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## Some properties of groups without the property $P_1$

V. LOSERT

Let  $G$  be a locally compact group with a left Haar measure  $dx$ . The group is said to have the property  $P_1$  if for every  $\epsilon > 0$  and every compact subset  $K$  of  $G$  there exists  $s \in L^1(G)$  with  $\|s\|_1 = 1$  and  $\sup_{x \in K} \int_G |s(xy) - s(y)| dy < \epsilon$ . (see [1] p. 168).

In [1] A. Derighetti introduced the quantity  $\rho_1$ , which is defined as the infimum of all non-negative real numbers  $\lambda$  such that for every compact subset  $K$  of  $G$  there exists  $s \geq 0$  with  $\|s\|_1 = 1$  and  $\sup_{x \in K} \int_G |s(xy) - s(y)| dy < \lambda$ . He proved that  $\rho_1 < 1$  implies property  $P_1$ .

In this paper we are able to show that  $\rho_1 < 2$  already implies property  $P_1$ . It follows that  $\rho_1$  can assume only two values,  $\rho_1 = 0$  if  $G$  has property  $P_1$  and  $\rho_1 = 2$  if not. Analogous relations are proved for the constants  $\rho_p$ , which are defined in the same manner as  $\rho_1$ , with  $L^1(G)$  replaced by  $L^p(G)$  ( $1 \leq p < \infty$ ). The same constants are obtained if the system of compact subsets of  $G$  is replaced by that of finite subsets.

More generally one can consider the case of a locally compact space  $X$  on which  $G$  acts continuously and which admits a quasi invariant measure  $\mu$  (see [5]). This defines a representation  $\pi$  of  $G$  on  $L^2(X, \mu)$ . One can define constants  $\rho_p$  as above and it turns out that again only two values for each  $\rho_p$  are possible, depending whether  $\pi$  weakly contains the one dimensional identity representation of  $G$  or not.

In the last section we show that the Fourier algebra  $A(G)$  (or more generally  $A_p(G)$ ) factorizes iff the group has property  $P_1$ . This generalizes a result proved in [7] 2.3 for the free group of two generators.

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## Notations

Let  $G$  be a locally compact group with unit element  $e$ ,  $X$  a locally compact topological space on which  $G$  acts continuously and which admits a quasi invariant Radon measure  $\mu$  with modular function  $\chi$  (cp. [5]).

If  $f$  is a function on  $X$  and  $a \in G$ , we define  $af(x) = f(ax)$  for  $x \in X$ .  $\mathcal{F}$  (resp.  $\mathcal{K}$ ) shall denote the family of all finite (resp. compact) subsets of  $G$ .

We have a representation  $\pi_p(x)s = \gamma(x^{-1}, \cdot)^{1/p} x^{-1s}$  for  $x \in G$ ,  $s \in L^p(X, \mu)$ . (We will write  $\pi$  instead of  $\pi_2$ .)

We define  $\rho(\pi_p) = \sup_{K \in \mathcal{K}} \inf \{ \sup_{x \in K} \|\pi(x)s - s\|_p : s \in L^p(X, \mu), s \geq 0, \|s\|_p = 1 \}$  ( $1 \leq p < \infty$ )

$\rho^*(\pi_p)$  is defined in the same manner with  $\mathcal{K}$  replaced by  $\mathcal{F}$ .

$$d(\pi) = \sup_{K \in \mathcal{K}} \inf_{x \in K} \{ \sup_{s \in L^2(X, \mu)} |(\pi(x)s, s) - 1| : s \in L^2(X, \mu) \}$$

$d^*(\pi)$  with  $\mathcal{F}$  instead of  $\mathcal{K}$ .

$\rho(\pi_2)$  coincides with the quantity  $\rho(\pi)$  of [1]. For  $X = G$ ,  $\mu$  Haar measure  $\rho(\pi_1)$  coincides with  $\rho_1$ .

If  $M$  is a mean on  $L^\infty(X, \mu)$  (a positive linear functional with  $M(1) = 1$ ) we put  $\alpha(M) = \sup \{ |M(x\phi) - M(\phi)| : x \in G, \phi \in L^\infty(X, \mu), \|\phi\|_\infty \leq 1 \}$ .

