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Periodic minimal surfaces

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Introduction

The interest in triply-periodic minimal surfaces in space seems to date from the work of H. A. Schwarz [11], beginning in 1865 with the construction of the first examples (see §7). All subsequent work known to us is restricted to these examples. We have found the work of Neovius [14] particularly beautiful and useful.

A triply-periodic minimal surface properly immersed in space corresponds to a minimal immersion f of a compact oriented surface X into a flat 3-torus T. With the induced conformal structure X is a compact Riemann surface and f is a conformal minimal immersion. Our object is then to study conformal minimal immersions of compact Riemann surfaces in flat 3-tori. This is also the point of view of Nagano-Smyth [6, 8] and Meeks [5]. The correct setting for this is the Jacobi variety of X and universality (see §1) plays an indispensable role.

The main question studied here is:
For a given compact Riemann surface X admitting some conformal minimal immersion f into a flat 3-torus T describe the set of all such immersions.

In this set there may be further immersions which are closely related to f and we call them associates of f (see §2). To describe these we first take a lift of f to universal covers, that is, $\tilde{f}:\tilde{X}\to R^3$. The classical Bonnet deformation gives a one-parameter family of (isometric) conformal minimal immersions $\tilde{f}_\theta:\tilde{X}\to R^3$ ($0\leqq\theta<\pi$). For certain values of θ these may be triply-periodic and project to conformal minimal immersions of X in flat 3-tori. These projections are the non-trivial associates of f. The great advantage of the Jacobi variety is that these associates are discernible directly from f and the structure of this variety. This can be found in §2.

The set of conformal minimal immersions of X in flat 3-tori divides into two sets; those having non-trivial associates and those which do not. Theorem 2

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describes the former set in terms of the codimension-3 abelian subvarieties of the Jacobi variety of X. It follows that *in genus* 3 *any two conformal minimal immersions are associate.*

Theorem 1 gives a criterion for existence of associates. When f is particularly nice, in that it has a high degree of symmetry (see §5) and a non-trivial associate exists, Theorem 3 shows that f has countably many non-trivial associates and that f is roughly the "real part" of a holomorphic immersion of X in a complex 3-torus $T_1 \times T_1 \times T_1$, where T_1 is some complex 1-torus. Essentially the same result was also obtained by Meeks [5]; in the form given here it generalises easily to minimal surfaces in flat n-tori. This and other features of the general case we treat in a separate paper. Theorem 3 serves as a good way of generating new periodic surfaces; for example the conditions of Theorem 3 are satisfied by the Schwarz surfaces of genus 3 and 9 (see §7) by the work of Neovius [14].

Theorem 4 is an extension of an earlier result [6] on the hyperelliptic case. The final section is given over to remarks on examples of compact minimal surfaces and some consequences of Theorems 2 and 3.

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1. The Jacobi map

Let X be a compact Riemann surface of genus p and complex structure denoted J. Let $\mathfrak{h}$ denote the complex p-dimensional vector space of all holomorphic one-forms on X. The natural map

$$H_1(X, Z) \times \mathfrak{h} \to C$$

given by integration of the forms in $\mathfrak{h}$ over the cycles in $H_1(X, Z)$ is additive in the first argument and complex linear in the other and so defines a cononical homomorphism

$$\tilde{a} : H_1(X, Z) \to \mathfrak{h}^*.$$

The image of $\tilde{a}$ is a free abelian group in $\mathfrak{h}^*$ and is well known to be a lattice Δ in $\mathfrak{h}^*$, and the quotient complex torus $A = \mathfrak{h}^*/\Delta$ is called the *Jacobi variety* of X. Fixing a point $x_0 \in X$ as origin and taking any path γ from x_0 to x in X, we see that the functional $\int_\gamma \in \mathfrak{h}^*$ depends only on x to within an element of Δ. The projection of this functional to A is then denoted $a(x)$ because it is independent of γ. By varying the end-point x it is easy to see that this map $a: X \to A$ is holomorphic. When $p > 0$ Riemann-Roch tells us that a is an immersion, that is, the differential a_* is nowhere singular. The map a is called the *Jacobi map* and is, by Abel's theorem, an imbedding (see Gunning [2]). The natural complex structure on A will also be denoted J.

Given a Riemann surface X, a map f of X into a real torus T will be called a *harmonic map* if every real linear one-form on T pulls back under f to the real part of a holomorphic one-form on X. Such a map is the Jacobi map and it is *universal* in this class of maps in the following sense: For any harmonic map $f: X \to T$, there corresponds a unique affine map $h: A \to T$ such that $f = h \circ a$. By an affine map we mean a composition of a homomorphism from A to T, and a translation in T in that order. In proving universality, suppose first $f(x_0) = e$, where x_0 is the base point for the Jacobi map a above; this we can always arrange by translation

V, the space of harmonic one-forms on T is identified with the dual vector space V^*. Then f induces a linear map $k: V^* \to \mathfrak{h}$, where $\mathfrak{h}$ is considered as a real vector space. The transpose of this real linear map k is a real linear map $h: \mathfrak{h}^* \to V$. It is straightforward to check that $h(\Delta) \subset L$, where Δ is the lattice of the Jacobi variety in $\mathfrak{h}^*$. Thus h induces a homomorphism from A to T, which we denote also h. From our construction, f and $h \circ a$ are harmonic maps agreeing on first cohomology, and it follows easily that they differ by a translation in T. When $f(x_0) \neq e$, there is the added translation by $-f(x_0)$ in T.

