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A note on groups with torsion-free abelianization and trivial multiplicator

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1. Introduction

1.1. A basic result on free groups F asserts that the factors $\{F_j/F_{j+1}\}_{1\leq j<\omega}$ of successive terms of the lower central series are free abelian (Magnus [4], Witt [11]). This can be proved using Lie algebra techniques and a proof (e.g. [1], pp. 35–39; or [7], LA 4.10–4.13) will then rely on three corner stones:

* The canonical Lie algebra homomorphism $\sigma : L_X \to \mathrm{Ass}_X$ from the free Lie $\mathbf{Z}$-algebra on the set X into the Lie algebra of the free associative $\mathbf{Z}$-algebra on X is *injective.*

* To every group G is associated a graded Lie algebra $\mathrm{gr}\, G$. Its underlying additive group is the direct sum $\bigoplus G_j/G_{j+1}$ of the factors of successive terms of the lower central series of G. Its Lie bracket is on homogeneous components induced by commutation in the group, sc.

$$(g \cdot G_{j+1}, h \cdot G_{k+1}) \mapsto (g^{-1}h^{-1}gh) \cdot G_{j+k+1} \qquad (g \in G_j,\ h \in G_k)$$

 and then extended linearly.

Reverting to free algebras and groups, let $\gamma : L_X \to \mathrm{gr}\, F_X$ denote the Lie algebra homomorphism defined by the assignments

$$X \ni x \mapsto x \cdot F_2 \in \mathrm{gr}^1 F_X \subseteq \mathrm{gr}\, F_X.$$

* There exists a Lie algebra homomorphism $\alpha : \mathrm{gr}\, F_X \to \mathrm{Ass}_X$. making the triangle

$$\begin{tikzcd} L_X \arrow[r, "\sigma"] \arrow[d, "\gamma"'] & \mathrm{Ass}_X \\ \mathrm{gr}\, F_X \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

 commute.

1.2. In this note it is shown that the proof sketched above can be adapted to the more general situation in which F_X gets replaced by a group G whose abelianization G_{ab} is torsion-free and whose multiplicator $H_2(G, \mathbf{Z})$, i.e. whose second homology group with integral coefficients, is a torsion group. Such a group will be referred to as being TFT. The place of the free associative $\mathbf{Z}$-algebra Ass_X will be taken by the tensor algebra $T\,G_{ab}$. The main problem is the existence of a Lie algebra homomorphism

$$\alpha : \mathrm{gr}\, G \to T\,G_{ab}$$

extending the identification $\mathrm{gr}\, G \xrightarrow{\sim} T\,G_{ab}$ in degree 1.

THEOREM 1. *If G is* TFT *then the identification* $\mathrm{gr}^1\, G \xrightarrow{\sim} T^1\, G_{ab}$, *taking* $g \cdot G_2 \in \mathrm{gr}\, G$ *to* $g \cdot G_2 \in T\,G_{ab}$, *extends (necessarily uniquely) to a Lie algebra isomorphism* $\alpha : \mathrm{gr}\, G \xrightarrow{\sim} T\,G_{ab}$ *from the graded Lie* $\mathbf{Z}$*-algebra* $\mathrm{gr}\, G$ *associated with* G *onto the Lie subalgebra of* $T\,G_{ab}$ *generated by* $G_{ab} = T^1\, G_{ab}$. *Moreover,* α *induces an isomorphism* $U\alpha : U(\mathrm{gr}\, G) \xrightarrow{\sim} T\,G_{ab}$ *between associative* $\mathbf{Z}$*-algebras, thus providing a model for the universal algebra of* $\mathrm{gr}\, G$.

1.3. Since the additive group underlying the tensor algebra $T\,G_{ab}$ is torsion-free if G_{ab} is so, Theorem 1 immediately entails the approved.

COROLLARY 1. *The factors* G_j/G_{j+1}, $j = 1, 2, \ldots$, *of the lower central series of a* TFT *group* G *are torsion-free.*

An application of Corollary 1 can be found in [10] (cf. *4.1.*).

The next corollary indicates that even in the special case of a TFT group the finer commutator structure gets lost in the passage from G to $\mathrm{gr}\, G$. (Examples testifying the loss will be given in *4.6.*ff.)

COROLLARY 2. *The Lie algebra* $\mathrm{gr}\, G$ *of a* TFT *group is determined by its first homogeneous component* G_{ab}.

