Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 54 (1979)

Artikel: Non-smoothable varieties.

Autor: Sommese, Andrew J.

DOI: https://doi.org/10.5169/seals-41566

Comment. Math. Helvetici 54 (1979) 140–146 Birkhäuser Verlag, Basel

Non-smoothable varieties

ANDREW J. SOMMESE

In this article I will give new examples of projective manifolds A in $\mathbf{P}_{\mathbf{C}}^N$ such that the cone CA on A from a point $x \in \mathbf{P}_{\mathbf{C}}^{N+1} - \mathbf{P}_{\mathbf{C}}^N$ is not smoothable [cf. §1 for precise definitions]. A sample result is:

PROPOSITION. *Let A be a projective manifold in $\mathbf{P}_{\mathbf{C}}^N$. The cone CA on A in $\mathbf{P}_{\mathbf{C}}^{N+1}$ is not smoothable if the first Betti number of A is zero and A is a product, $\prod_{i=1}^k A_i$, of projective manifolds A_i, such that either $k>2$ or $\dim_{\mathbf{C}} A_i \geq 2$ for each i.*

Examples of non-smoothable manifolds are not new [cf. 11, 8, 4], but the above example differs from the usual examples. Unlike Schlessinger's examples [11], one cannot compute the T_i. Unlike Hartshorne's examples [4], the manifold A can have very high codimension in $\mathbf{P}_{\mathbf{C}}^N$, indeed $N/\dim_{\mathbf{c}} A$ can be as large as we please.

The above examples are based on my earlier work [13] on manifolds that cannot be hyperplane sections in any projective manifold. The connection, Lemma (1.3), between this work and smoothability is that, if CA can be smoothed then some small deformation of A in $\mathbf{P}_{\mathbf{C}}^N$ is a hyperplane section of a projective manifold. In §1 I give definitions and background material. In §2 I present my examples. In §3 I ask a question and make some closing remarks.

This article was inspired by a reading of [10].

§1. Preliminaries

(1.1) DEFINITIONS. Let Y be an analytic subspace of a projective manifold X. A **deformation of Y in X** is a triple $(\tilde{Y}, \Delta, \pi)$ where Δ is the open unit disc in $\mathbf{C}$ and:

(a) $\tilde{Y}$ is an analytic subspace of $X \times \Delta$,

(b) the restriction $\pi: \tilde{Y} \to \Delta$, of the product projection $p: X \times \Delta \to \Delta$ is a proper, flat, surjection, and

(c) $\pi^{-1}(0) = Y$ where 0 is the origin of Δ. **Y is smoothable in X** if there exists a deformation as above with $\pi^{-1}(t)$ a submanifold of $X \times \{t\}$ for each $t \in \Delta - \{0\}$.

(1.2) *Remark.* It is easy to see by considering the Hilbert scheme of X, that the above definition of smoothing is equivalent to Hartshorne's definition [4, p. 241] over $\mathbf{C}$.

The results in this paper are stated for varieties with isolated singularities. The reader should specialize them to the case of a cone, CA, in $\mathbf{P}_{\mathbf{C}}^{N+1}$ on a submanifold, A, of $\mathbf{P}_{\mathbf{C}}^{N}$. Note in this case A is a hyperplane section of CA, i.e. $(CA)\cap\mathbf{P}_{\mathbf{C}}^{N}$.

(1.3) LEMMA. *Let Y be a subvariety of $\mathbf{P}_{\mathbf{C}}^{N}$. Let $\mathcal{H}$ be a smooth hypersurface of $\mathbf{P}_{\mathbf{C}}^{N}$ which intersects Y transversely in a submanifold H which doesn't contain any singularities of Y. Assume that given any deformation $(\tilde{H}, \Delta, \pi)$ of H in $\mathcal{H}$, there is a neighborhood U of 0 in Δ such that for each $t\in U$, $\pi^{-1}(t)$, considered as a submanifold of $\mathbf{P}_{\mathbf{C}}^{N}$, is not the transverse intersection of $\mathcal{H}$ with any projective submanifold of X in $\mathbf{P}_{\mathbf{C}}^{N}$. Then Y has isolated singularities and is not smoothable in $\mathbf{P}_{\mathbf{C}}^{N}$.*

Proof. First note that Y must have singularities, since otherwise we could take the trivial deformation $(\tilde{H}, \Delta, \pi)$ with $\tilde{H}=H\times\Delta$, and for each $t\in S$ use $X=Y$ to contradict the hypothesis. Further Y has at most isolated singularities since otherwise the singularities would be a positive dimensional subvariety of $\mathbf{P}_{\mathbf{C}}^{N}$ that was disjoint from a hypersurface $\mathcal{H}$.

