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Autor: Kaplan, Wilfred

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On the trajectories of a quadratic differential, *I*

WILFRED KAPLAN

Dedicated to Albert Pfluger on his Seventieth Birthday

1. Introduction

The nature of the family of trajectories of a quadratic differential on a Riemann surface has been the object of serious study for many years (see references at the end of this article). Although much has been learned about the possible structures of such families, one is far from a complete topological classification of all structures occurring. The presence of recurrent trajectories (nonclosed trajectories contained in their own omega limit sets) makes the classification exceptionally difficult.

It is our purpose here to develop some tools which are helpful in achieving the classification. In the second part of our paper [12] we shall illustrate the value of the tools in some special cases. In particular, we there consider the quadratic differentials $Q(z)z^{-2}\,dz^2$ and $P(z)z^{-2}\,dz$ on the sphere studied by Schaeffer and Spencer [15], give new proofs of some of their results, and provide a topological classification for the "generic" families arising. Further, we consider some cases of families with recurrent trajectories and present a method for generating such families on the sphere.

The principal tools presented here are first an extension of the index concept previously studied [11] to allow calculation of the total index for a region with singular points on the boundary and second a generalization of the Stoïlow theory of interior mappings [16] to cover the case of multiple-valued functions arising from integration of quadratic differentials.

Terminology

A *quadratic differential* on a Riemann surface $\mathfrak{R}$ is an expression $\phi(z)\,dz^2$, defined locally for each local complex coordinate z on $\mathfrak{R}$, where ϕ is meromorphic and $\phi(z)\,dz^2$ is invariant under change of coordinates. Here for the most part we take $\mathfrak{R}$ to be the sphere (equivalently, the extended complex plane) and a quadratic differential on $\mathcal{R}$ can be defined by giving $\phi(z)$ for a coordinate z defined except at one point of $\mathcal{R}$, which we usually take to be the point at ∞. By a *regular curve family* on $\mathcal{R}$, we mean a collection $\mathcal{F}$ of distinct curves on $\mathcal{R}$, whose union is an open set $D \subset \mathcal{R}$, such that each curve is one-to-one continuous image of a circle or of an open interval and such that locally the family has the topological structure of a family of parallel lines. By the last condition each point p of D has a neighborhood U such that there is a homeomorphism T of $\bar{U}$ onto the closure $\bar{S}$ of the open square $S: 0<x<1$, $0<y<1$ in the xy-plane, mapping the intersections of the curves of $\mathcal{F}$ with $\bar{U}$ onto the line segments $y = \text{const}$ in $\bar{S}$. We call U an r-neighborhood for $\mathcal{F}$. If E is an open subset of D, $\mathcal{F}_E$ denotes the curve family filling E formed of the intersections with E of the curves of $\mathcal{F}$. By a *cross-section* of $\mathcal{F}$ we mean an arc or simple closed curve γ contained in D such that for each point p of γ there is an r-neighborhood U of p such that each curve of $\mathcal{F}_U$ meets γ at most once. A point p of $\mathcal{R} - D$ is called a *singular point* of $\mathcal{F}$. By an *o-homeomorphism* we mean an orientation preserving homeomorphism of one oriented surface into another. Throughout the z-plane (xy-plane) will be given its usual orientation. For an arc ab, we denote by (ab) the arc ab minus its endpoints.

2. Extended index relation

We here consider a regular curve family $\mathcal{F}$ filling domain D of the plane as in [11]. As in Section 3 of that paper, the index $I(J)$ is defined for each admissible path J: that is, for each simple closed path J which is positively directed and is either a closed curve of $\mathcal{F}$, a closed cross-section of $\mathcal{F}$, or a path formed of a finite (even) number of arcs which are alternately arcs on curves of $\mathcal{F}$ and cross-section arcs. One has

$$I(J) = 2 + \tfrac{1}{2}(\rho - \lambda), \tag{1}$$

where ρ is the number of "right turns" on J and λ the number of "left turns."

We now extend the concept of admissible path J to allow for a finite number of isolated singular points on J.

DEFINITION 1. A positively directly simple closed path J in the z-plane is *semi-admissable* relative to the regular curve family $\mathcal{F}$ if J is formed of successive directed arcs $p_1p_2, p_2p_3, \ldots, p_{n-1}, p_np_{n+1}$ (with $p_{n+1} = p_1$), $n \geqq 2$, whereby for $i = 1, \ldots, n$ each p_i (called a *vertex* of J) is either in D or is an isolated boundary point of D, each open arc (p_ip_{i+1}) is in D and either (a) (p_ip_{i+1}) lies on a curve of $\mathcal{F}$ or (b) each closed subarc of (p_ip_{i+1}) is a cross-section of $\mathcal{F}$.

For a semi-admissible path J one can, by means of a suitable o-homeomorphism of the plane onto itself, introduce coordinates x, y such that for a particular i p_i is the origin $o:(0,0)$, a portion of J is given by a segment $-\varepsilon<x<\varepsilon$ of the x-axis, and the interior of J lies in the half-plane $Y>0$. With reference to these coordinates we now define a boundary index for the four cases which can occur at p_i.