1.4. Our second main result deals with subgroups of TFT groups. We state it as

THEOREM 2. *Let* $\varphi : G \to \bar{G}$ *be a homomorphism for which* $\varphi^1 : G_{ab} \to \bar{G}_{ab}$ *is injective. Suppose G is* TFT *and* $H_2(\bar{G}, \mathbf{Z})$ *is a torsion group. Then* $\mathrm{gr}\, \varphi : \mathrm{gr}\, G \to \mathrm{gr}\, \bar{G}$ *is injective. Put differently,* φ *induces injective homomorphisms* $\varphi_* : G/G_j \to \bar{G}/\bar{G}_j$ *for all* $j \geqq 2$.

If G and $\bar{G}$ in Theorem 2 are both free the claim reduces to a well-known result of Malcev on subgroups of free nilpotent groups ([5]; cf. [6], 42.51). Theorem 2 may also be compared with the following result:

THEOREM (Stallings [8], Stammbach [9]). *Let $\varphi : G \to \bar{G}$ be a homomorphism inducing an isomorphism $\varphi^1 : G_{ab} \to \bar{G}_{ab}$ and a surjection $H_2(\varphi) : H_2(G, \mathbf{Z}) \to H_2(\bar{G}, \mathbf{Z})$. Then $\mathrm{gr}\,\varphi : \mathrm{gr}\, G \to \mathrm{gr}\,\bar{G}$ is an isomorphism of graded Lie algebras.*

2. The proof of Theorem 1

2.1. Let R be a non-trivial commutative ring with 1. If G is a group, let RG denote its group algebra (over R) and $\varepsilon : RG \to R$ the associated augmentation, i.e. the R-algebra homomorphism sending every $g \in G$ to $1 \in R$. The kernel of ε is called the augmentation ideal $I = I(RG)$ and, as an R-module, it is freely generated by the elements $g-1$ ($g \in G \backslash \{e\}$). The powers $\{I^j\}_{0 \leq j < \omega}$ form an integral filtration of RG whose associated graded R-algebra will be denoted by $\mathrm{gr}\, RG$.

Define a descending chain of subsets of G by setting

$$D_R^j(G) = \{g \in G \mid g - 1 \in I^j\} \qquad (1 \leq j < \omega).$$

Then $D_R^1(G) = G$, each $D_R^j(G)$ is a (normal) subgroup and for every pair $(j, k) \in \mathbf{N}^2$ the commutator $[D_R^j(G), D_R^k(G)]$ is contained in $D_R^{j+k}(G)$ (see, e.g., [2], §4.5, Prop. 2, p. 42). Hence $\{D_R^j(G)\}_{1 \leqq j < \omega}$ is a central series of G and we can form the associated graded Lie $\mathbf{Z}$-algebra $\mathrm{gr}\,\{D_R(G)\}$. The function $g \mapsto g-1$ induces then an *injective* Lie algebra homomorphism

$$\beta : \mathrm{gr}\,\{D_R(G)\} \to \mathrm{gr}\, RG.$$

(It is clear that β is actually a natural transformation between functors from the category of groups to the category of graded Lie $\mathbf{Z}$-algebras.)

2.2. We specialize now to the case $R = \mathbf{Z}$. Then $D_{\mathbf{Z}}^2(G) = G_2 = G'$ and β gives an isomorphism $\beta^1 : G/G_2 \xrightarrow{\sim} I/I^2$, $gG_2 \mapsto (g-1) + I^2$. If TG_{ab} is the tensor algebra on $G_{ab} = G/G'$ the isomorphism β^1 will extend uniquely to a homomorphism $\mu : TG_{ab} \to \mathrm{gr}\,\mathbf{Z}G$ of graded associative $\mathbf{Z}$-algebras, given in degree j by

$$g_1G_2 \otimes g_2G_2 \otimes \cdots \otimes g_jG_2 \mapsto (g_1 - 1)(g_2 - 1) \cdots (g_j - 1) + I^{j+1}.$$

Clearly μ is always surjective. For TFT groups it is even bijective according to the following

LEMMA. *If G_{ab} is torsion-free and $H_2(G, \mathbf{Z})$ is a torsion group then $\mu : TG_{ab} \twoheadrightarrow \operatorname{gr} \mathbf{Z}G$ is an isomorphism of graded associative $\mathbf{Z}$-algebras.*

