mu_r^i : M_n(\mathscr{k}) \to \mathscr{k}$$

by $\mu_r^i(M) =$ the i-th coefficient of the characteristic polynomial of M. μ_r induces $\mu^* : P(\mathscr{k}^r) \to \operatorname{Inv} S\rho_r^*$, $P(\mathscr{k}^r)$ is the space of polynomials defined on $\mathscr{k}^r$, and ${}_r\pi^* \cdot \mu^*$ is an isomorphism, hence ${}_r\pi^*$ is surjective. To check that ${}_r^{\mathscr{k}}\pi^k$ is injective it suffices to show that ${}_r^{C}\pi^k$ is, since ${}_r^{Q}\pi^k$ and ${}_r^{R}\pi^k$ are restrictions of ${}_r^{C}\pi^k$; ${}_r^{C}\pi^k$ is injective because there exists an open dense set in $M_r(C)$ consisting of matrices which are conjugate to diagonal matrices. Consequently we have

$$\begin{array}{ccc} \operatorname{Inv} S^k\, \Theta_n & \xleftarrow{{}_n\pi^k} & \operatorname{Inv} \overset{k}{S}\, \rho_n^* \\ \uparrow {\scriptstyle S^k i_n^*} & & \uparrow {\scriptstyle S^k i_n^*} \\ \operatorname{Inv} \overset{k}{S}\, \Theta_{n+1} & \xleftarrow{{}_{n+1}\pi^k} & \operatorname{Inv} \overset{k}{S}\, \rho_{n+1}^* \end{array}$$

By A. Borel's theorem we know that for any k we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Inv} \overset{k}{S}\, \Theta_n^* & \xleftarrow{S\, i_n^*} & \operatorname{Inv} \overset{k}{S}\, \Theta_{n+1}^* \\ \downarrow {\scriptstyle \iota_n^k} & & \downarrow {\scriptstyle \iota_{n+1}^{\mathscr{k}}} \\ H^{2k}(BU(n) : \mathscr{k}) & \xleftarrow{i_n'} & H^{2k}(BU(n+1) : \mathscr{k}) \end{array}$$

with l_n^k isomorphisms; consequently if we take $q_n^k = ({}_n\bar{\pi}^k)\cdot(l_n^k)^{-1}$ (2) is proved q.e.d.

Passing to duals we obtain the commutative diagrams

$$\begin{array}{ccc} \overset{k}{\Lambda}\, M_n(\mathscr{k}) & \xrightarrow{(s_n{}^k)^*} & H_k(U(n):\mathscr{k}) \\ \Big\downarrow{\scriptstyle \Lambda^k i_n} & & \Big\downarrow{\scriptstyle i_{n*}} \\ \overset{k}{\Lambda}\, M_{n+1}(\mathscr{k}) & \xrightarrow{(s^{\mathscr{k}}_{n+1})^*} & H_k(U(n+1):\mathscr{k}) \end{array}$$

and

$$\begin{array}{ccc} \overset{k}{S}\, M_n(\mathscr{k}) & \xrightarrow{(q_n{}^k)^*} & H_{2k}(BU(n):\mathscr{k}) \\ \Big\downarrow{\scriptstyle S^k i_n} & & \Big\downarrow{\scriptstyle ('j_n)_*} \\ \overset{k}{S}\, M_{n+1}(\mathscr{k}) & \xrightarrow{(q^{\mathscr{k}}_{n+1})^*} & H_{2k}(BU(n+1):\mathscr{k}) \end{array}$$

which induce $(s_\infty^k)^*:\Lambda^k\, M_\infty(\mathscr{k})\to H_k(U(\infty):\mathscr{k})$ and $(q_\infty^k)^*:S^k\, M_\infty(\mathscr{k})\to H_{2k}(BU(\infty);\mathscr{k})$ $(s_n^k)^*$, $(q_n^k)^*$ restricted to $\operatorname{Inv}\Lambda^k\rho_n$ respectively $\operatorname{Inv} S^k\rho_n$ are isomorphisms therefore $(s_\infty^k)^*$ and $(q_\infty^k)^*$ are, since $\operatorname{Inv}\Lambda \overset{\mathscr{k}}{\rho}{}^*_\infty(k) = \lim_n \operatorname{Inv}(\Lambda^k \rho_n^*(\mathscr{k}))$ and $\operatorname{Inv} S^k(\rho_\infty^*(\mathscr{k})) = \lim_n \operatorname{Inv}(S^k\rho_n^*(\mathscr{k}))$.