DEFINITION 2. For each vertex p_i of the semi-admissible path J the boundary index $I_b(p_i)$ is defined as follows:

Case (*aa*): the arcs $p_{i-1}p_i$ and p_ip_{i+1} come under (a) of Definition 1. In terms of the xy-coordinates described above, we form a new curve family $\mathscr{F}'$ consisting of the intersections of the curves of $\mathscr{F}$ with the open half-plane $y>0$, the lines $y=\text{const}=c$ for $c<0$ and the segments $-\varepsilon<x<0$, $0<x<\varepsilon$ of the x-axis. $\mathscr{F}'$ is regular in a deleted neighborhood of o and we define $I_b(p_i)$ to be minus the index $I(o)$ relative to $\mathscr{F}'$.

Case (*bb*): the arcs $p_{i-1}p_i$ and p_ip_{i+1} come under (b) of Definition 1. We define $I_b(p_i)$ as for Case (aa) except that in the region $y\leqq 0$ minus the origin o each ray $x=c$, $-\varepsilon<c<\varepsilon$, lies on a curve of $\mathscr{F}'$.

Case (*ab*): the arc $p_{i-1}p_i$ comes under (a), p_ip_{i+1} comes under (b) of Definition 1. We define $I_b(p_i)$ to be $\frac{1}{2}-I(o)$, where $I(o)$ is defined as in Case (aa) except that in the region $y\leqq 0$ each semi-parabola $x/c=1-(y^2/c^2)$, $c=\text{const}$, $0<c<\varepsilon$, lies on a curve of $\mathscr{F}'$, as does the segment $y=0$, $-\varepsilon<x<0$.

Case (*ba*): the arc $p_{i-1}p_i$ comes under (b), p_ip_{i+1} comes under (a) of Definition 1. We define $I_b(p_i)$ to be $\frac{1}{2}-I(o)$, where $I(o)$ is defined as in Case (aa) except that in the region $y\leqq 0$ each semi-parabola $x/c=(y^2/c^2)-1$, $0<c<\varepsilon$, lies on a curve of $\mathscr{F}'$, as does the segment $y=0$, $0<x<\varepsilon$.

From Definition 2 one verifies that if p_i is in D, then in Cases (aa) and (bb), $I_b(p_i)=0$, and in Cases (ab) and (ba), $I_b(p_i)=\pm\frac{1}{2}$, with $+$ for a right turn at p_i, $-$ for a left turn. Furthermore, one verifies that $I_b(p_i)$ can be computed in every case by choosing an arc $\beta=p_i's_1\cdots s_kp_i''$ in $\bar{D}$ and lying inside J except for its endpoints p_i', p_i'', which lie on the arcs $(p_{i-1}p_i)$ and (p_ip_{i+1}) of J respectively, whereby the arcs $(p_{i-1}p_i')$, $p_i's_1$, $s_1s_2,\ldots,s_kp_i''$, $(p_i''p_{i+1})$ are alternately arcs on curves of $\mathscr{F}$ and cross-sections, and the simple closed curve formed of β plus the arcs $p_i''p_i$, p_ip_i' on J encloses no singular point. Then

$$I_b(p_i)=\tfrac{1}{2}(\rho_\beta-\lambda_\beta), \tag{2}$$

where ρ_β, λ_β are the total numbers of right and left turns on β at the successive points p_i', $s_1, \ldots, s_k$, p_i''. We call β a *bypassing arc*. The value $I_b(p_i)$, as given by (2), does not depend on the particular choice of β; this follows from the theorem [11, p. 304] that $I(J)=0$ for an admissible path J enclosing no singular points. The validity of (2) also shows that $I_b(p_i)$ as given by Definition 2 does not depend on the particular constructions used in this definition.

DEFINITION 3. For a semi-admissible path J relative to $\mathcal{F}$, the index $I(J)$ is defined in terms of the boundary indices of its vertices by the formula:

$$I(J) = 2 + \sum_{i=1}^{n} I_b(p_i). \tag{3}$$

If J happens to be admissible, all p_i are in D and only cases (ab) and (ba) arise as noted above, with $I_b(p_i) = \pm\frac{1}{2}$, so that (3) reduces to (1).

For an arbitrary semi-admissible path J, one can choose sufficiently short bypassing arcs β_i as above at all p_i so that, by replacing $p_i'p_ip_i''$ by β_i (directed from p_i' to p_i'') for all i, one obtains from J an admissible path J_0 enclosing the same singular points as J. But

$$I(J_0) = 2 + \frac{1}{2}\sum_{i=1}^{n} (\rho_{\beta_i} - \lambda_{\beta_i}) = 2 + \sum_{i=1}^{n} I_b(p_i) = I(J)$$

by (1), (2) and (3). With the aid of Theorem 1 of [11], we thus conclude:

THEOREM 1. *Let $\mathcal{F}$ be a regular curve family filling a domain D of the plane and let J be a semi-admissible path relative to $\mathcal{F}$. Then Definition 2 assigns to each vertex of J a unique boundary index and hence Definition 3 assigns to J a unique index, reducing to the previously defined index if J is admissible. If J encloses only a finite set $(z_1, \ldots, z_N)$ of singular points of $\mathcal{F}$, then $I(J) = \sum_{j=1}^{N} I(z_j)$.*

Case of trajectories of a quadratic differential. Here it is known that a pole of order n has index n, a zero of order m has index $-m$. To evaluate the boundary index at such a point one uses the known local structure of the family of trajectories [4, pp. 27–35]. For a zero p_i on J, there are h trajectories leading to p_i from D. For a pole of order $n \geqq 3$ there are $n-2$ "elliptic sectors" in all at the point [2, p. 163], of which k lie wholly or partly in D. In terms of these integers h, k, the values of the boundary index are given in Table 1. The pole of order 1 does not arise in Case (aa) for a semi-admissible path J but does arise for a quasi-admissible path as explained below.