2.3. Proof. For every $j \geq 0$ the short exact sequence $I^{j+1} \hookrightarrow I^j \xrightarrow{\pi} I^j/I^{j+1}$ of right G-modules induces a long exact sequence. In dimensions 2, 1 and 0 it looks like this:

$$\begin{tikzcd}
 & H_2(G, I^j/I^{j+1}) \arrow[dl, bend right, "\partial_2"'] & H_2(G, I^j) \arrow[l, "\pi_*"'] & \\
H_1(G, I^{j+1}) \arrow[r] & H_1(G, I^j) \arrow[r, "\pi_*"] & H_1(G, I^j/I^{j+1}) \arrow[d, bend left, "\partial_1"] & \\
0 & (I^j/I^{j+1}) \otimes_G Z \arrow[l] & I^j \otimes_G \mathbf{Z} \arrow[l, "\pi_*"'] & I^{j+1} \otimes_G \mathbf{Z} \arrow[l]
\end{tikzcd} \tag{1}$$

One readily verifies that the composite

$$\bar{\mu} : I/I^2 \otimes I^j/I^{j+1} = H_1(G, \mathbf{Z}) \otimes I^j/I^{j+1} \xrightarrow{\sim} \mathrm{H}_1(G, I^j/I^{j+1})$$

$$\xrightarrow{\partial_1} I^{j+1} \otimes_G \mathbf{Z} \xrightarrow{\sim} I^{j+1}/I^{j+2}$$

is the obvious multiplication map. Taking into account that $I^j \otimes_G \mathbf{Z} \to (I^j/I^{j+1}) \otimes_G \mathbf{Z}$ is an isomorphism and using the universal coefficient theorem, the sequence (1) can be rewritten as

$$\begin{tikzcd}
 & (H_2(G, \mathbf{Z}) \otimes I^j/I^{j+1} \oplus \operatorname{Tor}_1^{\mathbf{Z}}(G_{ab}, I^j/I^{j+1})) \arrow[dl, bend right, "\partial_2"'] & & & \\
H_1(G, I^{j+1}) \arrow[r] & H_1(G, I^j) \arrow[r] & I/I^2 \otimes I^j/I^{j+1} \arrow[r, "\bar{\mu}"] & I^{j+1}/I^{j+2} \arrow[r] & 0.
\end{tikzcd} \tag{2}$$

This exact sequence allows, first of all, to prove that all homology groups $H_1(G, I^j)$ $(0 \leq j < \omega)$ are torsion groups. To see this recall that $H_2(G, \mathbf{Z})$ is a torsion group by hypothesis and $\operatorname{Tor}_1^{\mathbf{Z}}(?, ?)$ by nature, and that $H_1(G, \mathbf{Z}G) = 0$; then use the exactness of (2). Secondly, (2) implies that all multiplication maps $\bar{\mu} : I/I^2 \otimes I^j/I^{j+1} \to I^{j+1}/I^{j+2}$ are bijective. As all $H_1(G, I^j)$ are torsion groups it will do to show inductively that $I/I^2 \otimes I^j/I^{j+1}$ is torsion-free. This follows from the hypothesis that $G_{ab} \cong I/I^2$ be torsion-free and the fact that the tensor product (over $\mathbf{Z}$) of torsion-free groups is again torsion-free. The proof is now easily completed.

2.4. The proof of Theorem 1. Assume G_{ab} is torsion-free and $H_2(G, \mathbf{Z})$ is a torsion group. By Lemma 2.2 the map $\mu: TG_{ab} \to \operatorname{gr} \mathbf{Z}G$ is bijective so that we can define a Lie algebra homomorphism α as the composite

$$\operatorname{gr} G \xrightarrow{\iota} \operatorname{gr}\{D_{\mathbf{Z}}(G)\} \xrightarrow{\beta} \operatorname{gr} \mathbf{Z}G \xleftarrow{\mu} TG_{ab}.$$

Here ι denotes the Lie algebra homomorphism stemming from the inclusions $G_j \subseteq D^j_{\mathbf{Z}}(G)$. Note that $\operatorname{gr} G$ is generated by its first homogeneous component and that $\alpha^1: \operatorname{gr}^1 G \to T^1 G_{ab}$ is the identity on G_{ab}. These facts, together with the universal property of TG_{ab}, imply that $\alpha: \operatorname{gr} G \to TG_{ab}$ is the canonical map of $\operatorname{gr} G$ into its universal algebra and so prove the addendum to Theorem 1.