Now to see that Y can't be smoothed, assume that it could be. Let $(\tilde{Y}, \Delta, \pi)$ be a smoothing. Then for $t\neq 0$ in a small enough subdisc, Δ', we have that $\pi^{-1}(t)$ is a submanifold of $\mathbf{P}_{\mathbf{C}}^{N}$ transverse to $\mathcal{H}$. Thus $(Z, \Delta', \pi|_Z)$ with $Z=\tilde{Y}\cap(\mathcal{H}\times\Delta')$ is a deformation of H in $\mathcal{H}$ of the sort we've hypothesized didn't exist.

The next two lemmas, which will be used in §2, show that certain properties are invariant under small deformations. The first is folk-lore, while the second is due to Kodaira [5].

(1.4) LEMMA. *Let A be a projective (Kaehler) manifold with the integral cohomology ring of an Abelian variety, e.g. A is diffeomorphic to an Abelian variety. Then A is an Abelian variety (a Kaehler torus).*

Proof. Let $\alpha: A\to ALB(A)$ be the Albanese map of A. If A is projective (Kaehler), $ALB(A)$ is an Abelian variety (a Kaehler torus) of complex dimension equal to half of $\dim_{\mathbf{C}} H^1(A,\mathbf{C})$. Now a basic property of $ALB(A)$ is that α induces an isomorphism of $H^1(A,\mathbf{Z})$ with $H^1(ALB(A),\mathbf{Z})$. This combined with the hypothesis about the integral cohomology ring of A immediately implies that α induces an isomorphism of $H^*(ALB(A),\mathbf{Z})$ with $H^*(A,\mathbf{Z})$. This implies that α is onto. Further the fibres of α are zero dimensional. If they weren't and F was a positive dimensional fibre then we conclude that the restriction of $H^2(A, \mathbf{Z})$ to F is 0 since it equals the pullback of $H^2(ALB(A), \mathbf{Z})$ by $\alpha|_F$ which is a constant map. This is absurd as a simple consideration of the Kaehler class of A shows, i.e. raise the Kaehler class to the $\dim_{\mathbf{C}} F$ power and restrict it to F. Now α is one to one, since otherwise α could not pull a generator of $H^{2a}(ALB(A), \mathbf{Z})$ back to a generator of $H^{2a}(A, \mathbf{Z})$ where $a = \dim_{\mathbf{C}} A$. Finally note that any one to one and onto map between complex manifolds is a biholomorphism [9, p. 86, Theorem 5].

(1.5) LEMMA (Kodaira). *Let $p: X \to \Delta$ be a proper, holomorphic surjection, of maximal rank from a connected complex manifold X onto the unit disc. Assume there is a maximal rank, holomorphic surjection, $q: p^{-1}(0) \to Y$ for some complex manifold Y. Assume that for each $y \in Y$, $H^1(q^{-1}(y), \mathcal{O}_{q^{-1}(y)}) = 0$ where $\mathcal{O}_{q^{-1}(y)}$ is the holomorphic structure sheaf of $q^{-1}(y)$. Then there exists a subdisc Δ' of Δ, a complex manifold $\tilde{Y}$, and holomorphic maximal rank surjections $\tilde{q}: p^{-1}(\Delta') \to \tilde{Y}$ and $\phi: \tilde{Y} \to \Delta'$ with $\phi \cdot \tilde{q} = p|_{p^{-1}(\Delta')}$, $\phi^{-1}(0) = Y$, and $\tilde{q}|_{p^{-1}(0)} = q$.*

Proof. See [3, p. 87, §2]. The idea of the proof is simply that for each $y \in Y$, the normal bundle $N_{q^{-1}(y)}$ of $q^{-1}(y)$ in X is a direct sum of some number of copies of $\mathcal{O}_{q^{-1}(y)}$. Thus the condition $H^1(q^{-1}(y), \mathcal{O}_{q^{-1}(y)}) = 0$ implies that $H^1(q^{-1}(y), N_{q^{-1}(y)}) = 0$, and thus there are no obstructions to deformation.

(1.6) *Remarks.* If $p^{-1}(0)$ is projective, then $H^1(q^{-1}(y), \mathbf{C}) = 0$ implies $H^1(q^{-1}(y), \mathcal{O}_{q^{-1}(y)}) = 0$ for $y \in Y$ by the Hodge decomposition theorem [6].

If $p^{-1}(0)$ was a product, $\prod_{i=1}^k Y_i$ with projections $q_i: p^{-1}(0) \to Y_i$ and if $H^1(q_i^{-1}(y), \mathcal{O}_{q_i^{-1}(y)}) = 0$ for all $y \in Y_i$ for all i, then (1.5) implies that $p^{-1}(\Delta')$ for some subdisc is a fibre product of holomorphic maximal rank surjections $\tilde{\phi}_i: \tilde{Y}_i \to \Delta'$ with $\tilde{\phi}_i(0) = Y_i$. By the last paragraph and the Kunneth theorem the vanishing first cohomology condition is satisfied for all $q_i^{-1}(y)$ if $H^1(p^{-1}(0), \mathbf{C}) = 0$ and $p^{-1}(0)$ is projective.

Finally one last lemma:

(1.7) LEMMA. *Let A be a connected projective manifold. Let $p: A \to Y$ be a maximal rank holomorphic surjection onto a compact complex manifold, Y. Then Y is projective.*

Proof. We assume $\dim_{\mathbf{C}} Y \geq 1$ or there is nothing to prove. Let ω be a closed, positive, integral (1, 1) form A, i.e. the Chern curvature form of a positive metric on a holomorphic line bundle on A. These exist since A is projective [cf. 6]. Let $f = \dim_{\mathbf{C}} A - \dim_{\mathbf{C}} Y$. We fibre integrate ω^{f+1} to get a closed, positive, integral (1, 1) form on Y; [7] is a good reference for the basic facts about fibre integration. Now Y is projective by the Kodaira embedding theorem [6].

§2. The examples

(2.1) PROPOSITION. *Let Y be an algebraic subvariety of $\mathbf{P}^N_{\mathbf{C}}$. Assume there is an hypersurface $\mathcal{H}$ of $\mathbf{P}^N_{\mathbf{C}}$ that intersects Y transversely in a submanifold H that does not meet the singular set of Y. Then Y has isolated singularities and is nonsmoothable in $\mathbf{P}^N_{\mathbf{C}}$ if H is any of the following:*

(2.1.1) *H is an Abelian variety of dimension greater than 1,*

(2.1.2) *the first Betti number of H is zero and H is a product $\prod_{i=1}^k H_i$ of projective manifolds H_i with either $k \geq 3$ or $\dim_{\mathbf{C}} H_i \geq 2$ for each i,*

(2.1.3) *there exists a surjective, maximal rank, holomorphic map $f: H \to Z$ where Z is a projective manifold, any fibre of f has its first Betti number zero, and either:*

(a) $2 + \dim_{\mathbf{C}} Z \leq \dim_{\mathbf{C}} H \leq 2\dim_{\mathbf{C}} Z - 2$

or,

(b) $\dim_{\mathbf{C}} H = 2\dim_{\mathbf{C}} Z - 1$ *or* $2\dim_{\mathbf{C}} Z$ *and a fibre of f doesn't have the Betti number's of projective space.*

Proof. Assume Y is smoothable. Then by (1.3) there exists a deformation (H, Δ, π) of H in $\mathcal{H}$, such that for each neighborhood U of 0 in Δ, there is a $t \in U$ such that $\pi^{-1}(t)$, considered as a submanifold of $\mathbf{P}^N_{\mathbf{C}}$ is the transverse intersection of $\mathcal{H}$ with a projective manifold X of $\mathbf{P}^N_{\mathbf{C}}$.

Now note that for small enough t, $\pi^{-1}(t)$ is diffeomorphic to $\pi^{-1}(0)$. This follows since the fact that $\pi^{-1}(0)$ is a manifold, and the fact that π is flat, imply that π is of maximal rank in a neighborhood of $\pi^{-1}(0)$. Thus the result will be shown if we show that for all t near 0, $\pi^{-1}(t)$ cannot be a hyperplane section of any projective manifold X. This will follow if we show that $\pi^{-1}(t)$ satisfies the same properties (2.1, 1–3) as $\pi^{-1}(0)$ for t near 0 and if we show that no projective manifold satisfying any of (2.1, 1–3) can be a hyperplane section of any projective manifold X. Now the latter follows from [13, Corollary I-A] for (2.1.1), from [13, Proposition IV] for (2.1.2), from [13, Proposition V] for (2.1.3). The former are an immediate consequence of (1.4), (1.5), and (1.6).