Table 1.
Values of index $I_b(p_i)$

Nature of p_i	Case (aa)	(bb)	(ab)	(ba)
zero	h	$h-1$	$h-\frac{1}{2}$	$h-\frac{1}{2}$
pole, order $\geqq 3$	$-k-1$	$-k$	$-k-\frac{3}{2}$	$-k-\frac{3}{2}$
pole, order 2	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
pole, order 1	(0)	-1	$-\frac{1}{2}$	$-\frac{1}{2}$

Case of nonsimple closed path. It is convenient to extend the definition of $I(J)$ further to allow for a nonsimple closed path which can be considered as the boundary of a simply connected domain D_1.

DEFINITION 4. A directed closed path J in the z-plane is *quasi-admissible* relative to the family $\mathscr{F}$ if J satisfies all the conditions of Definition 1 except for being a simple path, two arcs p_ip_{i+1} and p_jp_{j+1} $(i\neq j)$ either meet in no point, one common vertex or two common vertices or else coincide with reversal of direction, J forms the boundary of a simply connected domain D_1 and there is a continuous mapping Φ of the closed disc $\bar{D}_0:|w|\leqq 1$ in the complex w-plane into the z-plane with the following properties: Φ maps the open disc $D_0:|w|<1$ o-homeomorphically onto D_1; Φ maps successive arcs $P_1P_2, P_2P_3, \ldots, P_nP_{n+1}$ $(P_{n+1}=P_1)$ of the positively directed circle $|w|=1$ homeomorphically onto the correspondingly directed arcs $p_1p_2, p_2p_3, \ldots, p_np_{n+1}$ of J; Φ is a homeomorphism on each set $D_0\cup(P_iP_{i+1})$; Φ maps at most two points into each point of J other than a vertex.

From the definition it follows that $D_1\cup J$ is obtained from $\bar{D}_0$ by certain identifications of points among the $\{P_i\}$ and of certain pairs of arcs among the $\{P_iP_{i+1}\}$, with reversal of direction. One can also consider $\bar{D}_0$ as a representation of D_1 and its prime ends, and the mapping Φ can always be chosen to be a conformal mapping of D_0 onto D_1 (see the discussion of conformal mapping for domains with normal boundary points in [1, pp. 357–372]).

An example is given in Fig. 1. Here P_2P_3 is identified with P_6P_5, P_7 with P_9, $P_{10}P_{11}$ with $P_{12}P_{11}$.

The definition of the boundary index $I_b(p_i)$ can be extended to the vertices p_i of such a quasi-admissible path. It can also be found by bypassing arcs as above, so that (2) holds. In fact, the part of $\mathscr{F}$ in $\bar{D}_1-\bigcup_{i=1}^n\{p_i\}$ can be considered as the image of a family in $\bar{D}_0-\bigcup_{i=1}^n\{P_i\}$ under the mapping Φ. Definition 1 applies to this family and we can define $I_b(p_i)$ to equal $I_b(P_i)$ relative to the corresponding family. The result is equivalent to that obtained by the other methods, since Φ

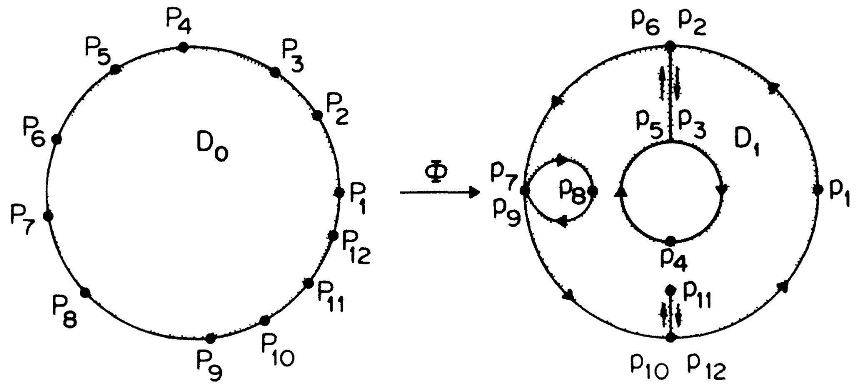

Figure 1. Quasi-admissible path J.

maps a bypassing arc for P_i onto a bypassing arc for p_i. As for Theorem 1, one proves:

THEOREM 2. *Let $\mathcal{F}$ be a regular curve family filling a domain D of the z-plane and let J be a quasi-admissible path relative to $\mathcal{F}$ as in Definition* 4. *Then the above definitions assign a unique boundary index $I_b(p_i)$ to each vertex of J and hence* (3) *assigns an index to J. This index reduces to the previously defined index if J is semi-admissible or admissible. If the domain D_1 of Definition* 4 *contains only the finite set $\{z_1, \ldots, z_N\}$ of singular points of $\mathcal{F}$, then $I(J) = \sum_{j=1}^{N} I(z_j)$.*

For the case of the trajectories of a quadratic differential, Table 1 can again be used for a quasi-admissible path, with h and k defined relative to one "side" of J at each p_i on a part of J traced twice. One new case now appears, that of a first order pole in case (aa), with boundary index 0.

In Figure 2 we give two examples. In the first, J is semi-admissible and the boundary indices are as shown, so that $I(J) = -2$. In the second, J is quasi-admissible, the boundary indices are as shown and $I(J) = -3$. In both cases we verify that $I(J) = \sum I(z_j)$ as in Theorems 1 and 2.