Let X be a fixed compact Riemann surface and $f: X \to T$ a conformal minimal immersion of X into a flat torus T of real dimension 3. Fixing any point $x_0 \in X$ we may assume, after a translation in T, that $f(x_0) = e$, the identity element of T. We will show that f is harmonic in the sense defined above. A linear one-form w on T is the differential of a local linear function ρ on T. Since f is minimal, $\rho \circ f$ is a locally defined function of X which is harmonic with respect to the induced Riemannian structure on X. Hence it is the real part of a (locally defined) function F on X which is holomorphic with respect to the induced complex structure–and this coincides with the original complex structure on X since f is conformal. Thus $f^*w = \mathrm{Re}\,(dF)$ on a domain in X, and it follows that f^*w is the real part of a holomorphic one-form on X. Now by universality of the Jacobi map (with base point x_0) of the Riemann surface X, we have a unique homomorphism $h: A \to T$ such that the left half of the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "a"'] & T & \\
& A \arrow[u, "h"] \arrow[r, "h_0"'] & A/(\mathrm{Ker}\, h)_0 = T_0 \arrow[ul, "\pi"']
\end{tikzcd}$$

commutes. If h is not surjective, $f(X)$ must be an affine subtorus of T, so X is itself a torus and f is an affine map. This case is really of no interest, so we assume that h is surjective. The compact subgroup $\operatorname{Ker} h$ may not be connected; however, denoting its identity component by $(\operatorname{Ker} h)_0$, we have the projection homomorphism h_0 into another 3-torus T_0 and the finite covering π which makes the other half of the diagram commute. Let $h_0 \circ a = f_0$ and consider the real 3-torus $T_0 = A/(\operatorname{Ker} h)_0$ with the flat metric lifted from T by means of the covering map π. Then

$$f_0 : X \to T_0$$

is a conformal minimal immersion of X into another flat 3-torus T_0. We call f_0 *the primitive of f.* It will sometimes be technically convenient to work with primitives. The results for non-primitives will be the same modulo finite covers.

Remark. In the case of genus $p=0$, a harmonic map f into T is constant. The case $p=1$ is special and trivial, for then f is just an affine map, from universality. So henceforth, we will only consider $p>1$. When $p>1$, the Gauss map of a minimal immersion $f:X\to T$ is a branched cover of the two-sphere of degree $p-1$. This is a simple consequence of the Gauss-Bonnet theorem, and we will use this fact in a moment. Consequently p cannot be 2 (cf. [6] or Lemma 1 below also).

As it is our main objective to describe the set of all conformal minimal immersions of a given compact Riemann surface X of genus $p>1$ in flat 3-tori, we should first explain when two such immersions are to be considered as only trivially distinct. Let $f_\alpha : X \to T_\alpha$ $(\alpha = 1,2)$ be two conformal minimal immersions of X in flat tori, normalised by $f_\alpha(x_0) = e$. If the corresponding homomorphisms $h_\alpha : A \to T_\alpha$ have the same kernel (connected or not), then $f_2 = \phi \circ f_1$, where $\phi : T_1 \to T_2$ is an isomorphism preserving the flat metrics to within a factor; the isomorphism ϕ is clear and that it preserves the flat metrics is a consequence of conformality and the fact that the Gauss maps of the f_α cover the sphere when $p>1$ (see the Remark above). A homomorphism (isomorphism) between flat tori preserving the metrics, to within a factor, will be called a *homothety* (*isothety*).

DEFINITION. Two conformal minimal immersions $f_\alpha : X \to T_\alpha$ $(\alpha=1,2)$ are *equivalent* if the kernels of the corresponding homomorphisms $h_\alpha : A \to T_\alpha$ have the same identity component.

If $f_\alpha : X \to T_\alpha$ $(\alpha = 1,2)$ are equivalent conformal minimal immersions then, dividing the kernels of both of the corresponding homomorphisms out of A, we obtain a conformal minimal immersion $f:X\to T$ into a flat torus T and homotheties $\pi_\alpha : T_\alpha \to T$ such that $f = \pi_\alpha \circ f_\alpha$. This f is uniquely determined (to within isothety) by f_1 and f_2 and can be thought of as the "lowest common multiple" of f_1 and f_2. We should add that each equivalence class of conformal minimal immersions contains a unique primitive. Each equivalence class determines then a unique real codimension 3 subtorus of A and, since a torus has but a countable number of subtori, we have: *The set of inequivalent conformal minimal immersions of X in flat 3-tori is at most countable.*

Let U denote the kernel of the differential h_* at the identity of A. