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Autor: Burghelea, Dan
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Some rational computations of the Waldhausen algebraic K theory

by DAN BURGHELEA*

Introduction

The purpose of this paper is to compute the rational part of the algebraic K-theory defined by Waldhausen [W] of the following type of topological (semisimplicial) rings R :

- (a) R is an associative topological (semisimplicial) ring with unit and $\Pi_0(R) = \mathbb{Z}$ (\mathbb{Z} denotes the ring of integers).
- (b) There exists a ring homomorphism $\iota: \Pi_0(R) \rightarrow R$ so that $\pi\iota = \text{id}$ where $\pi: R \rightarrow \Pi_0(R)$ ¹ is the canonical projection.
- (c) $\Pi_i(R) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $i \neq r \geq 1$ and $\Pi_r(R) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, where \mathbb{Q} is the ring of rationals.

These computations provide in particular the computation of the rational part of the Waldhausen's algebraic K-theory of a space X , in the case X has the rational homotopy type of a $K(\mathbb{Z}, 2r)$ ² and implicitly the computation of the rational homotopy type of ${}^{Pl}\text{Wh}(X)$, ${}^{Pl}\text{Wh} \pm(X)$, ${}^{Diff}\text{Wh}(X) \pm$ (see [B.L] for notations), for $K(\mathbb{Z}, 2r)$.

In this paper we give the results only for ${}^{Diff}\text{Wh}$, the problem of the computation of ${}^{Pl}\text{Wh}(X)$, ${}^{Pl}\text{Wh} \pm(X)$ will be contained in another paper on automorphisms of manifolds.

The methods of this paper allow the same computations for X of the rational homotopy type of a $K(G, 2r)$ but more "classical invariant theory" is necessary and the author has not yet worked it out.

The paper is organised as follows:

In section 1 we recall briefly the algebraic K-theory of Waldhausen for rings and for topological spaces and present the main results as a consequence of Theorem 3.1 of section 3. In section 2 we present the "invariant theory" necessary for the proof of Theorem 3.1 and in section 3 the proof of this theorem.

¹ $\Pi_0(R)$ denotes the ring of connected components and $\Pi_i(R)$ the homotopy groups of R with respect to the base point "0".

² $K(G, s)$ denotes the Eilenberg–MacLane space corresponding to G and s .

* INCREST—Bucharest, Bd. Păcui 220

This work has been done in the fall of 1977 while the author was visitor at the Institute for Advanced Study and the Princeton University. I am deeply indebted to the stimulating environment provided by these Institutes. I must also acknowledge the benefit I got from private discussions with W. C. Hsiang which has probably developed (in the meantime) parallel computations.*

§1

Let Ring' be the category of topological (semisimplicial) rings which are always assumed to be associative and with unit, and continuous (semisimplicial) ring homomorphisms which are assumed to be unit preserving. Let Top_* be the category of based pointed topological spaces (semisimplicial complexes) and based point preserving maps, Gr' be the category of topological (semisimplicial) groups and continuous (semisimplicial) homomorphisms and $\tilde{\Omega}$ the subcategory of Top_* consisting of ∞ -loopspaces and ∞ -loop space maps.

Following Waldhausen [W] one defines the algebraic K-theory as a functor $\mathbb{K} : \text{Ring}' \rightsquigarrow \tilde{\Omega}$ which is a homotopy functor in the sense that if f_1, f_2 are two homotopic morphisms (homotopic by “morphisms”) then $\mathbb{K}(f_1)$ and $\mathbb{K}(f_2)$ are homotopic in $\tilde{\Omega}$ and if $f : R. \rightarrow R.$ is k -connected then $\mathbb{K}(f)$ is $(k+1)$ -connected. Since we are not interested in the ∞ -loop space structure of \mathbb{K} we regard \mathbb{K} as a functor with values in Top_* whose definition is the following.

For any n let $\widetilde{\text{GL}}(R., n)$ be the space³ (semisimplicial complex) of $n \times n$ matrices $\{a_{ij}\}$, $a_{ij} \in R.$, with $\{\pi(a_{ij})\}$ invertible. The composition of matrices endows $\widetilde{\text{GL}}(R., n)$ with a structure of associative H -space and the inclusion $\widetilde{\text{GL}}(R., n) \rightarrow \widetilde{\text{GL}}(R., n+1)$ defined by $A \rightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ is a morphism of associative H -spaces. We take $\widetilde{\text{GL}}(R.) = \widetilde{\text{GL}}(R., \infty) = \lim_{\rightarrow} \widetilde{\text{GL}}(R., n)$ which is an associative H -space whose $\Pi_0(\widetilde{\text{GL}}(R.)) = \text{GL}(\pi_0(R.)) = \lim_{\rightarrow} \text{GL}(\Pi_0(R.); n)$. Applying the “classifying space” functor to $\widetilde{\text{GL}}(R.)$ one obtains $B\widetilde{\text{GL}}(R.)$ whose $\Pi_1(B\widetilde{\text{GL}}(R.)) = \text{GL}(\Pi_0(R.))$ has the commutator a perfect group [L]. Consequently one can apply the Kervaire–Quillen’s “+”-construction and the resulted space (semisimplicial complex) will be denoted by $B\widetilde{\text{GL}}(R.)_+$. We define $\mathbb{K}(R.) = \mathbb{Z} \times B\widetilde{\text{GL}}(R.)_+$ where \mathbb{Z} denotes the ring of integers. If $f : R. \rightarrow R'.$ is a morphism, it induces $\mathbb{K}(f) : \mathbb{K}(R.) \rightarrow \mathbb{K}(R'.)$ with the properties we have mentioned.