Remark. The concept of index of a path can be extended to paths J passing through the point at ∞. For example, if J is quasi-admissible, Definition 4 continues to have meaning and specifies a definite region D_1 bounded by J. A suitable o-homeomorphism of the extended plane onto itself takes D_1 to a bounded region, to which the previous analysis applies and yields a value for $I(J)$, independent of the particular o-homeomorphism chosen; Theorem 2 remains valid. There is a similar discussion for semi-admissible and admissible paths; in each case $I(J)$ is defined relative to one of the two domains into which J separates the extended plane.

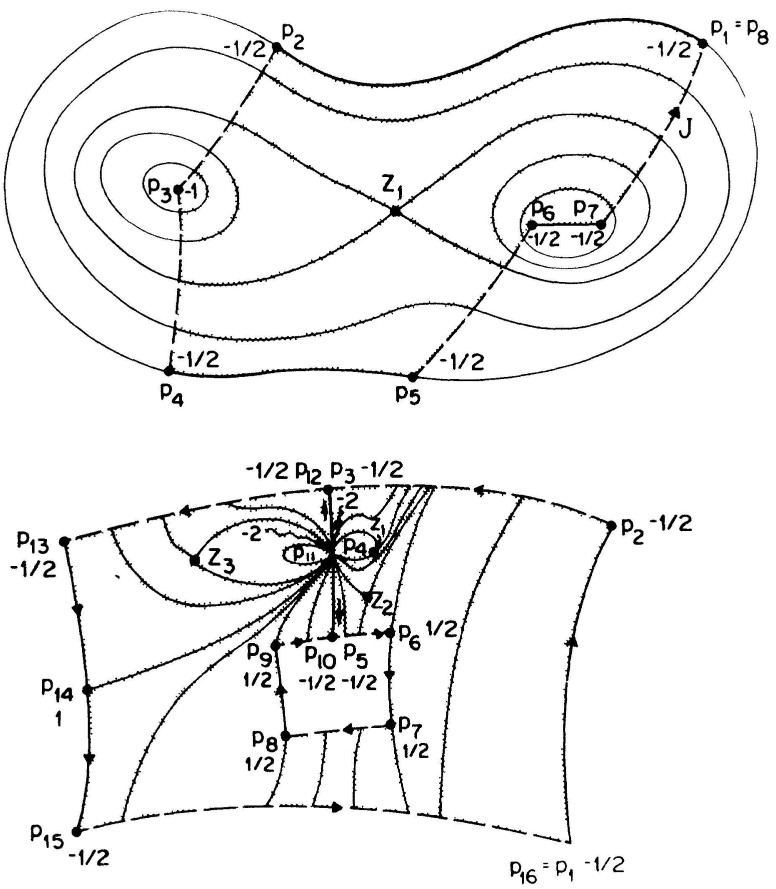

Figure 2. Application of index theorems.

3. A theorem on omega limit sets

For orientable curve families a curve C of the family having an omega limit point p^* which is regular must return repeatedly to the neighborhood of p^* and hence repeatedly cross an r-neighborhood U successively in the same direction. For the more general families considered here, this conclusion fails. However, for a recurrent curve C, we showed that C must have two successive crossings of an r-neighborhood in the same direction [11, Theorem 3]. We here extend this conclusion to an arbitrary curve having a regular omega limit point, but with a further restriction on $\mathcal{F}$.

THEOREM 3. *Let $\mathcal{F}$ be a regular curve family on the sphere having at most a finite number of singular points. Let C be a directed curve of $\mathcal{F}$ which is not closed and has a regular point p^* in its omega limit set. Then there exists an r-neighborhood U, relative to $\mathcal{F}$, which is crossed twice successively by C in the same direction.*

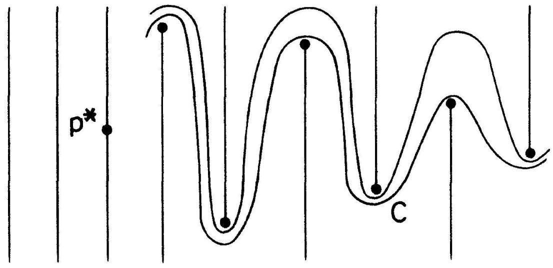

Figure 3. Example with infinitely many singular points.

Proof. By the above remarks, we can assume that C is not recurrent, so that $p^* \notin C$. As in [11, p. 306], we introduce local coordinates at p^* in terms of which the lines $y = \text{const}$ are on curves of $\mathcal{F}$ and label successive crossings of the neighborhood U^* of p^* by C by the corresponding y-coordinates. Let us suppose that the assertion fails and that, for every r-neighborhood, successive crossings by C are in opposite directions. Let $y_1, y_2, \ldots$ be successive crossings of U^* by C and let p^* have y-coordinate y^*. If $y_1 < y^* < y_2$, then there must be a first crossing in the interval $y_1 < y < y_2$ after y_2. This must be in the same direction as the crossing at y_1 or in the same direction as the crossing at y_2. Hence by restricting to a smaller r-neighborhood, excluding y_1 or y_2, we would have two successive crossings in the same direction, contrary to hypothesis. Therefore $y_1 < y^* < y_2$ is excluded, as is $y_2 < y^* < y_1$. Thus all crossings must be below y^* or else all must be above y^*. We consider the former case, the other case being similar, and suppose U^* restricted so that $y_1 < y_2 < y^*$. If now $y_3 < y_1 < y_2$, then an r-neighborhood excluding y_2 but including y_1 and y_3 is crossed successively by C at y_1 and y_3, contrary to hypothesis. There is similar difficulty if $y_1 < y_3 < y_2$, so that we must have $y_1 < y_2 < y_3$, and in general the sequence $\{y_k\}$ is strictly monotone. The arc $y_k y_{k+1}$ of C plus a cross-section arc $x = \text{const}$ in U^* then form an admissible path J_k which bounds an open region D_k on the sphere excluding p^* and intersecting U^* only between y_k and y_{k+1}. Hence the regions $D_1, D_2, \ldots$ are pairwise disjoint. Since $I(J_k) = 1$, it follows that each D_k contains a singular point z_k of $\mathcal{F}$; the points $z_1, z_2, \ldots$ provide infinitely many singular points of $\mathcal{F}$, contrary to hypothesis. Thus there must be at least two successive crossings in the same direction and the theorem is proved.