COROLLARY 2.2. *For any l and k*

(1) ${}_l m_\infty^k: H_l(GL(Z); \{\Lambda^k\, {}^{\mathscr{k}}\rho_\infty\}) \to H_l(GL(Z); H_k(U(\infty):\mathscr{k})$ *and*

(2) ${}_l m_\infty^k: H_l(GL(Z): S^k\, {}^{\mathscr{k}}\rho_\infty\})$—$H_l(GL(Z); H_{2k}(BU(\infty):\mathscr{k}))$[6] induced by $(s_\infty^k)^*$ respectively $(q_\infty^k)^*$ are isomorphisms.

Proof of Corollary 2.2. It is enough to prove the statement for $\mathscr{k} = R$. Since the proof of (1) and (2) are the same we give only the proof of (1). We observe that ${}_l m_\infty^k = \lim_n {}_l m_n^k$ with ${}_l m_n^k: H_l(GL(Z; n), \{\Lambda^k\, {}^R\rho_n\}) \to H_l(GL(Z, n); H_n(U(n):R))$ induced by $(s_n^k)^*$ hence it suffices to check that ${}_l m_n^k$ is an isomorphism for n big enough, for instance $(n-1)\geq 4l$. Let us recall that if τ is an GL(R, n) irreducible representation, it remains irreducible if restricted to SL(R, n); by Theorem 1.1 [F, H] $H_l(SL(Z, n), \{\tau\}) = 0$ if $l \leq (n-1)/4$, hence $H_l(GL(Z, n):\{\tau\}) = 0$ for

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[6] $H\cdots(G;\{\tau\})$ denotes the homology of G with coefficients in the G-module defined by the representation τ, $H\cdots(G; N)$ the homology of G with coefficients in the trivial G-module N.

$l<(n-1)/4$ (applying Lindon's spectral sequence [M] ch XI Theorem 10.1). Since $\Lambda^k \rho_n$ decomposes as sum of irreducible representations and the trivial representation on $\mathrm{Inv}\,\Lambda^{k\,\mathrm{R}}\rho_n = \mathrm{H}_k(\mathrm{U}(n):\mathrm{R})$ we conclude that ${}_l m_n^k$ is an isomorphism for $(n-1)\geq 4l$.

q.e.d.

Let V be a k-vector space, $\{V, r\}$ be the graded vector space with all but r-th components trivial (i.e. $\{V\,.\,r\}_i = 0$ if $i \neq r$) and the r-th component isomorphic to V. We denote by $L(\{V, r\})$ the k-graded commutative[7] algebra generated by the graded vector space $\{V, r\}$. Clearly $L(\{V, s\})_s = 0$ if $s \not\equiv 0 \pmod r$ and $L(\{V, r\})_{ir} = \Lambda^i V$, respectively $S^i V$ if r is odd respectively even. As algebra $L(\{V, r\})$ is isomorphic to an exterior respectively symmetric algebra if r is odd respectively even.

If $\rho: G \times V \to V$ is a representation of G on V, ρ induces the representation $L(\rho, r)$ of G on $L(\{V, r\})$; let $\mathrm{Inv}\,L(\rho, r)$ be subalgebra of $L(\rho, r)$ consisting of the invariant elements.

Clearly $\mathrm{Inv}\,L(\rho, r)$ is a k free algebra therefore $\mathrm{Inv}\,L(\rho, r) = L(\mathrm{W})$ where W is a k graded vector space. We are particularly interested to determine the graded vector space W in the case $\rho = \rho_\infty^*$. The result is contained in the following theorem:

THEOREM 2.3. $\mathrm{Inv}\,L({}^k\rho_n^*, r) = L(\mathrm{W})$ *where* W *is the following* k *graded vector space* $(n = 1, 2, 3, \ldots, \infty)$

$$\dim_k(\mathrm{W}_s) = \begin{cases} 0 & \text{if} \quad s \equiv 0 \pmod r \\ \dim \Pi_i(\mathrm{U}(n)) \otimes k & \text{if} \quad r \text{ is odd and } s = ri \\ \dim \Pi_{2i}(\mathrm{BU}(n)) \otimes k & \text{if} \quad r \text{ is even and } s = ri \end{cases}$$

Proof. Choose a graded preserving linear injective map $\iota: \tilde{\mathrm{W}} \to \mathrm{Inv}\,({}^k\rho_n^*, r)L(W)$ where $\tilde{\mathrm{W}}_s = 0$ if $s \not\equiv 0 \pmod r$ and $\tilde{\mathrm{W}}_{ir} = \Pi_i(\mathrm{U}(n) \otimes k)^*$ respectively $\Pi_{2i}(\mathrm{BU}(n) \otimes k)^*$ if r is odd respectively even and $\iota_{ir} = s_n^i \cdot o_i'$ respectively $\iota_{ir} = q_n^i \cdot o_{2i}$ where o_i' is a right inverse of the Hourewicz-homomorphism $H^i(\mathrm{U}(n): k) \to \Pi_i^*(\mathrm{U}(n)) \otimes k$. and o_{2i} a right inverse of the Hourewicz homomorphism $\mathrm{H}^{2i}(\mathrm{BU}(n): k) \to \Pi_{2i}^*(\mathrm{BU}(n)) \otimes k\, \iota_{ir}$ is injective because o_i' and o_{2i} are injective. Since ι is injective and $\mathrm{Inv}\,({}^k\rho_n^*, r)$ is free ι extends to $L(\iota): L(\tilde{\mathrm{W}}) \to \mathrm{Inv}\,({}^k\rho_n^*, r)$ which is injective. To prove it is an isomorphism it suffices to check it

[7] "Commutative" should be understood in "graded sense," namely $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ if a and b have pure degree.

is an isomorphism in any degree or else it suffices to show $\dim_{k} L(\tilde{W})_i = \dim \operatorname{Inv}({}^{k}\rho_n^*, r)_i$. Or, if r is odd then

$$\dim_{k} L(\tilde{W})_s = \begin{cases} 0 \quad \text{if} \quad s \not\equiv 0 \pmod r \\ \operatorname{card}\left\{(\alpha_1, \alpha_2, \ldots, \alpha_p) \Big/ \begin{matrix} 1 \le \alpha_1 < \alpha_2 \cdots < \alpha_p < n \\ 2\alpha_1 + \cdots + 2\alpha_p - p = i, p < n \end{matrix}\right\} \\ = \dim H^i(U(n) : k) \quad \text{if} \quad s = ir \end{cases}$$

and if r is even

$$\dim_{k} L(\tilde{W})_s = \begin{cases} 0 \quad \text{if} \quad s \not\equiv 0 \pmod r \\ \operatorname{card}\left\{(\alpha_1 \alpha_2 \cdots \alpha_n) \Big/ \begin{matrix} 0 \le \alpha_i \le n \\ \alpha_1 + 2\alpha_2 + \cdots n\alpha_n = 2i \end{matrix}\right\} \\ = \dim H^{2i}(BU(n) : k) \quad \text{if} \quad s = ir \end{cases}$$

For $n = \infty$ the result follows from the observation that for any fixed degree $\operatorname{Inv} L(\rho_n^*, r) \leftarrow \operatorname{Inv} L(\rho_{n+1}^*, r)$ is an isomorphism if n is big enough. This happens because of Theorem 2.1 and the stability property for the cohomology of $U(n)$ and $BU(n)$.

§ 3

The restriction of the adjoint representation ρ_∞ of GL(Q) on $M_\infty(Q)$ to the subgroup GL(Z) defines the action $\bar{\rho}_\infty$ of GL(Z) on $K(M_\infty(Q), r)$ and therefore the fibration $K(V, r) \to E \xrightarrow{\pi} BGL(Z)$, $V = M_\infty(Q)$. If $r > 1$ then $\Pi_1(E) = \Pi_1(BGL(Z)) = GL(Z)$ whose commutator is a perfect normal subgroup, hence one can apply the Quillen "+" construction.