The “loop space” functor $\Omega : \text{Top}_* \rightsquigarrow G'$ in the semisimplicial case is the Kan’s free group construction F and in topological case any “group type” construction

³ With the obvious topology.

* Added in proofs: Similar results have been independently obtained by Hsiang and Staffeldt; more recently, the author and Hsiang and Staffeldt have obtained upper bounds for $\dim \Pi_i(\mathcal{K}(X)) \otimes \mathbb{Q}$ for X 1-connected and with finite Betti numbers.

of the loop space for example the Milnor's construction $[M_2]$ or $X \rightsquigarrow |F(\text{Sing } X)|$ where Sing denotes the singular complex and $|\cdot|$ the "geometric realisation".

Let $\mathbf{Z}: G_r' \rightsquigarrow \text{Ring}'$ be the functor which associates with any topological (semisimplicial) group G the ring $\mathbf{Z}(G)$ the topological (semisimplicial) analogous of the group ring; in the semisimplicial context it is actually the group ring, in topological context we can take $|\mathbf{Z}(\text{Sing } G)|$ or any other functor from $G_r' \rightsquigarrow \text{Ring}'$ which is essentially the infinite symmetric product. The composition $\mathcal{K} = \mathbb{K} \circ \mathbf{Z} \circ \Omega$ produces a functor defined on Top_* with values in $\tilde{\Omega}$ which is a homotopy functor and has the property that $f: X \rightarrow Y$ κ -connected implies $\mathcal{K}(f)$ κ -connected.

As we have mentioned, our purpose is to compute $\mathbb{K}(R.)$ for rings which satisfy the properties (a), (b), (c) mentioned in Introduction and in fact we prove the following theorem.

THEOREM 1.1. *If $R.$ satisfies the conditions (a), (b), (c) then $\mathbb{K}(R.)$ has the rational homotopy of $\mathbb{K}(Z) \times T_{r+1}$ where $T_r = \prod_{i=1}^{\infty} K(Z, 2si)$ if $r = 2s$ and $T_r = \prod_{i=1}^{\infty} K(Z; (2s+1)(2i-1))$ if $r = 2s+1$.*

As a consequence one obtains.

THEOREM 1.2. *If X has the rational homotopy type of $K(Z, 2r)$, $r > 0$, then $\mathcal{K}(X)$ has the rational homotopy type of $\mathbb{K}(Z) \times \prod_{i=1}^{\infty} K(Z; 2ri)$.*

COROLLARY 1.3. *If X has the rational homotopy type of $K(Z, 2r)$, ${}^{\text{Diff}}\text{Wh}(X)$, ${}^{\text{Diff}}\text{Wh}_{\pm}(X)$ have the rational homotopy type of ${}^{\text{Diff}}\text{Wh}(pt)$ respectively ${}^{\text{Diff}}\text{Wh}_{\pm}(pt)$.*

Recall from [W] that ${}^{\text{Diff}}\text{Wh}(pt)^4$ has the rational homotopy type of $\prod_{i=1}^{\infty} K(Z; 4i)$ and from [F, H] that ${}^{\text{Diff}}\text{Wh}_+$ respectively ${}^{\text{Diff}}\text{Wh}_-$ have the rational homotopy type of ${}^{\text{Diff}}\text{Wh}(pt)$ respectively pt .

Proof of Theorem 1.1. Let $M_n(\mathring{R}.)$ respectively $\mathcal{M}_n(\mathring{R}.)$ be the space (semisimplicial complex) of $n \times n$ matrices with entries in $\mathring{R}.$, the connected component of "0", endowed with the composition law "+" respectively "*" given by $M * N = M + N + MN$.

Consider the diagram

$$\begin{array}{ccccc}
 \mathcal{M}_n(\mathring{R}.) & \xrightarrow{\sigma_n} & \widetilde{GL}(\mathring{R}., n) & \xrightarrow{\omega_n} & GL(\Pi_0(\mathring{R}.), n) \\
 (*) \swarrow i_n & & \swarrow i_n & & \swarrow i_n \\
 \mathcal{M}_{n+1}(\mathring{R}.) & \xrightarrow{\sigma_{n+1}} & \widetilde{GL}(\mathring{R}., n+1) & \xrightarrow{\omega_{n+1}} & GL(\Pi_0(\mathring{R}.), n+1)
 \end{array}$$

⁴ Our ${}^{\text{Diff}}\text{Wh}$ is the loop space of the one defined by Waldhausen in [W].

where $\sigma_n(M) = (M + I)$, \tilde{i}_n and \bar{i}_n are defined by $A \rightsquigarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, i_n by $M \rightsquigarrow \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$ and ω_n by $\omega_n\{a_{ij}\} = \{\pi a_{ij}\}$.