Remark. If one allows infinitely many singular points, the conclusion fails. In Figure 3 we suggest a family $\mathcal{F}$, curve C and regular point p^*, and there is no r-neighborhood which C crosses successively in the same direction.

4. Many-valued interior mappings and quadratic differentials

In [16, p. 121] Stoïlow showed that if a mapping f from an orientable 2-manifold M to the w-plane is "interior" (light and open in the terminology of

Whyburn [20]), then local complex coordinates can be introduced on M, in terms of which f becomes analytic.

This theorem can be extended in various ways. For example, if f is as above, if M is a covering surface of another surface M_0 and, for each covering transformation g, $f \circ g \circ f^{-1}$ is locally analytic in the w-plane, then complex coordinates can also be introduced on M_0, in terms of which f becomes a many-valued analytic function on M_0 whose Riemann surface is M. For if π is the projection of M on M_0, then the local mappings $f \circ \pi^{-1}$ provide such coordinates on M_0. The requirement of local analyticity of $f \circ g \circ f^{-1}$ forces f to be locally homeomorphic. One can add additional hypotheses to allow for points at which f is locally n to 1 (like $w = z^n$ at $z = 0$). Furthermore, one does not need a strict covering surface, but rather the sort of covering provided by Riemann surfaces.

Similar ideas are considered by Jenkins and Morse [7, pp. 123–128].

We are concerned here with a theorem, similar to Stoïlow's theorem, arising naturally in constructing a quadratic differential whose trajectories have a given topological structure. We shall phrase the theorem in terms of a continuation process, paralleling that for analytic functions. We confine attention to the extended plane, though the process can be extended to general orientable 2-manifolds.

In the following discussion, we consider interior mappings from domains in the extended z-plane into the finite w-plane ($z = x + iy$, $w = u + iv$). If f is such a mapping, with domain D, then by Stoïlow's theorem there is a homeomorphism h of D into the extended z-plane such that $f \circ h^{-1}$ is analytic. *We shall assume throughout that h is an o-homeomorphism.* This is equivalent to assuming that f preserves orientation locally, as does an analytic function.

DEFINITION 5. Let D_1 and D_2 be domains of the extended z-plane and let f_1 in D_1, f_2 in D_2 be interior mappings into the w-plane. We then say that f_1, f_2 are *direct interior continuations* of each other if $D = D_1 \cap D_2 \neq \varnothing$ and $f_1 | D = f_2 | D$.

Indirect interior continuation and interior continuation along a path are then defined as for analytic continuation. We also need the analogue of analytic continuation across an analytic curve (as in the Schwarz reflection principle).

DEFINITION 6. Let f_1 in D_1 and f_2 in D_2 be interior mappings and let a curve C (homeomorphic image of a circle or of an open interval) be contained in the common boundary of D_1 and D_2. Let D be a domain containing C and divided by C into two domains D_1', D_2', with $D_1' \subset D_1$ and $D_2' \subset D_2$. Let there exist an interior mapping f in D such that $f | D_1' = f_1 | D_1'$ and $f | D_2' = f_2 | D_2'$. Then f_1, f_2 are said to be *interior continuations of each other across C.*

THEOREM 4. *Let f_1, f_2, D_1, D_2, C, D, D_1', D_2' be as in Definition 6. Let $f_1 \mid D_1'$ be extendible to $D_1' \cup C$ to yield an o-homeomorphism f_1^* of this set, mapping C onto an interval of the u-axis and D_1' into the half-plane $v>0$; let $f_2 \mid D_2'$ be extendible to $D_2' \cup C$ to yield an o-homeomorphism f_2^* of this set, mapping C onto an interval of the u-axis and D_2' into the half-plane $v<0$. Let $f_1^* \mid C = f_2^* \mid C$. Then f_1, f_2 are indirect interior continuations of each other across C.*

Proof. Let f be defined in D by the requirements: $f \mid D_1' \cup C = f_1^*$, $f \mid D_2' \cup C = f_2^*$. By the hypotheses on f_1^* and f_2^*, f is an o-homeomorphism, hence interior, and f satisfies the requirements of Definition 6.

DEFINITION 7. Let $\mathcal{I}$ be a family of nonconstant interior mappings from domains of the extended z-plane to the w-plane. We call $\mathcal{I}$ an allowed family if (1) for each f in $\mathcal{I}$, with domain D, the restriction of f to a subdomain of D is also in $\mathcal{I}$; (2) for each pair f, g in $\mathcal{I}$ having the same domain, either $f=g$ or the equation $f(z)=g(z)$ is satisfied on a set with no limit point in D; (3) each two mappings in $\mathcal{I}$ are indirect interior continuations of each other within $\mathcal{I}$ – that is, one mapping can be obtained from the other by a finite sequence of direct interior continuations using only members of $\mathcal{I}$.