If  $x \in G$ ,  $\epsilon_x$  shall denote the point measure of mass one concentrated in  $x$ . We write  $L_{\mathbf{R}}^1(G)$  for the space of *real*-valued integrable functions on  $G$  and  $L_{\mathbf{R}}^0(G) = \{f \in L_{\mathbf{R}}^1(G) : \int f(x) dx = 0\}$ .

**LEMMA 1** Put  $B = \{\epsilon_x * f - \epsilon_y * f : x, y \in G, f \in L_{\mathbf{R}}^1(G), \|f\|_1 \leq 1\}$ . Then the closed absolutely convex hull of  $B$  in  $L_{\mathbf{R}}^1(G)$  coincides with  $\{f \in L_{\mathbf{R}}^0(G) : \|f\|_1 \leq 2\}$ .

*Proof.* We use the bipolar theorem [10] Th. 4 p. 35:

$$\begin{aligned} B^0 &= \{ \phi \in L^\infty(G) : |\int g(x)\phi(x) dx| \leq 1 \quad \forall g \in B \\ &= \{ \phi \in L_{\mathbf{R}}^\infty(G) : \|\phi - \phi_y\|_\infty \leq 1 \quad \forall x, y \in G \}. \end{aligned}$$

If  $\phi \in B^0$  is continuous this means that  $|\phi(x) - \phi(y)| \leq 1$  for all  $x, y \in G$ . It follows that  $\sup_{x \in G} \phi(x) - \inf_{x \in G} \phi(x) \leq 1$  ( $\phi$  is real valued) and so there exists some  $c \in \mathbf{R}$  such that  $\|\phi - c\|_\infty \leq \frac{1}{2}$ .  $B^0$  is by definition closed with respect to the topology  $\sigma(L^\infty, L^1)$ , it is convex and left-translation invariant. It follows that  $\phi \in B^0$ ,  $f \in L^1(G)$ ,  $f \geq 0$ ,  $\int f dx = 1$  implies  $f * \phi \in B^0$ .

If  $\phi \in B^0$  is arbitrary, we choose an appropriate approximate identity  $(u_i)$  in  $L^1(G)$  such that  $u_i * \phi \in B^0$  and  $u_i * \phi$  converges to  $\phi$  in the topology  $\sigma(L^\infty, L^1)$ . Since  $u_i * \phi$  is continuous we can find  $c_i \in \mathbf{R}$  such that  $\|u_i * \phi - c_i\|_\infty \leq \frac{1}{2}$ . Let  $c \in \mathbf{R}$  be a cluster point of the  $c_i$ , then some subset of  $u_i * \phi - c_i$  converges to  $\phi - c$  (for  $\sigma(L^\infty, L^1)$ ) and it follows that  $\|\phi - c\|_\infty \leq \sup \|u_i * \phi - c_i\|_\infty \leq \frac{1}{2}$ . This means that  $B^0 = \{ \phi \in L_{\mathbf{R}}^\infty(G) : \exists c \in \mathbf{R} : \|\phi - c\|_\infty \leq \frac{1}{2} \}$  (the converse inclusion is trivial).

If  $f \in L_{\mathbf{R}}^1(G)$ ,  $\int f dx = 0$ ,  $\|f\|_1 \leq 2$ ,  $\phi \in B^0$ ,  $\|\phi - c\|_\infty < \frac{1}{2}$  then  $|\int f(x)\phi(x) dx| = |\int f(x)(\phi(x) - c) dx| \leq 1$ .

Consequently  $f \in B^{00}$  and it is again trivial that any  $f$  in the closed absolutely convex hull of  $B$  satisfies  $\int f dx = 0$  and  $\|f\|_1 \leq 2$ .

*Remark.* The conclusion of Lemma 1 is not valid in the case of complex valued functions:

Let  $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$  be the roots of  $z^3 = 1$  and assume that  $\phi \in L^\infty(G)$  takes only the values  $3^{-1/2} \alpha_i$  ( $i = 1, 2, 3$ ) each on a set  $A_i$  of positive measure. Then  $\phi$  belongs to  $B^0$  but there does not exist a constant  $c \in \mathbf{C}$  such that  $\|\phi - c\|_\infty \leq \frac{1}{2}$ . Now one can easily construct a function  $f \in L^0(G)$  such that  $\|f\|_1 = 2$  and  $\arg f(x) = \bar{\alpha}_i$  for  $x \in A_i$ . Consequently  $\int f(x)\phi(x) dx = 3^{-1/2} \int |f(x)| dx = 2 \cdot 3^{-1/2} > 1$  and so  $f$  does not belong to the closed, absolutely convex hull of  $B$ .