Passing to the limit in diagram (*) one obtains the fibration

$$(**) \mathcal{M}_\infty(R.) \xrightarrow{\sigma_\infty} \widetilde{GL}(R.) \xrightarrow{\omega_\infty} GL(\Pi_0(R.))$$

with all terms associative H-spaces and σ_∞ respectively ω_∞ homomorphisms. Applying the “classifying space” functor to (**) one obtains the fibration

$$(***) B\mathcal{M}_\infty(\mathring{R}.) \xrightarrow{B\sigma_\infty} B\widetilde{GL}(R.) \xrightarrow{B\omega_\infty} BGL(\Pi_0(R.))$$

Assume now that the ring R satisfies our hypothesis (b), hence there exists a morphism $\iota: \Pi_0(R.) \rightarrow R$ so that $\Pi.\iota = \text{id}$; ι induces the group homomorphism $\bar{\iota}: GL(\Pi_0(R.)) \rightarrow \widetilde{GL}(R.)$ and consequently we can define the representation ρ_∞ of $GL(\Pi_0(R.))$ on the H-space $\mathcal{M}_\infty(\mathring{R}.)$ by $\rho_\infty(A; M) = \bar{\iota}(A) \cdot M \cdot \bar{\iota}(A)^{-1}$ for $A \in GL(\Pi_0(R.))$ $M \in \mathcal{M}_\infty(\mathring{R}.)$. Clearly

$$\rho_\infty(A; \dots): \mathcal{M}_\infty(\mathring{R}.) \rightarrow \mathcal{M}_\infty(\mathring{R}.)$$

is an H-space isomorphism, consequently one can apply the classifying space functor to $\rho_\infty(A; \dots)$ and obtain the action

$$B\rho_\infty: GL(\Pi_0(R.)) \times B\mathcal{M}_\infty(\mathring{R}.) \rightarrow B\mathcal{M}_\infty(\mathring{R}.).$$

PROPOSITION 1.4. *The fibration (***) is the fibration over $BGL(\Pi_0(R.))$ associated with the action $B\rho_\infty$.*

Proof of Proposition 1.4. Let us recall the definition of the semidirect product of $\mathcal{M}_\infty(\mathring{R}.) \times_{\rho_\infty} GL(Z)$. This is the associative H-space structure defined on $\mathcal{M}_\infty(\mathring{R}.) \times GL(Z)$ by the following composition law

$$(M', A') \# (M, A) = (\{\rho_\infty(A^{-1}; M)\} * M, A' \cdot A)$$

where $M, M' \in \mathcal{M}_\infty(\mathring{R}.)$ and $A, A' \in GL(Z)$. The natural projection $(M, A) \rightarrow A$ defines an homomorphism $p_2: \mathcal{M}_\infty(\mathring{R}.) \times_{\rho_\infty} GL(Z) \rightarrow GL(Z)$ whose kernel is exactly $\mathcal{M}_\infty(\mathring{R}.)$.

In order to prove Proposition 1.4 it is obviously enough to show that (**) is isomorphic to

$$\mathcal{M}_\infty(\mathring{R}.) \rightarrow \mathcal{M}_\infty(R.) \times_{\rho_\infty} GL(Z) \rightarrow GL(Z)$$

and this isomorphism is established by $\gamma: \widetilde{GL}(R.) \rightarrow \mathcal{M}_\infty(\mathring{R}.) \times_{\rho_\infty} GL(Z)$ defined by

$$\gamma(A) = (\sigma_\infty^{-1}\{\bar{\iota}(\omega(A)^{-1}) \cdot A - I\}, \omega(A)) \quad \text{which makes sense since}$$

$$\{\iota(\omega(A)^{-1}) \cdot A - I\} \in \sigma_\infty(M_\infty(R.))$$

q.e.d.

Remark. The same proof shows that

$B\mathcal{M}_n(\mathring{R}.) \rightarrow B\widetilde{GL}(R., n) \rightarrow BGL(\Pi_0(R.), n)$ (with the hypothesis (b) on $R.$) is the fibration induced by the representation $\rho_n: GL(\Pi_0(R.), n) \times \mathcal{M}_n(\mathring{R}.) \rightarrow \mathcal{M}_n(\mathring{R}.)$ defined by the same formula.

Let us observe that if conditions (a), (b), (c) are satisfied then

$$\Pi_i(B\mathcal{M}_\infty(\mathring{R}.)) \otimes_Z Q = \Pi_{i-1}(\mathcal{M}_\infty(\mathring{R}.)) \otimes_Z Q = M_\infty(\Pi_{i-1}(R.) \otimes_Z Q) = \begin{cases} 0 & \text{if } i \neq r+1 \\ M_\infty(Q) & \text{if } i = r+1 \end{cases}$$

and $\Pi_1(B\mathcal{M}_\infty(\mathring{R}.)) = \Pi_0(\mathcal{M}_\infty(\mathring{R}.)) = 0$. Consequently the fibrewise “0-localisation” of the fibration (***) is the fibration

$$K(M_\infty(Q), r+1) \rightarrow E \rightarrow BGL(Z)$$

associated with the action

$$\bar{\rho}_\infty: GL(Z) \times K(M_\infty(Q), r+1) \rightarrow K(M_\infty(Q), r+1);$$

this action is determined by the adjoint representations $\rho_\infty: GL(Z) \times M_\infty(Q) \rightarrow M_\infty(Q)$ given by $\rho_\infty(A: M) = A \cdot M \cdot A^{-1}$.

Warning. If (a) and (b) are satisfied and $B\mathcal{M}_\infty(\mathring{R}.)$ has trivial rational Postnikov invariants, we might be tempted to believe that the fibrewise “0-localisation” of the fibration (***) is the fibration with fibre $\prod_{s=2}^\infty (K(G_s \otimes Q), s)$ associated with the action $\prod_{s=2}^\infty s_{p_\infty}$ where s_{p_∞} is the action induced by the representation $s_{p_\infty}: GL(Z) \times M_\infty(G_s \otimes Q) \rightarrow M_\infty(G_s \otimes Q)$ defined by $s_{p_\infty}(A, M) = A \cdot M \cdot A^{-1}$ this is not always the case.

The proof of Theorem 1.1 follows now immediately from Theorem 3.1.