From (1) and (2) one proves as usual that interior continuation of an element f of $\mathcal{I}$ along a path within $\mathcal{I}$, when possible, is unique; further, that the monodromy theorem holds. Thus $\mathcal{I}$ resembles a family of analytic continuations of an analytic function. In the next theorem we characterize the families $\mathcal{I}$ which are in fact equivalent to such an analytic family arising from the integration of the square root of a quadratic differential on the sphere.

THEOREM 5. *Let G be the extended z-plane minus a finite nonempty set $S=\{z_1, \ldots, z_n\}$. Let $\mathcal{I}$ be an allowed family of interior mappings from subdomains of G to the finite w-plane. Let each mapping in $\mathcal{I}$ be locally a homeomorphism. Let each element of $\mathcal{I}$ be interiorly continuable within $\mathcal{I}$ on each path in G. If $D \subset G$ and f, g are two elements of $\mathcal{I}$ with domain D, let it follow that either $f(z)=g(z)+\text{const}$ in D or $f(z)=-g(z)+\text{const}$ in D. At each point z_k in S let $\mathcal{I}$ have the characteristics of the integral of the square root of a quadratic differential $\phi(z)\,dz^2$ having a zero, a pole or a regular point at z_k. That is, let one of the following cases arise when an element f of $\mathcal{I}$ is continued interiorly within $\mathcal{I}$ on all paths in a deleted neighborhood U of z_k contained in G:*

(i) *a single-valued mapping is obtained in U, with finite or infinite limit at z_k (as at a regular point of $\phi(z)\,dz^2$ or at a point where the differential has a zero of* *even order or a pole of even order greater than* 2 *at which the square root of the differential has zero residue*);

(ii) *a double-valued mapping* $c + g(z)$ *is obtained, where* c *is constant and* $g(z)$ *has limit* 0 *or* ∞ *at* z_k *(as at a zero or pole of odd order of* $\phi(z)\,dz^2$*)*;

(iii) *for each circuit of* z_k *in the positive direction,* f *is replaced by* $f+c$*, where* c *is a nonzero constant (so that after* m *such circuits* f *is replaced by* $f+mc$*) and on each path leading to* z_k *in* U $\operatorname{Im}(f/c)$ *has limit* ∞ *(as at a pole of order* 2 *of* $\phi(z)\,dz^2$*)*;

(iv) *for each circuit of* z_k *in the positive direction* f *is replaced by* $f+c$*, where* c *is a nonzero constant and* (iii) *fails but there is a simple path* α *leading to* z_k *in* U*, such that* $U-\alpha$ *is simply connected and such that continuation of* f *in* $U-\alpha$ *yields a single-valued interior mapping* f_0 *with the properties: for some integer* N*,* f_0 *takes each complex value at most* N *times in* $U-\alpha$ *and along* α $|\operatorname{Im}(f_0/c)| \to \infty$ *as* $z \to z_k$ *(as at a pole of* $\phi(z)\,dz^2$ *of even order greater than* 2 *at which the square root of the differential has nonzero residue).*

Then complex coordinates ζ *can be introduced in the extended plane in terms of which the elements of* $\mathscr{I}$ *become branches of an analytic function* $\int[\phi(\zeta)]^{1/2}\,d\zeta +$ const, *where* $\phi(\zeta)\,d\zeta^2$ *is a rational quadratic differential whose zeros and poles are contained in* S.

Proof. Let M be the universal covering surface of G. By uniqueness of interior continuation in $\mathscr{I}$ along paths and the monodromy theorem for such continuations, the given many-valued function in G represented by $\mathscr{I}$ can be lifted to M to provide a single-valued interior mapping F of M into the finite w-plane, a local homeomorphism. Since the various branches of $\mathscr{I}$ are related by the analytic relations $w_1 = \pm w + \text{const}$, complex coordinates can be introduced as above on G and on M, so that F becomes analytic on M and each mapping in $\mathscr{I}$ is analytic and is of form $F \circ \pi^{-1}$, restricted to a domain D, where π is the projection mapping from M to G.

It remains to introduce complex coordinates in (full) neighborhoods of the points of S. We consider the four cases of the theorem in turn.

(i) Here from an element of I defined in a subdomain of U we obtain a single-valued analytic function f in the *doubly connected* region U. U is conformally equivalent to an annulus $U_\zeta : r_1 < |\zeta| < r_2$ in the ζ-plane, where $0 \leqq r_1 < r_2 \leqq \infty$ and $z \to z_k$ in U is equivalent to $|\zeta| \to r_1$. The function f is locally one-to-one and hence becomes a nonconstant function $f_1(\zeta)$ in U_ζ with constant limit as $\zeta \to r_1$. It follows from the Riesz brothers' theorem [13, p. 197], for example, that $r_1 > 0$ cannot occur, so that $r_1 = 0$. We can thus use ζ in the disc $|\zeta| < r_2$ to determine local coordinates at z_k, and these coordinates are consistent with the previous ones at points of U. The element f can be defined at z_k so as to yield a function analytic in a neighborhood of z_k, except possibly for a pole at z_k. If a different initial element of $\mathcal{I}$ is chosen, we obtain the same function in the neighborhood, except possibly for a change of sign and addition of a constant. Thus, in terms of the coordinate ζ, f'^2 is a single-valued function in the neighborhood, independent of the branch chosen, and f'^2 is analytic except possibly for a pole at z_k.