LEMMA 2. If  $M$  is a mean on  $L^\infty(X, \mu)$ ,  $\phi \in L^\infty(X, \mu)$ ,  $f \in L^0_{\mathbf{R}}(G)$ , we have  $|M(f * \phi)| \leq 2^{-1} \alpha(M) \|f\|_1 \|\phi\|_\infty$ .

*Proof.* For fixed  $\phi \in L^\infty(X, \mu)$  the map  $f \rightarrow f * \phi$  is continuous from  $L^1(G)$  to  $L^\infty(X, \mu)$  (for the norm topology). For  $x, y \in G$ ,  $f \in L^1(G)$  we have by the definition of  $\alpha(M)$ :

$$|M(\epsilon_x * f * \phi - \epsilon_y * f * \phi)| \leq \alpha(M) \|f * \phi\|_\infty \leq \alpha(M) \|f\|_1 \|\phi\|_\infty.$$

The set  $B_1 = \{g \in L^1_{\mathbf{R}}(G) : |M(g * \phi)| \leq \alpha(M) \|\phi\|_\infty\}$  is a closed convex subset of  $L^1_{\mathbf{R}}(G)$  which contains the set  $0.B = \{\epsilon_x * f - \epsilon_y * f : x, y \in G, f \in L^1_{\mathbf{R}}(G), \|f\|_1 \leq 1\}$ . By Lemma 1  $B_1$  contains all functions  $g \in L^0_{\mathbf{R}}(G)$  with  $\|g\| \leq 2$  and so the result follows.

LEMMA 3. If  $M$  is a mean on  $L^\infty(X, \mu)$ ,  $C$  a compact subset of  $G$ ,  $\epsilon > 0$ , then there exists  $f \in L^1(X, \mu)$ ,  $f \geq 0$  with  $\int f(x) d\mu(x) = 1$  such that  $\|\pi_1(x)f - f\| \leq \alpha(M) + \epsilon$  for all  $x \in C$ .

*Proof.* Let  $h$  be an arbitrary function in  $L^1(G)$  with  $h \geq 0$ ,  $\int h(x) dx = 1$ . Then there exists some neighbourhood  $U$  of the unit element  $e$  in  $G$  such that  $\|\epsilon_x * h - h\|_1 \leq \epsilon/2$  for  $x \in U$ . Let  $F = \{x_1, \dots, x_n\}$  be a finite subset of  $G$  such that  $C \subseteq FU$ .

Now we consider the space  $Y$  which is defined as a product of  $n+1$ -copies of  $L^1_{\mathbf{R}}(X)$  with the norm  $\|(f_i)\| = \sup \|f_i\|_1$ . Put  $\alpha = \alpha(M)^{-1}$  (if  $\alpha(M) = 0$  the same proof works if  $\alpha$  is sufficiently large) and define a map  $u : L^1_{\mathbf{R}}(X, \mu) \rightarrow Y$  by

$$u(f) = (\alpha \pi_1(\epsilon_{x_1} * h - h)f, \dots, \alpha \pi_1(\epsilon_{x_n} * h - h)f, f).$$

(We write also  $\pi_1$  for the extension of the representation to  $L^1(G)$ .)  $u$  is clearly linear and satisfies  $\|f\|_1 \leq \|u(f)\| \leq 2\alpha \|f\|_1$ .

The dual space  $Y'$  of  $Y$  can be identified with the sum of  $(n+1)$ -copies of  $L^\infty_{\mathbf{R}}(X, \mu)$ , equipped with the norm  $\|(\phi_i)\| = \sum_{i=1}^n \|\phi_i\|_\infty$  and the dual map  $u'$



of  $u$  is then given by  $u((\phi_i)) = \sum_{i=1}^{n+1} \alpha[(\epsilon_{x_i} * h)^* * \phi_i - h^* * \phi_i] + \phi_{n+1} \cdot (h * (x) = h(x^{-1}) \wedge_G(x^{-1}))$ . Let  $K$  be the preimage of the unit ball of  $Y$  by  $u$ , then the polar  $K^0$  is the image of the unit ball of  $Y'$  by  $u'$ . If  $\phi \in K^0$ , it follows that there exist  $(\phi_i)_{i=1}^{n+1} \in Y'$  with  $\sum_{i=1}^{n+1} \|\phi_i\|_\infty \leq 1$  such that  $\phi = \sum_{i=1}^n \alpha[(\epsilon_{x_i} * h)^* * \phi_i - h^* * \phi_i] + \phi_{n+1}$ . By Lemma 2  $|M(\phi)| \leq 1$  and this means that  $M$  belongs to the bipolar  $K^{00}$  of  $K$  in  $L^1_{\mathbf{R}}(X, \mu)''$ . By the bipolar theorem  $K$  is dense in  $K^{00}$  for the topology  $\sigma(L^1_{\mathbf{R}}(X, \mu)'', L^\infty_{\mathbf{R}}(X, \mu))$ . Since  $M(1) = 1$  it follows that there exists  $f_0 \in K$  such that  $|\int_X f_0(x) d\mu(x) - 1| < \epsilon/8$ .  $f_0 \in K$  means that  $\|f_0\| \leq 1$  and  $\|\pi_1(\epsilon_{x_i} * h - h) f_0\|_1 \leq \alpha(M)$  for  $i = 1, \dots, n$ .

We decompose  $f_0$  into its positive and negative part:  $f_0 = f_0^+ - f_0^-$ . From  $\|f_0^+\|_1 + \|f_0^-\|_1 \leq 1$ ,  $|1 - \|f_0^+\|_1 + \|f_0^-\|_1| < \epsilon/8$  it follows that  $\|f_0 - f_0^+\|_1 = \|f_0^-\|_1 \leq \epsilon/16$  and  $\|f_0^+\|_1 \geq 1 - \epsilon/8$ . Put  $f = f_0^+ / \|f_0^+\|$ . Then  $\|\pi_1(\epsilon_{x_i} * h - h)f\|_1 \leq (\alpha(M) + \epsilon/8) (1 - \epsilon/8)^{-1} \leq \alpha(M) + \epsilon/2$ . (We assume  $\epsilon < 1$ .)

Finally if  $x \in U$  then  $\|\pi_1(\epsilon_{x,x} * h - h)f\|_1 \leq \epsilon/2 + \|\pi_1(\epsilon_{x_i} * h - h)f\|_1 \leq \alpha(M) + \epsilon$ . The following lemma, as well as its proof, is due to H. Rindler:

**LEMMA 4.**  $\rho(\pi_1) = 2$  if and only if  $\rho(\pi_2) = 2^{1/2}$   $\rho^*(\pi_1) = 2$  if and only if  $\rho^*(\pi_2) = 2^{1/2}$ .

*Proof.* Assume that  $x \in G$ ,  $f \in L^1(X, \mu)$ ,  $f \geq 0$ ,  $\|f\|_1 = 1$ ,  $\|\pi_1(x)f - f\|_1 \geq 2 - \epsilon$ . Put  $g = f^{1/2}$ .

Then

$$\begin{aligned} 2 - \epsilon &\leq \int_X |(\pi_2(x)g)^2 - g^2| d\mu \\ &= \int_X |\pi_2(x)g - g| |\pi_2(x)g + g| d\mu \\ &\leq \|\pi_2(x)g - g\|_2 \|\pi_2(x)g + g\|_2 \\ &= (2\langle g, g \rangle - 2\langle \pi_2(x)g, g \rangle)^{1/2} (2\langle g, g \rangle \\ &\quad + 8\langle \pi_2(x)g, g \rangle)^{1/2} \\ &= 2(1 - \langle \pi_2(x)g, g \rangle)^{1/2}. \end{aligned}$$

It follows that  $\langle \pi_2(x)g, g \rangle \leq (1 - (1 - \epsilon/2)^2)^{1/2} \leq \epsilon^{1/2}$  and

$$\|\pi_2(x)g - g\|_2^2 = 2 - 2\langle \pi_2(x)g, g \rangle \geq 2 \cdot (1 - \epsilon^{1/2}).$$

Conversely if  $\|\pi_2(x)g - g\|_2^2 \geq 2 - \epsilon$  it follows from the equation  $|a - b|^2 \leq |a^2 - b^2|$  ( $a, b \geq 0$ ) that  $\|\pi_1(x)f - f\|_1 \geq 2 - \epsilon$ .