2.5. We are left with proving that α is injective. If F_X is free on the set X then $(F_X)_{ab}$ is free-abelian and $H_2(F_X, \mathbf{Z}) = 0$. Hence α is defined and gives the classical Lie algebra homomorphism

$$\alpha: \operatorname{gr} F_X \to T(F_X)_{ab} \cong \mathrm{Ass}_X, \qquad x \cdot F_2 \mapsto x \qquad (x \in X).$$

The theory of basic sequences (see, e.g. [1]) or the Poincaré–Birkhoff–Witt theorem (see e.g. [7]) can then be used to prove that α is injective.

Now let $\varphi^1: F_{ab} \hookrightarrow G_{ab}$ be a finitely generated free-abelian subgroup of our torsion-free abelianization G_{ab}. Lift the inclusion to a group homomorphism $\varphi: F \to G$. The lift gives rise to the commutative square

$$\begin{tikzcd}
\operatorname{gr} F \arrow[r, "\alpha_F"] \arrow[d, "\operatorname{gr}\varphi"] & TF_{ab} \arrow[d, "T\varphi^1"] \\
\operatorname{gr} G \arrow[r, "\alpha_G"] & TG_{ab}\,.
\end{tikzcd}$$

In it α_F is injective, and because F_{ab} and G_{ab} are both torsion-free abelian groups and φ^1 is injective, $T\varphi^1$ is likewise injective. Consequently the restriction of α_G to the image of $\operatorname{gr} \varphi$ is injective. But $\operatorname{gr} G$ is generated by its first homogeneous component G_{ab} and G_{ab}, being torsion-free, is a union of finitely generated free-abelian subgroups. This proves that α is injective and establishes the claim of Theorem 1. The proofs of the corollaries present no problems.

2.7. Remark. The injectivity of α could also have been inferred from a (rather difficult) theorem of M. Lazard [3] asserting that the canonical map of a Lie R-algebra into its universal algebra is injective if R is a principal ideal domain.

3. The proof of Theorem 2

3.1. We first return to the set-up of Subsection 2.1 and choose R to be the rational numbers $\mathbf{Q}$. The commutative square

$$\begin{tikzcd} G/D^2_{\mathbf{Z}}(G) \arrow[r, "\beta^1_{\mathbf{Z}}"] \arrow[d, "\mathrm{can}"] & I/I^2 \arrow[d, "\mathrm{can}"] \\ G/D^2_{\mathbf{Q}}(G) \arrow[r, "\beta^1_{\mathbf{Q}}"] & \mathrm{gr}^1\,\mathbf{Q}G \cong I/I^2 \otimes \mathbf{Q} \end{tikzcd}$$

shows that $D^2_{\mathbf{Q}}(G)$ equals $\ker\{G \to G_{ab} \otimes \mathbf{Q}\}$ whence

$$\beta^1_{\mathbf{Q}} \otimes \mathbf{Q} : G_{ab} \otimes \mathbf{Q} \cong G/D^2_{\mathbf{Q}}(G) \otimes \mathbf{Q} \to \mathrm{gr}^1\,\mathbf{Q}G$$

is an isomorphism. It extends uniquely to a homomorphism

$$\mu_{\mathbf{Q}} : T(G_{ab} \otimes \mathbf{Q}) \to \mathrm{gr}\,\mathbf{Q}G$$

of graded associative $\mathbf{Q}$-algebras. Clearly $\mu_{\mathbf{Q}}$ is onto. An easy modification of the proof of Lemma 2.2 reveals that $\mu_{\mathbf{Q}}$ is also injective provided merely that $H_2(G, \mathbf{Z})$ is a torsion group. For a group G whose multiplicator is a torsion group one can therefore define a homomorphism

$$\alpha_{\mathbf{Q}} : \mathrm{gr}\,\{D_{\mathbf{Q}}(G)\} \xrightarrow{\beta^1_{\mathbf{Q}}} \mathrm{gr}\,\mathbf{Q}G \xleftarrow{\mu_{\mathbf{Q}}} T(G_{ab} \otimes \mathbf{Q})$$

of graded Lie $\mathbf{Z}$-algebras.

