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## The normality of closures of orbits in a Lie algebra

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*Abstract.* Let  $X$  be the closure of a  $G$ -orbit in the Lie algebra of a connected reductive group  $G$ . It seems that the variety  $X$  is always normal. After a reduction to nilpotent orbits, this is proved for some special cases. Results on determinantal schemes are used for  $GL_n$ . If  $X$  is small enough we use a resolution and Bott's theorem on the cohomology of homogeneous vector bundles. Our results are conclusive for groups of type  $A_1$ ,  $A_2$ ,  $A_3$  and  $B_2$ .

### 0. Introduction

Let  $G$  be a connected reductive algebraic group over an algebraically closed field  $k$  of characteristic zero.  $G$  has an adjoint action on its Lie algebra  $\mathfrak{g}$ . Let  $a \in \mathfrak{g}$  and let  $X$  be the closure of the  $G$ -orbit of  $a$ . If  $a$  is semi simple the orbit is closed so that  $X$  is a smooth variety. If  $a$  is regular  $X$  is normal cf. [17] Theorem 16.

**PROBLEM.** *Is the variety  $X$  always normal?*

This problem was brought to our attention by Walter Borho in the fall of 1975. A positive solution would have applications in the theory of the infinite dimensional representations of  $\mathfrak{g}$ , see [2] (2.6) and [3]. After a reduction we give two more cases where we have an (affirmative) answer. The method used in the second case is the more interesting one. It involves a resolution and some cohomology.

### 1. Reductions

We have the additive Jordan decomposition  $a = a_s + a_n$ . Let  $G'$  and  $\mathfrak{g}'$  be the centralizers of  $a_s$  in  $G$  and  $\mathfrak{g}$  respectively. Now  $a_n \in \mathfrak{g}'$  and  $\mathfrak{g}'$  is the Lie algebra of  $G'$  cf. [1] (9.1). Let  $X'$  be the closure of the  $G'$ -orbit of  $a_n$  in  $\mathfrak{g}'$ .

**PROPOSITION.** *The morphism  $f: G \times X' \rightarrow X$  given by  $f(g, x) = Ad(g)(a_s + x)$ , is a smooth surjective morphism.*

The proof is standard and may be omitted. The only assumption needed here is that  $G$  is a linear algebraic group.

By [9] (IV 17.5.7) normality of  $X$  is now equivalent to normality of  $X'$ . The group  $G'$  is connected and reductive, cf. [19] (3.11) and (3.7). So we may replace  $G, a, X$  by  $G', a_n, X'$ , i.e. we may assume that  $a$  is a nilpotent element of  $\mathfrak{g}$ .

It is easy to see that we may replace  $G$  by a reductive or semi simple group of the same type. As a product of normal varieties over  $k$  is normal we may assume that  $G$  has an irreducible root system.

2. *Case I.* Assume  $G = Gl(V)$  where  $V$  is a vector space of dimension  $n$ . Now  $\mathfrak{g} = \text{End}(V)$  and  $a$  is a nilpotent endomorphism of  $V$ . Let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be the partition of the blocks of the Jordan normal form of  $a$ . So  $\lambda_1 \geq \dots \geq \lambda_r \geq 1$ , there are  $e_1, \dots, e_r \in V$  such that the elements  $a^m e_i$  with  $0 \leq m < \lambda_i$  form a basis of  $V$  and that  $a^m e_i = 0$  if  $m \geq \lambda_i$ . Clearly  $n = \lambda_1 + \dots + \lambda_r$ .

