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The normality of closures of orbits in a Lie algebra

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Abstract. Let X be the closure of a G -orbit in the Lie algebra of a connected reductive group G . It seems that the variety X is always normal. After a reduction to nilpotent orbits, this is proved for some special cases. Results on determinantal schemes are used for GL_n . If X is small enough we use a resolution and Bott's theorem on the cohomology of homogeneous vector bundles. Our results are conclusive for groups of type A_1 , A_2 , A_3 and B_2 .

0. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field k of characteristic zero. G has an adjoint action on its Lie algebra \mathfrak{g} . Let $a \in \mathfrak{g}$ and let X be the closure of the G -orbit of a . If a is semi simple the orbit is closed so that X is a smooth variety. If a is regular X is normal cf. [17] Theorem 16.

PROBLEM. *Is the variety X always normal?*

This problem was brought to our attention by Walter Borho in the fall of 1975. A positive solution would have applications in the theory of the infinite dimensional representations of \mathfrak{g} , see [2] (2.6) and [3]. After a reduction we give two more cases where we have an (affirmative) answer. The method used in the second case is the more interesting one. It involves a resolution and some cohomology.

1. Reductions

We have the additive Jordan decomposition $a = a_s + a_n$. Let G' and \mathfrak{g}' be the centralizers of a_s in G and \mathfrak{g} respectively. Now $a_n \in \mathfrak{g}'$ and \mathfrak{g}' is the Lie algebra of G' cf. [1] (9.1). Let X' be the closure of the G' -orbit of a_n in \mathfrak{g}' .

PROPOSITION. *The morphism $f: G \times X' \rightarrow X$ given by $f(g, x) = \text{Ad}(g)(a_s + x)$, is a smooth surjective morphism.*

The proof is standard and may be omitted. The only assumption needed here is that G is a linear algebraic group.

By [9] (IV 17.5.7) normality of X is now equivalent to normality of X' . The group G' is connected and reductive, cf. [19] (3.11) and (3.7). So we may replace G, a, X by G', a_n, X' , i.e. we may assume that a is a nilpotent element of \mathfrak{g} .

It is easy to see that we may replace G by a reductive or semi simple group of the same type. As a product of normal varieties over k is normal we may assume that G has an irreducible root system.

2. Case I. Assume $G = \mathrm{Gl}(V)$ where V is a vector space of dimension n . Now $\mathfrak{g} = \mathrm{End}(V)$ and a is a nilpotent endomorphism of V . Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be the partition of the blocks of the Jordan normal form of a . So $\lambda_1 \geq \dots \geq \lambda_r \geq 1$, there are $e_1, \dots, e_r \in V$ such that the elements $a^m e_i$ with $0 \leq m < \lambda_i$ form a basis of V and that $a^m e_i = 0$ if $m \geq \lambda_i$. Clearly $n = \lambda_1 + \dots + \lambda_r$.

PROPOSITION. *If $\lambda_2 = 1$ then X is Cohen-Macaulay and normal.*

Proof. Put $q = \lambda_1$ so that $n = q + r - 1$. The dimension of X is $(q-1)(2n-q)$, cf. [10] (3.8). Let N be the variety of the nilpotent endomorphisms of V , let D be the variety of the endomorphisms of V of rank $< q$, and let X' be the schematic intersection of N and D . It follows from [10] (3.10) that $X = X'_{\mathrm{red}}$, i.e. that X is the reduced variety with the same points as X' . For $x \in \mathrm{End}(V)$ let

$$\det(x - T \cdot \mathrm{id}) = (-T)^n + \sum_{i=1}^n (-T)^{n-i} \sigma_i(x)$$

be its characteristic polynomial. The subvariety N of $\mathrm{End}(V)$ is defined by the ideal generated by $\sigma_1, \dots, \sigma_n$. As $\sigma_i|_D = 0$ for $i \geq q$, the subscheme X' of D is defined by the ideal generated by $\sigma_1, \dots, \sigma_{q-1}$. The variety D is Cohen-Macaulay of dimension $(q-1)(2n-q+1)$, cf. [7] Theorem 1 and [15] (4.13). So X' is Cohen-Macaulay by [9] (0_{IV} 16.5.6). Using the cross section of [10] (3.7) one verifies that the orbit of a is contained in the regular locus of X' , so that X' is non-singular in codimension one. By Serre's criterion [9] (IV 5.8.6) it follows that X' is normal and hence equal to X .

3. Some cohomology

The results in this section are due to Kempf [12], [13]. The language used is closer to [5] (1.5) and [11]. Let G be a connected reductive group and P a parabolic subgroup of G . Let E be a P -module, i.e. a finite dimensional vector

space with a given representation $P \rightarrow \text{Gl}(E)$. Consider the variety $Z = G \times^P E$ which is the quotient of $G \times E$ under the right P -action given by $(g, x)p = (gp, p^{-1}x)$. Let $\psi: Z \rightarrow G/P$ be given by $\psi(g, x)P = gP$, it is a locally trivial vector bundle. The locally free $\mathcal{O}_{G/P}$ -module $\mathcal{L}(E)$ is defined as the sheaf of sections of ψ . We write $H^n(E) = H^n(G/P, \mathcal{L}(E))$, these groups are G -modules.

LEMMA. *Let V be a G -module and E a completely reducible P -module. Let $\pi: V \rightarrow E$ be a surjective morphism of P -modules. Then $H^n(E) = 0$ for $n \geq 1$ and the canonical G -morphism $\pi': V \rightarrow H^0(E)$ is surjective.*

Proof. We may consider $H^0(E)$ as the G -module of the morphisms $f: G \rightarrow E$ satisfying $f(gp) = p^{-1}f(g)$. Now π' is given by $\pi'(v)(g) = \pi(g^{-1}v)$. Clearly $\pi = q \circ \pi'$ where $q: H^0(E) \rightarrow E$ is given by $q(f) = f(1)$. Write $E = \bigoplus_i E_i$ where each E_i is an irreducible P -module. As q is surjective we have $H^0(E_i) \neq 0$ for all i . Now Bott's theorem, cf. [16] (6.4), which holds in our algebraic situation by theorem 5 of [4] exp. II, implies that $H^n(E) = 0$ for all $n \geq 1$ and that the G -modules $H^0(E_i)$ are irreducible. The image of π' has a non-zero intersection with each $H^0(E_i)$, so π' is surjective.

Construction. Let V be a G -module and E a P -invariant subspace. Put $Z = G \times^P E$. Let $\tau: Z \rightarrow V$ be given by $\tau(g, x)P = gx$. The group G acts on Z and τ is G -equivariant. Identifying Z with the closed subvariety of $(G/P) \times V$ of the pairs (gP, x) with $g^{-1}x \in E$, one verifies that τ is a projective morphism. So the image of τ is the irreducible closed subvariety of V defined by the ideal $\ker(\tau^0)$ where $\tau^0: \Gamma(V, \mathcal{O}_V) \rightarrow \Gamma(Z, \mathcal{O}_Z)$ is the comorphism.

THEOREM. (Kempf [12]). *If E is a completely reducible P -module then $H^n(Z, \mathcal{O}_Z) = 0$ for $n \geq 1$, and τ^0 is surjective.*

Proof. The ring $\Gamma(V, \mathcal{O}_V)$ is the graded symmetrical algebra $\bigoplus_{m \geq 0} S_m(V^*)$ on the dual V^* of V . As $\psi_*(\mathcal{O}_Z) = \bigoplus_m \mathcal{L}(S_m(E^*))$, we have $H^n(Z, \mathcal{O}_Z) = \bigoplus_m H^n(S_m(E^*))$ for all $n \geq 0$ by [9] (III 1.3.3) and [8] chap. II (3.10). A P -module F is completely reducible if and only if the unipotent radical of P acts trivially on F . So the P -modules $S_m(E^*)$ are completely reducible. Now the assertions follow from the lemma applied on the projections from $S_m(V^*)$ to $S_m(E^*)$.

4. The resolution

Let G be connected and reductive with an irreducible root system. Let a be a non-zero nilpotent element of \mathfrak{g} . There is a uniquely determined parabolic subgroup P of G associated to a , see [18] (III, 4). The closure of the P -orbit of a

is a normal subalgebra, called u_2 , of the Lie algebra \mathfrak{p} of P . We form $Z = G \times^P u_2$ and $\tau: Z \rightarrow \mathfrak{g}$ as above.

PROPOSITION. *The morphism τ induces a G -equivariant, projective, birational and surjective morphism $\tau: Z \rightarrow X$. The variety X is normal if and only if the comorphism $\tau^0: \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(Z, \mathcal{O}_Z)$ is bijective.*

Proof. Consider $b = (1, a)P$ in Z . The centralizer of a in G is contained in P , cf. loc. cit., and hence equal to the centralizer of b . So τ induces a bijection between the orbits of b and a . Using [18] (I, 5.6) and [1] (6.7) one shows that this bijection is an isomorphism. The orbits of b and a are dense and open in Z and X respectively, so $\tau: Z \rightarrow X$ is birational. The other properties of τ follow immediately. Since the variety Z is regular and the morphism τ is proper and birational, the ring $\Gamma(Z, \mathcal{O}_Z)$ is the integral closure of $\Gamma(X, \mathcal{O}_X)$ in its field of fractions. This concludes the proof.