3.2. Now let G be TFT, let $\bar{G}$ be a group with $H_2(\bar{G}, \mathbf{Z})$ a torsion group and let $\varphi : G \to \bar{G}$ be a group homomorphism. Then the canonical maps $\alpha(G)$, $\alpha_{\mathbf{Q}}(G)$ and $\alpha_{\mathbf{Q}}(\bar{G})$ are all three defined and they combine to produce the following commutative diagram

$$\begin{tikzcd} \mathrm{gr}\,G \arrow[r, "\alpha(G)"] \arrow[d, "\iota"] & T\,G_{ab} \arrow[d, "T\kappa"] \\ \mathrm{gr}\,\{D_{\mathbf{Q}}(G)\} \arrow[r, "\alpha_{\mathbf{Q}}(G)"] \arrow[d, "\mathrm{gr}_{\mathbf{Q}}\,\varphi"] & T(G_{ab} \otimes \mathbf{Q}) \arrow[d, "T(\varphi^1 \otimes \mathbf{Q})"] \\ \mathrm{gr}\,\{D_{\mathbf{Q}}(\bar{G})\} \arrow[r, "\alpha_{\mathbf{Q}}(\bar{G})"] & T(\bar{G}_{ab} \otimes \mathbf{Q}) \end{tikzcd}$$

In it ι denotes the canonical Lie algebra homomorphism stemming from the inclusions $G_j \subseteq D^j_{\mathbf{Q}}(G)$, and $\kappa : G_{ab} \to G_{ab} \otimes \mathbf{Q}$ is the obvious canonical $\mathbf{Z}$-module homomorphism.

By assumption G_{ab} is torsion-free. Therefore κ and $T\kappa$ are injective. By Theorem 1 the same is true for $\alpha(G)$. If, as is required in the hypotheses of Theorem 2, $\varphi^1: G_{ab} \to \bar{G}_{ab}$ is injective $T(\varphi^1 \otimes \mathbf{Q})$ will also be injective. Hence the composite $\iota \circ \mathrm{gr}_{\mathbf{Q}}\, \varphi : \mathrm{gr}\, G \to \mathrm{gr}\, \{D_{\mathbf{Q}}(G)\}$ is seen to be injective and the claim of Theorem 2 follows upon noting that $\iota \circ \mathrm{gr}_{\mathbf{Q}}\, \varphi$ factors through $\mathrm{gr}\, \varphi : \mathrm{gr}\, G \to \mathrm{gr}\, \bar{G}$.

4. Examples and counter-examples

4.1. **E**-*groups.* Let G be a group having torsion-free abelianization and trivial multiplicator. If G_{ab} is even free-abelian the Stallings–Stammbach theorem quoted in 1.4 applies and proves that each G_j/G_{j+1} is isomorphic with the corresponding factor F_j/F_{j+1} of a suitable free group F and so, in particular, torsion-free.

This argument breaks down if G_{ab} is not free abelian, as it usually happens when G is the derived group of a knot group or, more generally, when G is an **E**-group in the sense of [10]. A group G is there called an **E**-group if G_{ab} is torsion-free and if the G-trivial module $\mathbf{Z}$ admits a $\mathbf{Z}G$-projective resolution $\cdots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \to \mathbf{Z} \to 0$ for which the induced differential $\mathbf{1} \otimes \partial_2 : \mathbf{Z} \otimes_G P_2 \to \mathbf{Z} \otimes_G P_1$ is injective. The condition on $\mathbf{1} \otimes \partial_2$ implies that $H_2(G, \mathbf{Z})$ is zero; the converse, however, is false (see *4.2*).

E-groups have the following stability property: if $G \in \mathbf{E}$ and $N \lhd G$ is a normal subgroup with torsion-free, abelian factor group then $N \in \mathbf{E}$. In particular, the terms of the derived series of an **E**-group are **E**-groups and so are the terms of the lower central series.

4.2. Groups G with G_{ab} torsion-free, $H_2(G, \mathbf{Z}) = 0$ but $G \notin \mathbf{E}$. It suffices to prove that G does not have the stability property enjoyed by **E**-groups. Let A be an abelian group possessing an automorphism τ for which $\tau - \mathbf{1} : A \to A$ is bijective and $\tau \wedge \tau - \mathbf{1} \wedge \mathbf{1} : A \wedge A \to A \wedge A$ is onto. Let $C = (t)$ be an infinite cyclic group and define G to be the split extension $A \rtimes C$ where t induces on A the given τ. Then $A = G_2$, $G_{ab} \cong \mathbf{Z}$ and $H_2(G, \mathbf{Z}) = 0$, although A is in general neither torsion-free nor has it trivial multiplicator (take e.g. $A = (\mathbf{Z}/5\mathbf{Z}) \oplus (\mathbf{Z}/5\mathbf{Z})$ and let τ operate by multiplication by 2).