Proof of Theorem 1.2. If $X = K(Z, 2r)$ then $\Omega X = K(Z, 2r-1)$ and consequently $Z\Omega(X)$ has as homotopy groups the homology groups of $\Omega(X)$ since

$\mathbf{Z}\Omega(X)$ is essentially the infinite symmetric product of ΩX . This makes clear that $\mathbf{Z}\Omega(X)$ satisfies (c) since X is 1-connected (a) is also satisfied and (b) is trivially satisfied since $\mathbf{Z}pt = \mathbf{Z}$. Consequently the theorem is true by Theorem 1.1 for $K(\mathbf{Z}, 2r)$. The construction of the functor \mathcal{K} implies immediately that if X and Y are rationally homotopy equivalent then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are.

q.e.d

Proof of Corollary 1.3. In [W] Waldhausen defines two natural transformation $\mathcal{K}(\dots) \rightarrow \mathcal{K}^s(\dots)$ where $\mathcal{K}^s(\dots)$ is the stabilized functor associated with \mathcal{K} , which is an unreduced homology theory, and $h(\dots; \mathcal{K}(pt)) \rightarrow \mathcal{K}(\dots)$ where $h(\dots; \mathcal{K}(pt))$ is the reduced homology theory produced by the ∞ -loop space $\mathcal{K}(pt)$.

The composition $h(\dots; \mathcal{K}(pt)) \rightarrow \mathcal{K}^s(\dots)$ is a natural transformation of homology theory and because $\mathcal{K}(pt) \rightarrow \mathcal{K}^s(pt)$, is rationally homotopy surjective, $h(X; \mathcal{K}(pt)) \rightarrow \mathcal{K}^s(X)$ is rationally homotopy surjective. On the other side $B^{Diff}Wh(X)$ is the fibre of $\mathcal{K}(X) \rightarrow \mathcal{K}^s(X)$. Consequently $\mathcal{K}(X)$ and $B^{Diff}Wh(X) \times \mathcal{K}^s(X)$ are rationally homotopy equivalent. (Waldhausen claims a much stronger fact namely $A(X)$ and $B^{Diff}Wh(X) \times A^s(X)$ are homotopy equivalent which will imply the mentioned rational homotopy equivalence).

§2

Let \mathcal{K} be one of the fields $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ of rational, real or complex numbers, and $M_n(\mathcal{K})$ be the Lie algebra of $GL(\mathcal{K}, n)$, i.e. the Lie algebra of $n \times n$ matrices over \mathcal{K} .

We denote by ${}^{\mathcal{K}}\rho_n$ or ρ_n , using ${}^{\mathcal{K}}\rho_n$ only when we are interested to explicit the field, the adjoint representation of $GL(\mathcal{K}, n)$ on $M_n(\mathcal{K})$ defined by $\rho_n(A, M) = A \cdot M \cdot A^{-1}$ for $A = GL(\mathcal{K}, n)$ and $M \in M_n(\mathcal{K})$, by ρ_n^* its dual representation and by $\Lambda^k \rho_n, S^k \rho_n, \Lambda^k \rho_n^*, S^k \rho_n^*$ the k -times exterior respectively symmetric power of ρ_n respectively ρ_n^* . Denote by $\text{Inv } \xi$ the fixed point subspace of the representation ξ . The following theorem contains well known facts; since the present formulation is not easy to be found in literature we enclose the proof.

THEOREM 2.1. (1) *There exists an injective linear map $s_n^k: H^k(U(n); \mathcal{K}) \rightarrow \Lambda^k M_n^*(\mathcal{K})$ with $s_n^k(H^k(U(n); \mathcal{K})) = \text{Inv } \Lambda^k({}^{\mathcal{K}}\rho_n^*)$ such that the following diagram is commutative.*

$$\begin{array}{ccc} H^k(U(n); \mathcal{K}) & \xrightarrow{s_n^k} & \Lambda^k M_n^*(\mathcal{K}) \\ \uparrow j_n^* & & \uparrow \Lambda^k i_n^* \\ H^k(U(n+1); \mathcal{K}) & \xrightarrow{s_{n+1}^k} & \Lambda^k M_{n+1}^*(\mathcal{K}) \end{array}$$

where j_n^* is induced by the canonical inclusion $U(n) \xrightarrow{i_n} U(n+1)$ and i_n is the canonical Lie algebra inclusion $i_n(M) = \begin{vmatrix} M & 0 \\ 0 & 0 \end{vmatrix}$.

(2) There exists an injective linear map

$$q_n^k: H^{2k}(BU(n): \mathcal{L}) \rightarrow {}^k S M_n^*(\mathcal{L})$$

with

$$q_n^k(H^{2k}(BU(n): \mathcal{L})) = \text{Inv } {}^k S \mathcal{L} \rho_n^*$$

such that the following diagram is commutative

$$\begin{array}{ccc} H^{2k}(BU(n): \mathcal{L}) & \xrightarrow{q_n^k} & {}^k S M_n^*(\mathcal{L}) \\ \uparrow j_n^* & & \uparrow s^k i_n^* \\ H^{2k}(BU(n+1): \mathcal{L}) & \xrightarrow{q_{n+1}^k} & {}^k S M_{n+1}^*(\mathcal{L}) \end{array}$$

where j_n^* is induced by the canonical inclusion $BU(n) \xrightarrow{i_n} BU(n+1)$.

Proof. Since ${}^R \rho_n^*$ is a real form of ${}^C \rho_n^*$ it is clear that the proof for $\mathcal{L} = \mathbb{R}$ implies the result for $\mathcal{L} = \mathbb{C}$.