Consider the following cases.

Case II. The P -module u_2 is completely reducible.

Case III. The nilpotent element a is regular.

THEOREM. *In the cases II and III the variety X is normal and $H^n(Z, \mathcal{O}_Z) = 0$ for $n \geq 1$.*

Remark. So in these cases X has rational singularities cf. [14] p. 51.

Proof. Case II is immediate from the above proposition and the theorem in 3. For case III see [17] theorem 16 and [11] theorem A.

5. Applications

We follow [18] (III, 4). There are $h, b \in \mathfrak{g}$ with $[h, a] = 2a$, $[h, b] = -2b$, $[a, b] = h$. For $i \in \mathbb{Z}$ put $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. We have $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$, $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, $u_2 = \bigoplus_{i \geq 2} \mathfrak{g}(i)$. Let T be a maximal torus which leaves each $\mathfrak{g}(i)$ invariant. Let R be the root system of G with respect to T . For $\alpha \in R$ let d_α be given by $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(d_\alpha)$. Let S be a set of simple roots with $d_\alpha \geq 0$ for all $\alpha \in S$. Then $d_\alpha \in \{0, 1, 2\}$ for all $\alpha \in S$. The G -orbit of a is characterized by the numbers $d_\alpha, \alpha \in S$, attached to the corresponding nodes of the Dynkin diagram. Let $\sum_{\alpha \in S} n_\alpha \alpha$ be the highest root. As the unipotent radical of P has Lie algebra

$u_1 = \bigoplus_{i \geq 1} g(i)$, we obtain

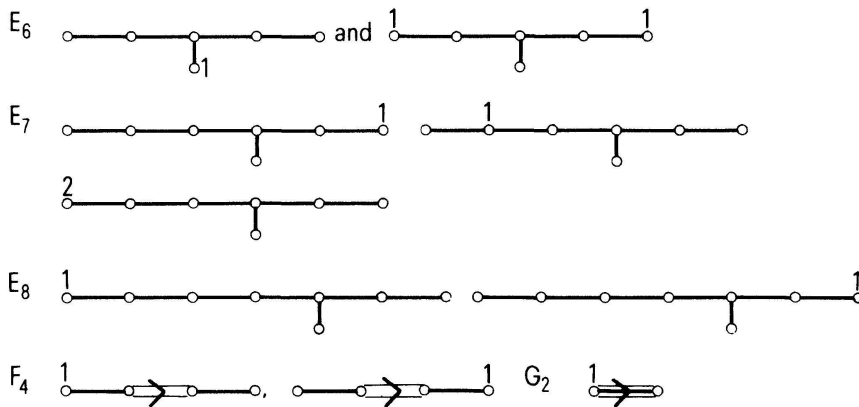
Criterion 1. *Case II applies if and only if $\sum_{\alpha \in S} n_{\alpha} d_{\alpha} \leq 2$.*

Let G be one of the classical groups GL_n , Sp_n , SO_n and let $\rho : G \hookrightarrow GL(V)$ be its usual representation in a vector space V of dimension n . Let λ be the partition of the nilpotent endomorphism $d\rho(a)$ of V , cf section 2. Using [18] (IV 1.13 and 2.32) we obtain

Criterion 2. *If G is GL_n or Sp_n then case II applies if and only if $\lambda_1 \leq 2$. If G is SO_n then case II applies if and only if $\lambda_1 + \lambda_2 \leq 4$.*

Remark 1. By inspection of the tables in [10] (4.9) it follows that X is normal if G is of type A_1 , A_2 , A_3 , B_2 and that X has rational singularities if G is of type A_1 , A_2 , B_2 .

Remark 2. For the exceptional groups inspection of the tables 16–20 in [6] yields that case II applies for nilpotent elements with the following weighted Dynkin diagrams (here the numbers $d_{\alpha} = 0$ are suppressed).



Remark 3. Let k be a field of positive characteristic p . The propositions in **1** and **2** still hold. For the reductions in **1**, the proposition in **4** and the normality of X in case III we need some restrictions on p , cf. [19], [18], [20]. Although the theorems fail, a case-by-case analysis shows that X is normal if $p \neq 2, 3$ and G is of type A_1 , A_2 , A_3 and B_2 .

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