**THEOREM 1.** *The following statements are equivalent*

- (i)  $\pi$  does not contain weakly  $i_G$
- (ii)  $\rho(\pi_p) = 2^{1/p}$  for  $1 \leq p < \infty$
- (iii)  $\rho^*(\pi_p) = 2^{1/p}$  for  $1 \leq p < \infty$
- (iv)  $d(\pi) = 1$
- (v)  $d^*(\pi) = 1$

*Proof.* If  $\pi$  contains  $i_G$  weakly then all the quantities in (ii)–(v) are zero and so (ii)–(v) implies (i).

If  $\rho^*(\pi_2) < 2^{1/2}$  then by Lemma 4  $\rho^*(\pi_1) < 2$ . The same argument as in [1] Prop. 1 shows that there exists a mean  $M$  on  $L^\infty(X, \mu)$  with  $\alpha(M) < 2$ . By Lemma 3  $\rho(\pi_1) < 2$  and by Lemma 4  $\rho(\pi_2) < 2^{1/2}$ . If  $\|\pi_2(x)s - s\|_2^2 < \alpha < 2$  then  $\pi_2(x)s, s \gg 1 - \alpha/2$ . It follows that  $d(\pi) < 1$  and by [1] Cor. 14  $\pi$  contains  $i_G$  weakly. This shows that (i) implies (ii), (iii) for  $p = 1, 2$  and (iv).

If  $d^*(\pi) = 1$  a simple computation shows that  $\rho^*(\pi_2) = 2^{1/2}$ . It remains to show that  $\rho(\pi_2)^2 = 2$  implies  $\rho(\pi_p) = 2^{1/p}$  for  $1 \leq p < \infty$  (analogously for  $\rho^*(\pi_p)$ ). By the inequality  $|a - b|^p \leq a^p + b^p$  for  $a, b \geq 0, p \geq 1$  it follows that  $\rho(\pi_p) \leq 2^{1/p}$ . Assume that  $\|\pi_2(x)s - s\|_2^2 \geq 2 - 2\epsilon^2$  i.e.  $\langle \pi_2(x)s, s \rangle \leq \epsilon^2$  and that  $s \geq 0, \|s\|_2 = 1$ . Put  $t = \pi_2(x)s, A_1 = \{x \in X : s(x) \leq \epsilon t(x)\}, A_2 = \{x \in X : t(x) \leq \epsilon s(x)\}$  (we assume  $\epsilon < 1$ ). Then it follows that  $\epsilon \int_{X/A_2} s(x)^2 d\mu(x) \leq \int_{X/A_2} s(x)t(x) d\mu(x) \leq \epsilon^2$ .

Consequently  $\int_{X \setminus A_2} s(x)^2 d\mu(x) \leq \epsilon$  and similarly  $\int_{X \setminus A_1} t(x)^2 d\mu(x) \leq \epsilon$ . For  $1 \leq p < \infty$  we get:

$$\begin{aligned} & \left( \int_{A_1} |s(x)^{2/p} - t(x)^{2/p}|^p d\mu(x) \right)^{1/p} \\ & \geq \left( \int_{A_1} t(x)^2 d\mu(x) \right)^{1/p} - \left( \int_{A_1} s(x)^2 d\mu(x) \right)^{1/p} \geq (1 - \epsilon)^{1/p} - \epsilon^{1/p}. \end{aligned}$$

The same estimate holds for  $(\int_{A_2} |s(x)^{1/p} - t(x)^{2/p}|^p d\mu(x))/p$ . This gives combined  $\int_X |s(x)^{2/p} - t(x)^{2/p}| d\mu(x) \geq 2 - \delta(\epsilon)$  where  $\delta(\epsilon) \rightarrow 0$  for  $\epsilon \rightarrow 0$ . If one puts  $s_1 = s^{2/p}$  then  $\|\pi_p(x)s_1 - s_1\|_p^p \geq 2 - \delta(\epsilon)$  and it follows that  $\rho(\pi_p) \geq 2^{1/p}$ .