4.3. We give next two examples demonstrating that $\alpha : \mathrm{gr}\, G \to TG_{ab}$ need not exist if the hypotheses of Theorem 1 are weakened. Consider first an *abelian group A*. Then $\mathrm{gr}\, A$ is a *commutative* graded Lie algebra concentrated in degree 1 and its universal algebra is the symmetric algebra SA of A. Hence $\alpha : \mathrm{gr}\, A \to TA$ can only exist if TA is commutative. The commutativity of $\otimes^2 A$, in turn, is equivalent with the vanishing of the exterior square $\Lambda^2 A \cong H_2(A, \mathbf{Z})$; for the canonical map $A \wedge A \to A \otimes A$ taking $a \wedge b$ to $a \otimes b - b \otimes a$ is injective. For a torsion-free abelian group we thus get the following conclusion: The identification $\mathrm{gr}^1 A \xrightarrow{\sim} T^1 A$ extends to a Lie algebra homomorphism $\alpha : \mathrm{gr}\, A \to TA$ if and only if $H_2(A, \mathbf{Z}) = 0$.

4.4. Groups G with $H_2(G, \mathbf{Z}) = 0$ but G_{ab} not torsion-free. The exact sequence

$$H_2(G, \mathbf{Z}) \longrightarrow I/I^2 \otimes I/I^2 \xrightarrow{\bar{\mu}} I^2/I^3 \longrightarrow 0$$

(cf. sequence (2) in *2.3.*) shows that $\mu^2 : \otimes G_{ab} \xrightarrow{\sim} I^2/I^3$ is bijective. Consequently the identification $\mathrm{gr}^1 G \xrightarrow{\sim} T^1 G_{ab}$ extends to

$$\alpha^2 : G_2/G_3 \longrightarrow I^2/I^3 \xleftarrow{\mu^2} \otimes^2 G_{ab}$$

taking $[g, h] \cdot G_3$ to $g \cdot G_2 \otimes h \cdot G_2 - h \cdot G_2 \otimes g \cdot G_2$. (The existence of α^2 can also be deduced from the 5-term sequence associated with the extension $G_2 \lhd G \twoheadrightarrow G_{ab}$, namely

$$H_2(G, \mathbf{Z}) \to H_2(G_{ab}, \mathbf{Z}) \xrightarrow{\chi} G_2/G_3 \to G_{ab} \xrightarrow{\sim} G_{ab} \to 0, \tag{3}$$

and from the facts that $H_2(G_{ab}, \mathbf{Z}) \cong G_{ab} \wedge G_{ab}$, that under this isomorphism χ becomes the obvious commutator map and that $\Lambda^2 G_{ab}$ maps canonically into $\otimes^2 G_{ab}$.)

However, it is in general not possible to extend the identification $\alpha^1 : \mathrm{gr}^1 G \xrightarrow{\sim} T^1 G_{ab}$ to a Lie algebra homomorphism

$$\alpha_* : G/G_2 \oplus G_2/G_3 \oplus G_3/G_4 \to G_{ab} \oplus \otimes^2 G_{ab} \oplus \otimes^3 G_{ab}$$

of nilpotent Lie algebras of class two. To see this let G be a one-relator group of the form

$$G = \langle a, t; t^{-1}at = a^m \rangle = \langle a, t; [a, t] = a^{m-1} \rangle \qquad (m \in \mathbf{Z} \setminus \{0, 1, 2\}).$$

Then $G_{ab} = gp(aG_2) \times gp(tG_2) \cong (\mathbf{Z}/|m-1|\,\mathbf{Z}) \times \mathbf{Z}$ and $H_2(G, \mathbf{Z}) = 0$. The iterated commutator $[a, [a, t]]$ represents the trivial element in G_3/G_4, whereas the corresponding Lie bracket in $\otimes^3 G_{ab}$, namely

$$[aG_2, [aG_2, tG_2]] = aG_2 \otimes aG_2 \otimes tG_2 - 2 \cdot aG_2 \otimes tG_2 \otimes aG_2 + tG_2 \otimes aG_2 \otimes aG_2$$

has order $|m-1| > 1$.