Proof of (1). ($\mathcal{L} = \mathbb{R}, \mathbb{Q}$). Let η_n be the adjoint representation of $U(n)$ on its Lie algebra; η_n^* is a real form of ${}^R \rho_n^*$. Analogously let ${}^Q \eta_n$ be the adjoint representation of the group

$${}^Q U(n) = \{A = \{a_{ij} = \alpha_{ij} + i\beta_{ij}\} \mid A \in U(n), \alpha_{ij}, \beta_{ij} \in \mathbb{Q}\}$$

on the \mathbb{Q} -Lie algebra

$$\mathfrak{m} = \{M = \{m_{ij} = \alpha_{ij} + i\beta_{ij}\} \mid m_{ji} + \bar{m}_{ij} = 0, \alpha_{ij}, \beta_{ij} \in \mathbb{Q}\}$$

given by ${}^Q \eta_n(A, M) = A \cdot M \cdot A^{-1}$.

⁵ For a \mathcal{L} vector we denote by V^* its dual.

Clearly it is enough to prove (1) for $R \eta_n^*$ respectively $Q \eta_n^*$ in order to have it proved for $R \rho_n^*$ respectively $Q \rho_n^*$.

Let us recall that de Rham theory permits to associate with any closed differential form on a differentiable manifold a precise singular cohomology class with coefficients in R .

Therefore we have the linear map $t_n: \text{Inv}(\Lambda^k \eta_n^*) \rightarrow H^k(U(n); R)$ constructed as follows; an element of $\text{Inv}(\Lambda^k \eta_n^*)$ is regarded as a k -form on the Lie algebra of $U(n)$ which by translation is extended to a k -differential form on the compact Lie group $U(n)$; since the element we started with is in $\text{Inv}(\Lambda^k \eta_n^*)$ the obtained differential form is biinvariant therefore closed.

It is well known (for any compact Lie group) that t_n is an isomorphism. Moreover $t_n/\Lambda^k Q \eta_n^*$ factors through $H^k(U(n); Q)$ since a form in $\text{Inv}(\Lambda^k \eta_n^*)$ with rational coefficients (with respect to the canonical base) produces a cohomology class with rational periods on all integral cycles. Consequently we have the commutative diagram

$$\begin{array}{ccc} \text{Inv } \Lambda^k R \eta_n^* & \xrightarrow{t_n^k} & H^k(U(n); R) \\ \uparrow \bar{v} & & \uparrow \bar{v} \\ \text{Inv } \Lambda^k Q \eta_n^* & \xrightarrow{Q t_n^k} & H^k(U(n); Q) \end{array}$$

which implies $Q t_n^k$ is an isomorphism.

We also observe that $\Lambda^k i_n^*: \Lambda^k \eta_{n+1}^* \rightarrow \Lambda^k \eta_n^*$ sends $\text{Inv } \Lambda^k \eta_{n+1}^*$ into $\text{Inv } \Lambda^k \eta_n^*$ where i_n is the canonical inclusion of the Lie algebra of $U(n)$ into the Lie algebra of $U(n+1)$, (analogously $\Lambda^{kQ} i_n^*$ sends $\text{Inv } \Lambda^{kQ} \eta_{n+1}^*$ into $\text{Inv } \Lambda^{kQ} \eta_n^*$) and the following diagram is commutative

$$\begin{array}{ccc} \Lambda^k \eta_{n+1}^* & \longrightarrow & \Lambda^k \eta_n^* \\ \downarrow i_{n+1}^k & & \searrow t_n^k \\ H^k(U(n+1); R) & \longrightarrow & H^k(U(n); R) \end{array}$$

since the correspondence “biinvariant forms” \rightsquigarrow “cohomology” is a functorial isomorphism for the category of compact Lie groups. If we take $s_n^k = (t_n^k)^{-1}$ and $Q s_n^k = (Q t_n^k)^{-1}$, (1) is proved.

Proof of 2. ($\ell = Q, R$). Let us consider $c_n(\ell)$ the Lie subalgebra of $M_n(\ell)$ consisting of the diagonal matrices and ${}^\ell \Theta_n$ the representation of the symmetric group on $c_n(\ell)$.

$S^k M_n^*(\ell)$ respectively $S^k c_n^*(\ell)$ can be identified to the vector space of the degree k homogeneous polynomials on $M_n(\ell)$ respectively on $c_n(\ell)$; let ${}_n\pi^k: S^k M_n^*(\ell) \rightarrow S^k c_n^*(\ell)$ be the linear map defined by "restriction to $c_n(\ell)$." Clearly we have the commutative diagram

$$\begin{array}{ccc} S^k M_n^*(\ell) & \xleftarrow{S i_n^*} & S^k M_{n+1}^*(\ell) \\ \downarrow {}_n\pi^k & & \downarrow {}_{n+1}\pi^k \\ S^k c_n^*(\ell) & \xleftarrow{S i_n^*} & S^k c_{n+1}^*(\ell) \end{array}$$