(2.2) PROPOSITION. *Let Y be an algebraic subvariety of $\mathbf{P}^N_{\mathbf{C}}$. Assume there is an hypersurface of $\mathbf{P}^N_{\mathbf{C}}$ that intersects Y transversely in a submanifold H of complex dimension at least two that doesn't meet the singular set of Y. Let $c_1(N_H)$ and $c_1(T_H)$ be the first Chern classes of the holomorphic normal bundle of H in Y and the holomorphic tangent bundle of H respectively. Assume $c_1(N_H) = \lambda g$ and $c_1(T_H) = \mu g$ where λ is a positive integer and $g \in H^2(H, \mathbf{Z})$. Then Y has isolated singularities and is not smoothable if $\mu + \lambda > \dim_{\mathbf{C}} H + 2$.*

Proof. If Y is smoothable then by (1.3) there exists a projective manifold X in $\mathbf{P}^N_{\mathbf{C}}$ which $\mathcal{H}$ is transverse to and such that $H' = \mathcal{H} \cap X$ satisfies the same relations as H, i.e. $c_1(N_{H'}) = \lambda g$ and $c_1(T_{H'}) = \mu g$ where $\lambda > 0$ and $g \in H^2(H', \mathbf{Z})$. This is because the conditions are topological and are easily seen to be preserved by the deformation from H to H'.

Now there exists an element g' of $H^2(X, \mathbf{Z})$ that restricts to g. To see this first note that $[H']$, the holomorphic line bundle on X associated to the divisor H', restricts to $N_{H'}$ on H'. Thus $\lambda g = c_1(N_{H'}) = c_1([H'])|_{H'}$ extends to X; the extension being $c_1([H'])$. Next note that since $\dim_{\mathbf{C}} H' \geq 2$, the first Lefschetz theorem [1, 2] says that the cokernel of the image under restriction of $H^2(X, \mathbf{Z})$ in $H^2(H', \mathbf{Z})$ has no torsion.

Now by the adjunction formula we have $(K_X|_{H'}) \otimes_{\mathbf{C}} [H'] = K_{H'}$. Thus since $K_{H'} = \det T^*_{H'}$ we have that $c_1(K_X) = (\lambda + \mu) g'$. Now let L be a holomorphic line bundle on X such that $L^{-(\lambda+\mu)} = K_X$. This is possible as we see from the Kummer sequence:

$$0 \to \mathbf{Z}_{\lambda+\mu} \to \mathcal{O}^*_X \to \mathcal{O}^*_X \to 0.$$

Now L has a Hermitian metric whose curvature form is positive. To see this note that $c_1(L^\lambda) = \lambda g' = c_1([H'])$. Now $[H']$ is the restriction of a power of the hyperplane bundle or projective space and thus $c_1([H'])$ is represented by a closed positive (1.1) form. Thus [3, p. 17, Lemma (1.13)], L^λ possesses a Hermitian metric whose curvature form is positive. By taking the λ^{th} root we get the desired metric on L.

By the Kodaira embedding theorem [6], some power L^A of L has enough global holomorphic sections to give an embedding of X into projective space. This implies that the Hilbert polynomial:

$$p(n) = \sum_{j=0}^{\dim_{\mathbf{C}} X} (-1)^j \dim_{\mathbf{C}} H^j(X, L^n)$$

has degree $\dim_{\mathbf{C}} X$, and that the coefficient of the term of degree $\dim_{\mathbf{C}} X$ is non zero.

Note that if $-(\lambda+\mu) < -\dim_{\mathbf{C}} X - 1$ then $L^r \otimes_{\mathbf{C}} K_X^{-1} = L^{r+\lambda+\mu}$ has a positive metric for $r > -(\lambda+\mu)$, since L has a positive metric. Thus by the Kodaira vanishing theorem [6], $H^i(X, L^r) = 0$ for $r > -(\lambda+\mu)$ and $i > 0$. Further $H^0(X, L^r) = 0$ for $r < 0$. To see this use Serre duality to get $H^0(X, L^r) \approx H^{\dim_{\mathbf{C}} X}(X, K_X \otimes_{\mathbf{C}} L^{-r}) = H^{\dim_{\mathbf{C}} X}(X, L^{-r-\lambda-\mu})$, note that $-r-\lambda-\mu > -(\lambda+\mu)$ for $r < 0$, and use the Kodaira vanishing theorem again.