4.5. Groups G, $\bar{G}$ with trivial multiplicator, $\varphi : G \to \bar{G}$ with φ^1 injective but G_{ab} not torsion-free. Our goal is to show that $\varphi^2 : G_2/G_3 \to \bar{G}_2/\bar{G}_3$ is not always injective. Let G be the one-relator group $\langle a, t; t^{-1}at = a^m\rangle$ considered before and let $\bar{G}$ arise out of G by adjoining a k^{th} root of t, i.e.

$$\bar{G} = G \underset{t=u^k}{*} (u) = \langle a, u; u^{-k}au^k = a^m\rangle \qquad (k \geq 2),$$

and let $\varphi : G \to \bar{G}$ be the canonical injection. Then $H_2(G, \mathbf{Z}) = H_2(\bar{G}, \mathbf{Z}) = 0$ and $\varphi^1 : G_{ab} \to \bar{G}_{ab}$ is injective. The map $\varphi^2 : G_2/G_3 \to \bar{G}_2/\bar{G}_3$ can be identified with the exterior square $\Lambda^2\varphi^1 : \Lambda^2 G_{ab} \to \Lambda^2 \bar{G}_{ab}$ (consult (3) above). Both $\Lambda^2 G_{ab}$ and $\Lambda^2 \bar{G}_{ab}$ are cyclic of order $|m-1|$ and $\Lambda^2\varphi^1$ takes the generator $aG_2 \wedge tG_2$ to $aG_2 \wedge u^kG_2 = k(aG_2 \wedge uG_2)$. Hence φ^2 is injective if and only k and m are relatively prime.

This example shows that the conclusion of Theorem 2 becomes false if G_{ab} is not assumed to be torsion-free, everything else remaining unchanged. It is clear that a strong assumption on $H_2(\bar{G}, \mathbf{Z})$ is necessary to exclude cases like the abelianization $\varphi : F \to F_{ab}$ of a free group. But I have not been able to determine to what extent the hypothesis on $H_2(G, \mathbf{Z})$ could be weakened without jeopardizing the claim. (The theorem of Stallings–Stammbach quoted in *1.4.* bears also on the issue.)

4.6. A family of $2^{\aleph_0}$ non-isomorphic groups with trivial multiplicator having all the same torsion-free abelianization. Let $\{{}_kF\}_{k \in \mathbf{N}}$ be a sequence of free groups of rank two, say ${}_kF$ is free on x_k and y_k, and let $ab : {}_kF \twoheadrightarrow ({}_kF)_{ab}$ be the abelianizations. If

$$\varphi = \{\varphi_k : ({}_kF)_{ab} \to ({}_{k+1}F)_{ab}\}_{k \in \mathbf{N}}$$

is a given sequence of homomorphisms it can be lifted to a sequence

$$\Phi = \{\Phi_j : {}_kF \to {}_{k+1}F\}_{k \in \mathbf{N}}$$

so as to produce a commutative ladder

$$\begin{array}{ccccccccc}
{}_1F & \xrightarrow{\Phi_1} & {}_2F & \xrightarrow{\Phi_2} & {}_3F & \xrightarrow{\Phi_3} & {}_4F & \longrightarrow & \cdots \\
\downarrow{\scriptstyle ab} & & \downarrow{\scriptstyle ab} & & \downarrow{\scriptstyle ab} & & \downarrow{\scriptstyle ab} & & \\
({}_1F)_{ab} & \xrightarrow{\varphi_1} & ({}_2F)_{ab} & \xrightarrow{\varphi_2} & ({}_3F)_{ab} & \xrightarrow{\varphi_3} & ({}_4F)_{ab} & \longrightarrow & \cdots .
\end{array}$$

If the φ_k are injective the lifts Φ_k are likewise injective, e.g. because of Theorem 2 and the residual nilpotency of free groups. The direct limit $G_\Phi = \operatorname{colim} \Phi$ is therefore a locally free group with trivial multiplicator and torsion-free abelianization $(G_\Phi)_{ab} = \operatorname{colim} \varphi$; and gr G_Φ is isomorphic to the Lie algebra of $T(G_\Phi)_{ab} \cong T(\operatorname{colim} \varphi)$ generated by its first homogeneous component colim φ. In particular, gr G_Φ depends only on φ and not on the choice of the lift Φ.

Next let P be an infinite set of odd rational primes and let $\lambda : \mathbf{N} \rightarrow P$ be an enumeration of P. Define the sequence $\varphi = \{\varphi_k\}$ by

$$\varphi_k : x_k \cdot ({}_kF)_2 \mapsto x_{k+1}^{\lambda_k} \cdot ({}_{k+1}F)_2 \quad \text{and} \quad y_k \cdot ({}_kF)_2 \mapsto y_{k+1}^{\lambda_k} \cdot ({}_{k+1}F)_2.$$