(ii) Here we again assign local coordinates in U, so that the continuations of our given element of $\mathcal{I}$ in U provide a 2-valued analytic function f in U; that is, a 1-valued analytic function on a 2-sheeted Riemann surface M_U covering U. Again U is conformally equivalent to an annulus $U_\zeta : r_1 < |\zeta| < r_2$, which implies at once that the Riemann surface M_U is also conformally equivalent to an annulus, of inner and outer radii $r_1^{1/2}$, $r_2^{1/2}$. The hypothesis that f has a limit as $z \to z_k$ (and proper choice of the coordinate ζ) again imply that $r_1 = 0$ and hence that ζ, in the disk $|\zeta| < r_2$, provides the desired local complex coordinate at z_k; f becomes a function on M_U with a regular point or a pole at $\zeta = 0$. Since the two branches $f_1 = c + g(z)$, $f_2 = c - g(z)$ of f have constant sum $2c$, f'^2 is again single-valued in U_ζ, analytic except possibly for a pole at z_k.

(iii) Here we proceed as in (ii) to obtain U_ζ and the Riemann surface M_U, but now M_U is the universal covering surface of U_ζ and hence is conformally equivalent to the infinite strip $\log r_1 < \operatorname{Re} t < \log r_2$ in a t-plane. Our element and its continuations provide a single-valued analytic function $g(t)$ in this strip such that $\operatorname{Im} g(t)/c \to \infty$ as $\operatorname{Re} t \to \log r_1$. It follows again from the Riesz brothers' theorem that the case $r_1 > 0$ cannot arise. Thus again we can use ζ as a local complex coordinate in a full neighborhood of z_k. Our element generates a many-valued analytic function $f(\zeta)$ in $U_\zeta : 0 < |\zeta| < r_2$ and since f increases by c in one circuit, f' is single-valued in the neighborhood. Thus f' has an isolated singularity at $\zeta = 0$, so that

$$f'(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n, \qquad f(\zeta) = a_{-1} \log \zeta + p(\zeta) + q(\zeta), \tag{4}$$

where

$$p(\zeta) = \sum_{n=-\infty}^{-1} a_{n-1} \zeta^n / n, \qquad q(\zeta) = \sum_{n=1}^{\infty} a_{n-1} \zeta^n / n + b_0. \tag{5}$$

Since f increases by c on one circuit, $a_{-1} = c/(2\pi i)$. Hence

$$\operatorname{Im}\left(\frac{f}{c}\right) = \frac{1}{2\pi} \log \frac{1}{|\zeta|} + \operatorname{Im}\left(\frac{p(\zeta)}{c}\right) + \operatorname{Im}\left(\frac{q(\zeta)}{c}\right) \tag{6}$$

and, since $\operatorname{Im}(f/c) \to \infty$ as $\zeta \to 0$,

$$\lim_{\zeta \to 0} \left(\frac{1}{2\pi} \log \frac{1}{|\zeta|} + \operatorname{Im} \frac{p(\zeta)}{c} \right) = \infty \tag{7}$$

If we let $G(z) = \exp[2\pi i p(1/z)/c]$, then $G(z)$ is an entire function and (7) implies that, with $M(r) = \max |G(z)|$ for $|z| = r$,

$$\lim_{r \to \infty} \log \frac{r}{M(r)} = \infty.$$

Hence $M(r)/r \to 0$, so that $G(z)$ and hence $p(\zeta)$ reduce to constants. Thus $f'(\zeta)$ has a first order pole at $\zeta = 0$ so that f'^2 has a second order pole.

(iv) Here, as in Case (iii), we again have the coordinate ζ in terms of which our element generates a many-valued analytic function f in the annulus $U_\zeta : r_1 < |\zeta| < r_2$ and $f'(\zeta)$ is again single-valued in U_ζ, so that (4), (5), (6) again hold, with $a_{-1} = c/(2\pi i)$. By (6), $\operatorname{Im}(f/c)$ is single-valued in U_ζ. In the coordinate ζ, α becomes a curve α_ζ joining the two boundaries $|\zeta| = r_j$ of U_ζ and $U_\zeta - \alpha_\zeta$ is a simply connected domain in which we have a single-valued branch $f_0(\zeta)$ of f, as described in the hypotheses. Since $\operatorname{Im}(f/c)$ is single-valued in U, so is $\operatorname{Im}(f_0/c)$ and by hypothesis $|\operatorname{Im}(f_0/c)| \to \infty$ as ζ approaches the circle $|\zeta| = r_1$ along α_ζ. Also

$$\operatorname{Re}\left(\frac{f_0}{c}\right) = \frac{\arg \zeta}{2\pi} + \operatorname{Re}\left(\frac{p(\zeta)}{c}\right) + \operatorname{Re}\left(\frac{q(\zeta)}{c}\right), \tag{8}$$

for an appropriate single-valued branch of $\arg \zeta$ in $U_\zeta - \alpha_\zeta$.