THEOREM 3.1. $E_+ = BGL(Z)_+ \times T_r$, π_+ *is the projection on* $BGL_\infty(Z)_+$ *where* T_r *has the homotopy type of* $\prod_{i=1}^\infty K(Q; (2s+1)(2i-1))$ *if* $r = 2s+1$ *and of* $\prod_{i=1}^\infty K(Q; 2si)$ *if* $r = 2s$.

Proof. The proof will be given in two steps. In step 1 we will produce an explicit construction of F_1, F_2, $f: F_1 \to F_2$, F_1, F_2 CW-complexes (semisimplicial complexes), f continuous (semisimplicial) map, together with a continuous (semisimplicial) action $\mu: GL(Z) \times F_1 \to F_1$ so that the following properties are satisfied

(a) F_1 is homotopy equivalent to $K(M_\infty(Q), r)$ and F_2 to T_r;

(b) The action μ induces on the r-th homotopy group of F_1 the representation ρ_∞.

(c) If F_2 is endowed with the trivial action of GL(Z) then f is equivariant.

(d) The minimal model (in the sense of Sullivan) [S] of F_1 is the commutative graded algebra $L(\rho^*_\infty, r)$ endowed with the differential 0, the minimal model of F_2 is the graded commutative algebra $\operatorname{Inv} L(\rho^*_\infty, r) = L(W)$ with differential 0 and the morphism induced by f is the inclusion $L(W) = \operatorname{Inv} L(\rho^*_\infty, r) \subset L(\rho^*_\infty, r)$.

To construct F_1, F_2, f, μ we use the "spatial realisation" functor $\langle\ \rangle$ of D. Sullivan [S][8] and take $F_1 = \langle L(M^*_\infty(Q), r),\ d = 0\rangle$, $F_2 = \langle \operatorname{Inv} L(\rho^*_\infty, r),\ d = 0\rangle$ $f = \langle$inclusion of $\operatorname{Inv} L(\rho^*_\infty, r)$ into $L(\rho^*_\infty, r) = L(M^*_\infty(Q), r)\rangle$ and $\mu_A : F_1 \to F_1$ for any $A \in GL(Z)$ is $\langle L(\rho^*_\infty(A) : M^*_\infty(Q) \to M^*_\infty(Q), r)\rangle$. (a), (b), (c) are trivially satisfied and (d) follows simply remarking that $(L(\rho^*_\infty, r), d = 0)$ and $(\operatorname{Inv} L(\rho^*_\infty, r), d = 0)$ are actually minimal models. We recall from Sullivan's theory of minimal models that a 1-connected space X has trivial rational Postnicov invariants iff the differential in the minimal model is trivial, hence F_2 is a product of Eilenberg MacLane's.

Theorem 2.3 gives the homotopy equivalence of F_2 and T_r.

Step 2. We consider the diagram

$$\begin{tikzcd} F_2 \arrow[r] & E_2 \arrow[r] & BGL(Z) \\ F_1 \arrow[r] \arrow[u, "f"] & E_1 \arrow[r] \arrow[u, "f_E"] & BGL(Z) \arrow[u, "id"] \end{tikzcd}$$

with horizontal lines the fibrations induced by the action and the trivial action of GL(Z) on F_2 and observe that $F_1 \to E_1 \to BGL(Z)$ is actually the fibration $K(V, r) \to E \to BGL(Z)$ while $F_2 \to E_2 \to BGL(Z)$ is the trivial fibration with F_2 homotopy equivalent to $T_r \cdot (f,\ f_E,\ id)$ induces a morphism of the spectral sequence (in homology) of the first fibration in the spectral sequence of the second and Corollary 2.2 claims that this morphism is an isomorphism for $E^2 \ldots$, hence f_E induces an isomorphism on integral homology and on Π_1 hence $f_{E_+} : E_{1_+} \to E_{2_+}$ is a homotopy equivalence; this proves Theorem 3.1. q.e.d.

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[8] Recall that the "spatial realisation" of a 1-connected a differential graded algebra A is the geometric realisation of the semisimplicial complex whose k-simplexes are *d.g.a.* maps from A to the Q-de Rham algebra of the standard simplex. The degenaracies and the face operators are obviously defined.

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