The direct limit colim φ can be identified with the direct sum $A_x \oplus A_y$ of two copies of the subgroup of the rationals generated by the elements $1/p$ $(p \in P)$. For each $S \subseteq \mathbf{N}$ define a lift $\Phi(S)$ of φ by the formulae

$$\Phi_k(S) : x_k \mapsto \begin{Bmatrix} x_{k+1}^{\lambda_k} & \text{if} \quad k \in S \\ x_{k+1}^{\lambda_k}[y_{k+1}, x_{k+1}] & \text{if} \quad k \notin S \end{Bmatrix} \quad \text{and} \quad y_k \mapsto y_{k+1}^{\lambda_k}.$$

We shall prove that colim $\Phi(S)$ and colim $\Phi(S')$ are isomorphic if and only if the symmetric difference of S and S' is finite. Since $\mathbf{N}$ can be written as a disjoint union of infinitely many infinite subjects this will imply that there are $2^{\aleph_0}$ many non-isomorphic locally free groups whose associated graded Lie $\mathbf{Z}$-algebras are isomorphic.

4.7. If the symmetric difference of S and S' is finite then clearly colim $\Phi(S)$ and colim $\Phi(S')$ are isomorphic. The converse will be established by showing that, up to a finite error, S can be recovered from the nilpotent quotient of class two $G_{\Phi(S)}/(G_{\Phi(S)})_3$.

Let F be free on x and y. The elements of $H = F/F_3$ can be parametrized by the lattice points $\mathbf{Z}^3$ via

$$\mathbf{Z}^3 \ni (a, b, c) \leftrightarrow x^a y^b (y^{-1}x^{-1}yx)^c \cdot F_3 \in F/F_3 = H.$$

The resulting group multiplication on $\mathbf{Z}^3$ is then given by

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', ba' + c + c').$$

Note that this group multiplication has an obvious extension to points of $\mathbf{Q}^3$.

For positive powers and roots of elements of $H = H_\mathbf{Z} \subseteq H_\mathbf{Q}$ one gets

$$(a, b, c)^m = \left(ma, mb, mc + \binom{m}{2} \cdot a \cdot b\right)$$

$$(a, b, c)^{1/m} = (a/m, b/m, c/m - \tfrac{1}{2} \cdot (m-1) \cdot (a/m) \cdot (b/m))$$

It follows that an element of $H_{\mathbf{Z}}$ is an m^{th} power (m an *odd* integer) if and only if all three entries are integral multiples of m.

The endomorphism $\Phi^{\in}$ of H corresponding to the lifts Φ_k with $k \in S$ has the parametric description

$$(a, b, c)\Phi^{\in} = (\lambda_k \cdot a, \lambda_k \cdot b, (\lambda_k)^2 \cdot c).$$

It has the property that the image of an element of H which is an m^{th} power is at least a $(\lambda_k \cdot m)^{\text{th}}$ power and that the image of an element which is not a q^{th} power ($q \neq \lambda_k$ odd prime) is still not a q^{th} power.

The endomorphism $\Phi^{\notin}$ of H corresponding to the lifts Φ_k with $k \notin S$ has the description

$$(a, b, c)\Phi^{\notin} = (\lambda_k \cdot a, \lambda_k \cdot b, (\lambda_k)^2 \cdot c + a).$$

If $q \neq \lambda_k$ is an odd prime then the image under $\Phi^{\notin}$ of an element which is not a q^{th} power is still not a q^{th} power. Moreover, if $(a, b, c)\Phi^{\notin}$ is a $\lambda_k{}^{\text{th}}$ power then $\lambda_k \mid a$.

4.8. Now let $S \subseteq \mathbf{N}$ and construct the group $G_{\boldsymbol{\Phi}(S)} = \operatorname{colim} \boldsymbol{\Phi}(S)$. Then the nilpotent group $N(S) = G_{\boldsymbol{\Phi}(S)}/(G_{\boldsymbol{\Phi}(S)})_3$ is the direct limit of the obvious chain

$${}_1H \xrightarrow{\Phi_{1*}} {}_2H \xrightarrow{\Phi_{2*}} {}_3H \xrightarrow{\Phi_{3*}}$$

where each ${}_kH$ is isomorphic with the free nilpotent group H discussed above. The isolators $I(n) = \{n' \in N(S) \mid n = (n')^j \text{ some } j \in \mathbf{Z}\}$ of an element $n \in N(S)$ are of two types: if n stems from an element $(a_k, b_k, c_k) \in {}_kH$ with $a_k \neq 0$, – note the choice of k does not matter – then $I(n) \cong gp\{1/p \mid p \in \lambda(S)\}$, otherwise $I(n) \cong gp\{1/p \mid p \in P\}$. The claim then follows from the classification of isomorphism types of subgroups of the rationals.

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