**PROPOSITION.** *If  $\lambda_2 = 1$  then  $X$  is Cohen-Macaulay and normal.*

*Proof.* Put  $q = \lambda_1$  so that  $n = q + r - 1$ . The dimension of  $X$  is  $(q-1)(2n-q)$ , cf. [10] (3.8). Let  $N$  be the variety of the nilpotent endomorphisms of  $V$ , let  $D$  be the variety of the endomorphisms of  $V$  of rank  $< q$ , and let  $X'$  be the schematic intersection of  $N$  and  $D$ . It follows from [10] (3.10) that  $X = X'_{\text{red}}$ , i.e. that  $X$  is the reduced variety with the same points as  $X'$ . For  $x \in \text{End}(V)$  let

$$\det(x - T \cdot \text{id}) = (-T)^n + \sum_{i=1}^n (-T)^{n-i} \sigma_i(x)$$

be its characteristic polynomial. The subvariety  $N$  of  $\text{End}(V)$  is defined by the ideal generated by  $\sigma_1, \dots, \sigma_n$ . As  $\sigma_i|_D = 0$  for  $i \geq q$ , the subscheme  $X'$  of  $D$  is defined by the ideal generated by  $\sigma_1, \dots, \sigma_{q-1}$ . The variety  $D$  is Cohen-Macaulay of dimension  $(q-1)(2n-q+1)$ , cf. [7] Theorem 1 and [15] (4.13). So  $X'$  is Cohen-Macaulay by [9] (0<sub>IV</sub> 16.5.6). Using the cross section of [10] (3.7) one verifies that the orbit of  $a$  is contained in the regular locus of  $X'$ , so that  $X'$  is non-singular in codimension one. By Serre's criterion [9] (IV 5.8.6) it follows that  $X'$  is normal and hence equal to  $X$ .

### 3. Some cohomology

The results in this section are due to Kempf [12], [13]. The language used is closer to [5] (1.5) and [11]. Let  $G$  be a connected reductive group and  $P$  a parabolic subgroup of  $G$ . Let  $E$  be a  $P$ -module, i.e. a finite dimensional vector

space with a given representation  $P \rightarrow Gl(E)$ . Consider the variety  $Z = G \times^P E$  which is the quotient of  $G \times E$  under the right  $P$ -action given by  $(g, x)p = (gp, p^{-1}x)$ . Let  $\psi: Z \rightarrow G/P$  be given by  $\psi(g, x)P = gP$ , it is a locally trivial vector bundle. The locally free  $\mathcal{O}_{G/P}$ -module  $\mathcal{L}(E)$  is defined as the sheaf of sections of  $\psi$ . We write  $H^n(E) = H^n(G/P, \mathcal{L}(E))$ , these groups are  $G$ -modules.

**LEMMA.** *Let  $V$  be a  $G$ -module and  $E$  a completely reducible  $P$ -module. Let  $\pi: V \rightarrow E$  be a surjective morphism of  $P$ -modules. Then  $H^n(E) = 0$  for  $n \geq 1$  and the canonical  $G$ -morphism  $\pi': V \rightarrow H^0(E)$  is surjective.*

*Proof.* We may consider  $H^0(E)$  as the  $G$ -module of the morphisms  $f: G \rightarrow E$  satisfying  $f(gp) = p^{-1}f(g)$ . Now  $\pi'$  is given by  $\pi'(v)(g) = \pi(g^{-1}v)$ . Clearly  $\pi = q \circ \pi'$  where  $q: H^0(E) \rightarrow E$  is given by  $q(f) = f(1)$ . Write  $E = \bigoplus_i E_i$  where each  $E_i$  is an irreducible  $P$ -module. As  $q$  is surjective we have  $H^0(E_i) \neq 0$  for all  $i$ . Now Bott's theorem, cf. [16] (6.4), which holds in our algebraic situation by theorem 5 of [4] exp. II, implies that  $H^n(E) = 0$  for all  $n \geq 1$  and that the  $G$ -modules  $H^0(E_i)$  are irreducible. The image of  $\pi'$  has a non-zero intersection with each  $H^0(E_i)$ , so  $\pi'$  is surjective.

*Construction.* Let  $V$  be a  $G$ -module and  $E$  a  $P$ -invariant subspace. Put  $Z = G \times^P E$ . Let  $\tau: Z \rightarrow V$  be given by  $\tau(g, x)P = gx$ . The group  $G$  acts on  $Z$  and  $\tau$  is  $G$ -equivariant. Identifying  $Z$  with the closed subvariety of  $(G/P) \times V$  of the pairs  $(gP, x)$  with  $g^{-1}x \in E$ , one verifies that  $\tau$  is a projective morphism. So the image of  $\tau$  is the irreducible closed subvariety of  $V$  defined by the ideal  $\ker(\tau^0)$  where  $\tau^0: \Gamma(V, \mathcal{O}_V) \rightarrow \Gamma(Z, \mathcal{O}_Z)$  is the comorphism.

**THEOREM.** (Kempf [12]). *If  $E$  is a completely reducible  $P$ -module then  $H^n(Z, \mathcal{O}_Z) = 0$  for  $n \geq 1$ , and  $\tau^0$  is surjective.*

*Proof.* The ring  $\Gamma(V, \mathcal{O}_V)$  is the graded symmetrical algebra  $\bigoplus_{m \geq 0} S_m(V^*)$  on the dual  $V^*$  of  $V$ . As  $\psi_*(\mathcal{O}_Z) = \bigoplus_m \mathcal{L}(S_m(E^*))$ , we have  $H^n(Z, \mathcal{O}_Z) = \bigoplus_m H^n(S_m(E^*))$  for all  $n \geq 0$  by [9] (III 1.3.3) and [8] chap. II (3.10). A  $P$ -module  $F$  is completely reducible if and only if the unipotent radical of  $P$  acts trivially on  $F$ . So the  $P$ -modules  $S_m(E^*)$  are completely reducible. Now the assertions follow from the lemma applied on the projections from  $S_m(V^*)$  to  $S_m(E^*)$ .

#### 4. The resolution

Let  $G$  be connected and reductive with an irreducible root system. Let  $a$  be a non-zero nilpotent element of  $\mathfrak{g}$ . There is a uniquely determined parabolic subgroup  $P$  of  $G$  associated to  $a$ , see [18] (III, 4). The closure of the  $P$ -orbit of  $a$

is a normal subalgebra, called  $\mathfrak{u}_2$ , of the Lie algebra  $\mathfrak{p}$  of  $P$ . We form  $Z = G \times^P \mathfrak{u}_2$  and  $\tau: Z \rightarrow \mathfrak{g}$  as above.

**PROPOSITION.** *The morphism  $\tau$  induces a  $G$ -equivariant, projective, birational and surjective morphism  $\tau: Z \rightarrow X$ . The variety  $X$  is normal if and only if the comorphism  $\tau^0: \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(Z, \mathcal{O}_Z)$  is bijective.*

*Proof.* Consider  $b = (1, a)P$  in  $Z$ . The centralizer of  $a$  in  $G$  is contained in  $P$ , cf. loc. cit., and hence equal to the centralizer of  $b$ . So  $\tau$  induces a bijection between the orbits of  $b$  and  $a$ . Using [18] (I, 5.6) and [1] (6.7) one shows that this bijection is an isomorphism. The orbits of  $b$  and  $a$  are dense and open in  $Z$  and  $X$  respectively, so  $\tau: Z \rightarrow X$  is birational. The other properties of  $\tau$  follow immediately. Since the variety  $Z$  is regular and the morphism  $\tau$  is proper and birational, the ring  $\Gamma(Z, \mathcal{O}_Z)$  is the integral closure of  $\Gamma(X, \mathcal{O}_X)$  in its field of fractions. This concludes the proof.

Consider the following cases.

*Case II.* The  $P$ -module  $\mathfrak{u}_2$  is completely reducible.

*Case III.* The nilpotent element  $a$  is regular.

**THEOREM.** *In the cases II and III the variety  $X$  is normal and  $H^n(Z, \mathcal{O}_Z) = 0$  for  $n \geq 1$ .*

*Remark.* So in these cases  $X$  has rational singularities cf. [14] p. 51.

*Proof.* Case II is immediate from the above proposition and the theorem in 3. For case III see [17] theorem 16 and [11] theorem A.