where $S^k i_n^*(\text{Inv } S^k \rho_{n+1}^*) \subset \text{Inv } S^k \rho_n^*$ and $S^k i_n^*(\text{Inv } S^k \Theta_{n+1}^*) = \text{Inv } S^k (\Theta_n^*)$, ${}_r\pi^k(\text{Inv } S^k \rho_r^*) \subset \text{Inv } S^k \Theta_r^*$; $\text{Inv } S^k \Theta_n^*$ is the fixed point subspace of $S^k \Theta_n^*$. Let ${}_r\bar{\pi}^k: \text{Inv } (S^k \rho_r^*) \rightarrow \text{Inv } S^k \Theta_r^*$ be the same map as ${}_r\pi^k$ with the target restricted to $\text{Inv } S^k \Theta_r^*$. We will prove that ${}_r\bar{\pi}^k$ is surjective checking that ${}_r\bar{\pi}^* = \bigoplus_{k=0}^{\infty} {}_r\bar{\pi}^k$, ${}_r\bar{\pi}^*: \text{Inv } S \rho_r^* \rightarrow \text{Inv } S \Theta_r^*$ where $S \cdots = \bigoplus_k S^k \cdots$ is. For this purpose we define

$$\mu_r: M_r(\ell) \rightarrow \ell^r \quad \mu_r = (\mu_r^1, \dots, \mu_r^r) \quad \text{with} \quad \mu_r^i: M_n(\ell) \rightarrow \ell$$

by $\mu_r^i(M) =$ the i -th coefficient of the characteristic polynomial of M . μ_r induces $\mu^*: P(\ell^r) \rightarrow \text{Inv } S \rho_r^*$, $P(\ell^r)$ is the space of polynomials defined on ℓ^r , and ${}_r\pi^* \cdot \mu^*$ is an isomorphism, hence ${}_r\pi^*$ is surjective. To check that ${}_r\pi^k$ is injective it suffices to show that ${}_r\pi^k$ is, since ${}_r\pi^k$ and ${}_r\pi^k$ are restrictions of ${}_r\pi^k$; ${}_r\pi^k$ is injective because there exists an open dense set in $M_r(\mathbb{C})$ consisting of matrices which are conjugate to diagonal matrices. Consequently we have

$$\begin{array}{ccc} \text{Inv } S^k \Theta_n & \xleftarrow{{}_n\pi^k} & \text{Inv } S^k \rho_n^* \\ \uparrow S^k i_n^* & & \uparrow S^k i_n^* \\ \text{Inv } S^k \Theta_{n+1} & \xleftarrow{{}_{n+1}\pi^k} & \text{Inv } S^k \rho_{n+1}^* \end{array}$$

By A. Borel's theorem we know that for any k we have the commutative diagram

$$\begin{array}{ccc} \text{Inv } S^k \Theta_n^* & \xleftarrow{S i_n^*} & \text{Inv } S^k \Theta_{n+1}^* \\ \downarrow l_n^k & & \downarrow l_{n+1}^k \\ H^{2k}(\text{BU}(n): \ell) & \xleftarrow{i_n'} & H^{2k}(\text{BU}(n+1): \ell) \end{array}$$

with l_n^k isomorphisms; consequently if we take $q_n^k = ({}_n\bar{\pi}^k) \cdot (l_n^k)^{-1}$ (2) is proved q.e.d.

Passing to duals we obtain the commutative diagrams

$$\begin{array}{ccc} \Lambda^k M_n(\ell) & \xrightarrow{(s_n^k)^*} & H_k(U(n): \ell) \\ \downarrow \Lambda^k i_n & & \downarrow i_{n*} \\ \Lambda^k M_{n+1}(\ell) & \xrightarrow{(s_{n+1}^\ell)^*} & H_k(U(n+1): \ell) \end{array}$$

and

$$\begin{array}{ccc} S^k M_n(\ell) & \xrightarrow{(q_n^k)^*} & H_{2k}(BU(n): \ell) \\ \downarrow S^k i_n & & \downarrow (i_n)_* \\ S^k M_{n+1}(\ell) & \xrightarrow{(q_{n+1}^\ell)^*} & H_{2k}(BU(n+1): \ell) \end{array}$$

which induce $(s_\infty^k)^*: \Lambda^k M_\infty(\ell) \rightarrow H_k(U(\infty): \ell)$ and $(q_\infty^k)^*: S^k M_\infty(\ell) \rightarrow H_{2k}(BU(\infty); \ell)$
 $(s_n^k)^*$, $(q_n^k)^*$ restricted to $\text{Inv } \Lambda^k \rho_n$ respectively $\text{Inv } S^k \rho_n$ are isomorphisms
 therefore $(s_\infty^k)^*$ and $(q_\infty^k)^*$ are, since $\text{Inv } \Lambda^k \rho_\infty(k) = \lim_n \text{Inv } (\Lambda^k \rho_n^*(\ell))$ and
 $\text{Inv } S^k (\rho_\infty^*(\ell)) = \lim_n \text{Inv } (S^k \rho_n^*(\ell))$.

COROLLARY 2.2. *For any l and k*

- (1) ${}_l m_\infty^k: H_l(\text{GL}(Z); \{\Lambda^k \rho_\infty\}) \rightarrow H_l(\text{GL}(Z); H_k(U(\infty): \ell))$ and
- (2) ${}_l m_\infty^k: H_l(\text{GL}(Z); \{S^k \rho_\infty\}) \rightarrow H_l(\text{GL}(Z); H_{2k}(BU(\infty): \ell))$ ⁶ induced by $(s_\infty^k)^*$ respectively $(q_\infty^k)^*$ are isomorphisms.