For  $f \in L^1(G), x \in G$  put  $A_x f = A_G(x)f_x$ . If  $H$  is a closed normal subgroup we write  $T_H$  for the canonical map  $L^1(G) \rightarrow L^1(G/H)$  and  $\mathcal{A}$  for the convex hull of  $\{A_x : x \in H\}$ . It is well known (see [8] p. 174) that if  $H$  has the property  $P_1$  then

$$\inf_{A \in \mathcal{A}} \|Af\|_1 = \|T_H f\|_1 \text{ for all } f \in L^1(G).$$

The next proposition shows that a converse similar to Th. 1 holds for this characterization:

**PROPOSITION 1.** *If  $H$  does not have the property  $P_1$ , then for any  $\lambda < 1$  there exists  $f \in L^1(G)$  such that  $\|f\|_1 = 1, T_H f = 0$  and  $\inf_{A \in \mathcal{A}} \|Af\|_1 > \lambda$ .*

*Proof.* If  $\lambda < 1$  and for any  $f \in L^1(G)$  with  $T_H f = 0$  there exists  $A \in \mathcal{A}$  such that  $\|Af\| \leq \lambda \|f\|$ , one can iterate this procedure to get  $A \in \mathcal{A}$  such that  $\|Af\| \leq \lambda^n \|f\|$ . An analogous argument as in [8] Ch. 8, 4.5 p. 176 shows that for  $f_1, \dots, f_m \in L^1(G)$  with  $T_H f_i = 0$  and  $\epsilon > 0$  there exists  $A \in \mathcal{A}$  such that  $\|Af_i\|_1 < \epsilon$  for  $i = 1, \dots, m$ . Let  $\{x_1, \dots, x_n\}$  be a finite subset of  $H$ ,  $\epsilon > 0$ . If  $f \in L^1(G)$  with  $f \geq 0$ ,  $\|f\|_1 = 1$  is arbitrary put  $f_i = \epsilon_{x_i} * f - f$ . Then  $T_H f_i = 0$  and consequently there exists  $A \in \mathcal{A}$  such that  $\|Af_i\|_1 < \epsilon$ . Then  $g = Af$  satisfies  $g \geq 0$ ,  $\|g\|_1 = 1$  and  $\|\epsilon_{x_i} * g - g\|_1 < \epsilon$  for  $i = 1, \dots, n$ . This means that  $(G, H)$  has the property  $P_*(G, H)$  (see [9] §3 p. 12) and by Prop. 1 p. 13 of [9],  $H$  has the property  $P_1$ .

2. In the last section we consider some function algebras on a locally compact group, whose factorization properties depend on the property  $P_1$ . The proof of this result was obtained independently by A. Derighetti.

Let  $A_p(G)$  be the algebra of all functions  $\sum \check{f}_n * g_n$ , where  $f_n \in L^p(G)$ ,  $g_n \in L^q(G)$ ,  $1/p + 1/q = 1$ ,  $\check{g}_n(x) = g_n(x^{-1})$  (see [6]). – for  $p = 2$  one gets the ordinary Fourier algebra of  $G$ . Let  $B_p(G)$  be the algebra generated by those positive definite functions which are subordinate to the left regular representation of  $G$  on  $L^2(G)$ . [2] An algebra  $A$  is said to *factorize weakly*, if it coincides with the linear span of  $A \cdot A$ .

**PROPOSITION 2.** *If one of the algebras  $A_p(G)$  ( $1 \leq p < \infty$ ) or  $B_p(G)$  factorizes weakly, then  $G$  has the property  $P_1$ .*

*Remark.* If  $G$  has the property  $P_1$  then  $B_p(G)$  has a unit element and each  $A_p(G)$  has a bounded approximate identity (see e.g. [6] p. 121).

*Proof.* If a function  $f$  belongs to one of these algebras, then its complex conjugate does also and has the same norm. By Th. 2.3 of [3] there exists a constant  $K_1 > 0$  such that for each compact subset  $C$  of  $G$  there exists  $f \in A$  such that  $f \geq 1$  on  $C$ ,  $f \geq 0$  on  $G$  and  $\|f\| \leq K_1$  ( $A$  denotes of the algebras cited above.) In the case of  $A_p(G)$  one can now use the same argument as given in [6] Th. 6 to see that  $G$  has property  $P_1$ . An analogous argument holds for  $B_p(G)$  since this is the dual of the  $C^*$ -algebra on  $L^2(G)$  generated by left convolution operators from  $L^1(G)$  ([2] p. 192).

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