We now let

$$F_0(z) = \frac{1}{c} f_0\left(\frac{1}{z}\right), \quad P(z) = \frac{1}{c} p\left(\frac{1}{z}\right), \quad Q(z) = \frac{1}{c} q\left(\frac{1}{z}\right), \tag{9}$$

so that $F_0(z)$ is analytic in a region $V_z - \alpha_z$, where V_z is the annulus $r_2^{-1} < |z| < r_1^{-1}$ and in this region

$$F_0(z) = \frac{i}{2\pi} (\log |z| + i\theta) + P(z) + Q(z) \tag{10}$$

for $\theta = \arg z$, an appropriate single-valued branch of $\arg z$. Here $P(z)$ is analytic for $|z| < r_1^{-1}$, $Q(z)$ for $|z| > r_2^{-1}$. We choose R so that $r_2^{-1} < R < r_1^{-1}$ and, for an appropriate analytic function ψ in the region $\Delta_r : R \leqq |z| \leqq r$, we let $A_\psi(r)$ be the spherical area of the image $\psi(\Delta_r)$: that is, as in [13, p. 163],

$$A_\psi(r) = \frac{1}{\pi} \int_0^{2\pi} \int_R^r \frac{|\psi'|^2}{(1+|\psi|^2)^2} \rho \, d\rho \, d\theta, \tag{11}$$

in polar coordinates ρ, θ. We can even apply this with $\psi = F_0(z)$ and with $\psi = \log|z| = \log z + i\theta$, with $\theta = \arg z$ as above; in both cases ψ is discontinuous along α_z, but the integral has meaning. By the hypotheses on f_0, $|F_0(z)| \to \infty$ uniformly as $|z| \to r_1^{-1}$ and F_0 takes each complex value at most N times in $V_z - \alpha_z$; we conclude that $A_{F_0}(r)$ is bounded for $R < r < r_1^{-1}$. Also $A_{\log z}(r)$ and $A_Q(r)$ are bounded for $R < r < r_1^{-1}$; the first because our branch of $\log z$ is univalent, the second because Q is analytic for $R \leqq |z| \leqq r_1^{-1}$. Now one shows easily that

$$A_{\psi_1+\psi_2}(r) \leqq 2A_{\psi_1}(r) + 2A_{\psi_2}(r).$$

Thus it now follows from (10) that $A_P(r)$ is also bounded for $R < r < r_1^{-1}$. We observe that

$$T_P(r) = \int_R^r \frac{A_P(t)}{t} \, dt + \text{const} \tag{12}$$

is the characteristic function of P [13, p. 167].

If now $r_1 > 0$, then $P(z)$ is an analytic function in the disc $|z| < r_1^{-1}$ and by (12) and the boundedness of $A_P(r)$, $P(z)$ has a bounded characteristic function in the disc. It follows that $P(z)$ has finite radial limits almost everywhere on $|z| = r_1^{-1}$.

Now from the properties of f_0 it follows that $|\mathrm{Im}\, F_0(z)| \to \infty$ as z approaches the circle $|z| = r_1^{-1}$ on α_z. By (10) and the fact that $Q(z)$ is analytic for $|z| < r_2^{-1}$, we conclude that $|\mathrm{Im}\, P(z)|$ has limit infinity as z approaches the circle $|z| = r_1^{-1}$ along α_z. Now α_z has as limit set on $|z| = r_1^{-1}$ a connected set β, which may reduce to a point. If β is not a point, then the fact that $|\mathrm{Im}\, P(z)| \to \infty$ as z approaches the circle along α_z would imply that $P(z)$ could not have a finite radial limit almost everywhere, so that this case cannot occur. Hence β is a point. But then in our branch $\log z = \log|z| + i\theta$ of $\log z$, $\theta = \arg z$ is bounded for $|z|$ close to r_1^{-1}, so that $|\log z|$ is itself bounded. Also $|Q(z)|$ is bounded for $|z|$ near r_1^{-1}. But in (10) the left side has infinite limit as $|z| \to r_1^{-1}$ along every radius except perhaps one, while the right side remains bounded along almost all radii. This is a contradiction. Therefore $r_1 = 0$.

Now $P(z)$ is an entire function with $A_P(r)$ bounded for $R<r<r_1^{-1}=\infty$ and by (12) $T_P(r)\leqq k\log r+\text{const}$, with constant k. This implies that $P(z)$ is a polynomial [13, p. 209], [14, p. 2, no. 9].

Thus finally, for some integer m

$$f'(\zeta)=\sum_{n=-m}^{\infty} a_n\zeta^n.$$

If $m\leqq 0$ here, the hypotheses on f_0 cannot be satisfied, since $f_0(\zeta)$ would be analytic at $\zeta=0$. Thus $f'(\zeta)$ has a pole at $\zeta=0$ and $\phi=f'^2$ has a pole of even order. For a pole of ϕ of order 2, (iii) is satisfied and since (iii) is ruled out by hypothesis, we have a pole of ϕ of even order at least 4, so that $m\geqq 2$.

Thus in all four cases complex coordinates can be introduced at z_k and we have our local complex coordinates everywhere in the extended plane. Hence we can represent the extended plane as the extended ζ-plane, in terms of a single complex coordinate ζ, in terms of which our given elements of $\mathcal{F}$ become analytic functions of ζ, all obtainable from a single function by analytic continuation. Under the hypotheses on $\mathcal{F}$, these functions f are such that f'^2 is single-valued, say $f'^2(\zeta)=\phi(\zeta)$, where $\phi(\zeta)$ is analytic and non-vanishing except at the n points of S. The analysis of the four cases shows that ϕ has rational character at these points, so that ϕ is a rational function of ζ. Therefore $\phi(\zeta)\,d\zeta^2$ defines a rational quadratic differential in the extended plane, and the branches f are representable as $\int[\phi(\zeta)]^{1/2}\,d\zeta+\text{const}$ as asserted.

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University of Michigan
Ann Arbor, Michigan 48109
U.S.A.

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