Proof of Corollary 2.2. It is enough to prove the statement for $\ell = \mathbb{R}$. Since the proof of (1) and (2) are the same we give only the proof of (1). We observe that ${}_l m_\infty^k = \lim_n {}_l m_n^k$ with ${}_l m_n^k: H_l(\text{GL}(Z; n), \{\Lambda^k \rho_n\}) \rightarrow H_l(\text{GL}(Z, n); H_n(U(n): \mathbb{R}))$ induced by $(s_n^k)^*$ hence it suffices to check that ${}_l m_n^k$ is an isomorphism for n big enough, for instance $(n-1) \geq 4l$. Let us recall that if τ is an $\text{GL}(\mathbb{R}, n)$ irreducible representation, it remains irreducible if restricted to $\text{SL}(\mathbb{R}, n)$; by Theorem 1.1 [F, H] $H_l(\text{SL}(Z, n), \{\tau\}) = 0$ if $l \leq (n-1)/4$, hence $H_l(\text{GL}(Z, n); \{\tau\}) = 0$ for

⁶ $H \cdots (G; \{\tau\})$ denotes the homology of G with coefficients in the G -module defined by the representation τ , $H \cdots (G; N)$ the homology of G with coefficients in the trivial G -module N .

$l < (n-1)/4$ (applying Lindon's spectral sequence [M] ch XI Theorem 10.1). Since $\Lambda^k \rho_n$ decomposes as sum of irreducible representations and the trivial representation on $\text{Inv } \Lambda^k \rho_n = H_k(U(n):R)$ we conclude that m_n^k is an isomorphism for $(n-1) \geq 4l$.

q.e.d.

Let V be a \mathcal{K} -vector space, $\{V, r\}$ be the graded vector space with all but r -th components trivial (i.e. $\{V, r\}_i = 0$ if $i \neq r$) and the r -th component isomorphic to V . We denote by $L(\{V, r\})$ the \mathcal{K} -graded commutative⁷ algebra generated by the graded vector space $\{V, r\}$. Clearly $L(\{V, s\})_s = 0$ if $s \not\equiv 0 \pmod{r}$ and $L(\{V, r\})_{ir} = \Lambda^i V$, respectively $S^i V$ if r is odd respectively even. As algebra $L(\{V, r\})$ is isomorphic to an exterior respectively symmetric algebra if r is odd respectively even.

If $\rho: G \times V \rightarrow V$ is a representation of G on V , ρ induces the representation $L(\rho, r)$ of G on $L(\{V, r\})$; let $\text{Inv } L(\rho, r)$ be subalgebra of $L(\rho, r)$ consisting of the invariant elements.

Clearly $\text{Inv } L(\rho, r)$ is a \mathcal{K} free algebra therefore $\text{Inv } L(\rho, r) = L(W)$ where W is a \mathcal{K} graded vector space. We are particularly interested to determine the graded vector space W in the case $\rho = \rho_\infty^*$. The result is contained in the following theorem:

THEOREM 2.3. $\text{Inv } L(\rho_n^*, r) = L(W)$ where W is the following \mathcal{K} graded vector space ($n = 1, 2, 3, \dots, \infty$)

$$\dim_{\mathcal{K}}(W_s) = \begin{cases} 0 & \text{if } s \not\equiv 0 \pmod{r} \\ \dim \Pi_i(U(n)) \otimes \mathcal{K} & \text{if } r \text{ is odd and } s = ri \\ \dim \Pi_{2i}(BU(n)) \otimes \mathcal{K} & \text{if } r \text{ is even and } s = ri \end{cases}$$

Proof. Choose a graded preserving linear injective map $\iota: \tilde{W} \rightarrow \text{Inv } (\rho_n^*, r) L(W)$ where $\tilde{W}_s = 0$ if $s \not\equiv 0 \pmod{r}$ and $\tilde{W}_{ir} = \Pi_i(U(n) \otimes \mathcal{K})^*$ respectively $\Pi_{2i}(BU(n) \otimes \mathcal{K})^*$ if r is odd respectively even and $\iota_{ir} = s_n^i \cdot o'_i$ respectively $\iota_{ir} = q_n^i \cdot o_{2i}$ where o'_i is a right inverse of the Hourewicz-homomorphism $H^i(U(n): \mathcal{K}) \rightarrow \Pi_i^*(U(n)) \otimes \mathcal{K}$ and o_{2i} a right inverse of the Hourewicz homomorphism $H^{2i}(BU(n): \mathcal{K}) \rightarrow \Pi_{2i}^*(BU(n)) \otimes \mathcal{K}$. ι_{ir} is injective because o'_i and o_{2i} are injective. Since ι is injective and $\text{Inv } (\rho_n^*, r)$ is free ι extends to $L(\iota): L(\tilde{W}) \rightarrow \text{Inv } (\rho_n^*, r)$ which is injective. To prove it is an isomorphism it suffices to check it

⁷ "Commutative" should be understood in "graded sense," namely $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ if a and b have pure degree.

is an isomorphism in any degree or else it suffices to show $\dim_{\mathbb{K}} L(\tilde{W})_i = \dim \operatorname{Inv}(\rho_n^*, r)_i$. Or, if r is odd then

$$\begin{aligned} & 0 \quad \text{if } s \not\equiv 0 \pmod{r} \\ \dim_{\mathbb{K}} L(\tilde{W})_s &= \operatorname{card} \left\{ (\alpha_1, \alpha_2, \dots, \alpha_p) \middle/ \begin{array}{l} 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_p < n \\ 2\alpha_1 + \dots + 2\alpha_p - p = i, p < n \end{array} \right\} \\ &= \dim H^i(U(n): \mathbb{K}) \quad \text{if } s = ir \end{aligned}$$

and if r is even

$$\begin{aligned} & 0 \quad \text{if } s \not\equiv 0 \pmod{r} \\ \dim_{\mathbb{K}} L(\tilde{W})_s &= \operatorname{card} \left\{ (\alpha_1 \alpha_2 \dots \alpha_n) \middle/ \begin{array}{l} 0 \leq \alpha_i \leq n \\ \alpha_1 + 2\alpha_2 + \dots + n\alpha_n = 2i \end{array} \right\} \\ &= \dim H^{2i}(BU(n): \mathbb{K}) \quad \text{if } s = ir \end{aligned}$$

For $n = \infty$ the result follows from the observation that for any fixed degree $\operatorname{Inv} L(\rho_n^*, r) \leftarrow \operatorname{Inv} L(\rho_{n+1}^*, r)$ is an isomorphism if n is big enough. This happens because of Theorem 2.1 and the stability property for the cohomology of $U(n)$ and $BU(n)$.