Now we have shown that $p(n)=0$ if $-(\lambda+\mu)<n<0$. If $-(\lambda+\mu)< -\dim_{\mathbf{C}} X-1$ this implies there are at least $\dim_{\mathbf{C}} X+1$ zeros which is incompatible with $p(n)$ being a non-zero polynomial of degree equal to $\dim_{\mathbf{C}} X$.

(2.3) COROLLARY. *Let A be a connected projective embedded into $P_{\mathbf{C}}^N$ by global sections of K_A^r for some r where K_A is the canonical bundle. Assume* $\dim_{\mathbf{C}} A \geq 2$. *Then the cone CA in $P_{\mathbf{C}}^{N+1}$, on A from a point $x \in P_{\mathbf{C}}^{N+1}-P_{\mathbf{C}}^N$ cannot be smoothed if $r>\dim_{\mathbf{C}} A+3$.*

Proof. $c_1(K_A^r)=rc_1(K_A)=-rc_1(T_A)$.

§3. Closing remarks

(3.1) QUESTION. *Let A be a submanifold of an Abelian variety. Assume that the holomorphic tangent bundle of A splits into a direct sum of proper holomorphic subbundles, e.g. A is a product of submanifolds of Abelian varieties. Let A be embedded in $\mathbf{P}_{\mathbf{C}}^N$ and let CA be the cone in $\mathbf{P}_{\mathbf{C}}^{N+1}$ on A from a point $x \in \mathbf{P}_{\mathbf{C}}^{N+1}-\mathbf{P}_{\mathbf{C}}^N$. Then CA cannot be smoothed in $\mathbf{P}_{\mathbf{C}}^{N+1}$.*

This is made plausible since such an A cannot be a hyperplane section in any projective manifold X [13, Proposition I]. Unfortunately neither the conditions on A or the properties that are drawn from the conditions on A are stable under small deformations.

It should be noted that by considering cones on submanifolds of $\mathbf{P}_{\mathbf{C}}^N$ from $\mathbf{P}_{\mathbf{C}}^k \subseteq \mathbf{P}_{\mathbf{C}}^{N+k+1}-\mathbf{P}_{\mathbf{C}}^N$, one can construct subvarieties with k dimensional singular sets and such that small deformations still have k dimensional singular sets.

Finally I would like to call attention to [12], where there are some generalizations of the results of [13]. Also I would like to mention that T. Fujita has informed me of some nice progress he has made on a number of questions of [13], in particular the question of blowing down, and [13, Question III-B] which he answered in the negative.

BIBLIOGRAPHY

[1] ANDREOTTI, A. and FRANKEL, T., *The Lefschetz theorem on hyperplane sections*, Ann. of Math. *69* (1959), 713–717.

[2] BOTT, R., *On a theorem of Lefschetz*, Mich. Math. J. *6* (1957), 211–216.

[3] GRIFFITHS, P. A., *Entire holomorphic mappings in one and several complex variables*, Ann. of Math. Study, No. 85, Princeton University Press, Princeton (1976).

[4] HARTSHORNE, R., *Topological conditions for smoothing algebraic singularities*, Topology, *13* (1974), 24–253.
[5] KODAIRA, K., *On stability of compact submanifolds of complex manifolds*, Amer. J. Math., *85* (1963), 79–94.
[6] —— and MORROW, J., *Complex manifolds*, Holt, Rinehart, and Winston, New York (1971). (1971).
[7] MATSUSHIMA, Y. and STOLL, W., *Ample vector bundles on compact complex spaces*, Rice University Study Vol. 59, No. 2, 71–107, Rice University Press, Houston (1973).
[8] MUMFORD, D., *A remark on the paper of M. Schlessinger*, Rice University Study, Vol. 59, No. 1, 113–118, Rice University Press, Houston (1973).
[9] NARASIMHAM, R., *Several complex variables*, Chicago Lecture Notes, University of Chicago Press, Chicago (1971).
[10] REES, E. and THOMAS, E., *Cobordism obstructions to deforming isolated singularities*, Math. Ann. *232* (1978), 33–54.
[11] SCHLESSINGER, M., *On rigid singularities*, Rice University Study, Vol. 59, No. 1, 147–162, Rice University Press, Houston (1973).
[12] SILVA, A., *Relative vanishing theorems I: applications to ample divisors*, Comm. Math. Helv. *52* (1978), 483–490.
[13] SOMMESE, A., *On manifolds that cannot be ample divisors*, Math. Ann., *221* (1976), 55–72.

Dept. of Mathematics, White Hall
Cornell University
Ithaca, N.Y. 14853
U.S.A.

Received April 10, 1978