## 5. Applications

We follow [18] (III, 4). There are  $h, b \in \mathfrak{g}$  with  $[h, a] = 2a$ ,  $[h, b] = -2b$ ,  $[a, b] = h$ . For  $i \in \mathbf{Z}$  put  $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$ . We have  $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ ,  $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$ ,  $\mathfrak{u}_2 = \bigoplus_{i \geq 2} \mathfrak{g}(i)$ . Let  $T$  be a maximal torus which leaves each  $\mathfrak{g}(i)$  invariant. Let  $R$  be the root system of  $G$  with respect to  $T$ . For  $\alpha \in R$  let  $d_\alpha$  be given by  $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(d_\alpha)$ . Let  $S$  be a set of simple roots with  $d_\alpha \geq 0$  for all  $\alpha \in S$ . Then  $d_\alpha \in \{0, 1, 2\}$  for all  $\alpha \in S$ . The  $G$ -orbit of  $a$  is characterized by the numbers  $d_\alpha, \alpha \in S$ , attached to the corresponding nodes of the Dynkin diagram. Let  $\sum_{\alpha \in S} n_\alpha \alpha$  be the highest root. As the unipotent radical of  $P$  has Lie algebra

$u_1 = \bigoplus_{i \geq 1} \mathfrak{g}(i)$ , we obtain

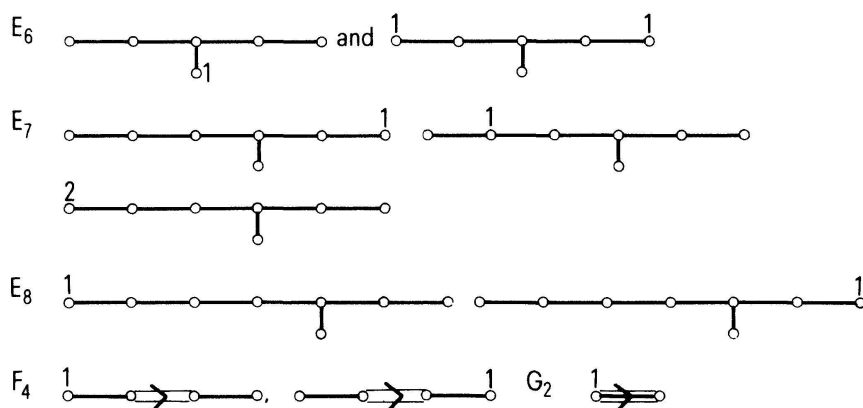
**Criterion 1.** *Case II applies if and only if  $\sum_{\alpha \in S} n_\alpha d_\alpha \leq 2$ .*

Let  $G$  be one of the classical groups  $GL_n, Sp_n, SO_n$  and let  $\rho : G \hookrightarrow GL(V)$  be its usual representation in a vector space  $V$  of dimension  $n$ . Let  $\lambda$  be the partition of the nilpotent endomorphism  $d\rho(a)$  of  $V$ , cf section 2. Using [18] (IV 1.13 and 2.32) we obtain

**Criterion 2.** *If  $G$  is  $GL_n$  or  $Sp_n$  then case II applies if and only if  $\lambda_1 \leq 2$ . If  $G$  is  $SO_n$  then case II applies if and only if  $\lambda_1 + \lambda_2 \leq 4$ .*

*Remark 1.* By inspection of the tables in [10] (4.9) it follows that  $X$  is normal if  $G$  is of type  $A_1, A_2, A_3, B_2$  and that  $X$  has rational singularities if  $G$  is of type  $A_1, A_2, B_2$ .

*Remark 2.* For the exceptional groups inspection of the tables 16–20 in [6] yields that case II applies for nilpotent elements with the following weighted Dynkin diagrams (here the numbers  $d_\alpha = 0$  are suppressed).



*Remark 3.* Let  $k$  be a field of positive characteristic  $p$ . The propositions in **1** and **2** still hold. For the reductions in **1**, the proposition in **4** and the normality of  $X$  in case III we need some restrictions on  $p$ , cf. [19], [18], [20]. Although the theorems fail, a case-by-case analysis shows that  $X$  is normal if  $p \neq 2, 3$  and  $G$  is of type  $A_1, A_2, A_3$  and  $B_2$ .

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