§ 3

The restriction of the adjoint representation ρ_{∞} of $GL(Q)$ on $M_{\infty}(Q)$ to the subgroup $GL(Z)$ defines the action $\bar{\rho}_{\infty}$ of $GL(Z)$ on $K(M_{\infty}(Q), r)$ and therefore the fibration $K(V, r) \rightarrow E \xrightarrow{\pi} BGL(Z)$, $V = M_{\infty}(Q)$. If $r > 1$ then $\Pi_1(E) = \Pi_1(BGL(Z)) = GL(Z)$ whose commutator is a perfect normal subgroup, hence one can apply the Quillen “+” construction.

THEOREM 3.1. $E_+ = BGL(Z)_+ \times T_r$, π_+ is the projection on $BGL_{\infty}(Z)_+$ where T_r has the homotopy type of $\prod_{i=1}^{\infty} K(Q; (2s+1)(2i-1))$ if $r = 2s+1$ and of $\prod_{i=1}^{\infty} K(Q; 2si)$ if $r = 2s$.

Proof. The proof will be given in two steps. In step 1 we will produce an explicit construction of $F_1, F_2, f: F_1 \rightarrow F_2$, F_1, F_2 CW-complexes (semisimplicial complexes), f continuous (semisimplicial) map, together with a continuous (semisimplicial) action $\mu: GL(Z) \times F_1 \rightarrow F_1$ so that the following properties are satisfied

- (a) F_1 is homotopy equivalent to $K(M_{\infty}(Q), r)$ and F_2 to T_r ;
- (b) The action μ induces on the r -th homotopy group of F_1 the representation ρ_{∞} .

(c) If F_2 is endowed with the trivial action of $GL(Z)$ then f is equivariant.

(d) The minimal model (in the sense of Sullivan) [S] of F_1 is the commutative graded algebra $L(\rho_\infty^*, r)$ endowed with the differential 0, the minimal model of F_2 is the graded commutative algebra $\text{Inv } L(\rho_\infty^*, r) = L(W)$ with differential 0 and the morphism induced by f is the inclusion $L(W) = \text{Inv } L(\rho_\infty^*, r) \subset L(\rho_\infty^*, r)$.

To construct F_1 , F_2 , f , μ we use the “spatial realisation” functor $\langle \rangle$ of D. Sullivan [S]⁸ and take $F_1 = \langle L(M_\infty^*(Q), r), d = 0 \rangle$, $F_2 = \langle \text{Inv } L(\rho_\infty^*, r), d = 0 \rangle$ $f = \langle \text{inclusion of } \text{Inv } L(\rho_\infty^*, r) \text{ into } L(\rho_\infty^*, r) = L(M_\infty^*(Q), r) \rangle$ and $\mu_A: F_1 \rightarrow F_1$ for any $A \in GL(Z)$ is $\langle L(\rho_\infty^*(A): M_\infty^*(Q) \rightarrow M_\infty^*(Q), r) \rangle$. (a), (b), (c) are trivially satisfied and (d) follows simply remarking that $(L(\rho_\infty^*, r), d = 0)$ and $(\text{Inv } L(\rho_\infty^*, r), d = 0)$ are actually minimal models. We recall from Sullivan’s theory of minimal models that a 1-connected space X has trivial rational Postnikov invariants iff the differential in the minimal model is trivial, hence F_2 is a product of Eilenberg MacLane’s.

Theorem 2.3 gives the homotopy equivalence of F_2 and T_r .

Step 2. We consider the diagram

$$\begin{array}{ccccc} F_2 & \longrightarrow & E_2 & \longrightarrow & BGL(Z) \\ \uparrow f & & \uparrow f_E & & \uparrow id \\ F_1 & \longrightarrow & E_1 & \longrightarrow & BGL(Z) \end{array}$$

with horizontal lines the fibrations induced by the action and the trivial action of $GL(Z)$ on F_2 and observe that $F_1 \rightarrow E_1 \rightarrow BGL(Z)$ is actually the fibration $K(V, r) \rightarrow E \rightarrow BGL(Z)$ while $F_2 \rightarrow E_2 \rightarrow BGL(Z)$ is the trivial fibration with F_2 homotopy equivalent to T_r . (f, f_E, id) induces a morphism of the spectral sequence (in homology) of the first fibration in the spectral sequence of the second and Corollary 2.2 claims that this morphism is an isomorphism for $E^2 \dots$, hence f_E induces an isomorphism on integral homology and on Π_1 hence $f_{E_+}: E_{1+} \rightarrow E_{2+}$ is a homotopy equivalence; this proves Theorem 3.1. q.e.d.

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⁸ Recall that the “spatial realisation” of a 1-connected a differential graded algebra A is the geometric realisation of the semisimplicial complex whose k -simplexes are $d.g.a.$ maps from A to the Q -de Rham algebra of the standard simplex. The degeneracies and the face operators are obviously defined.

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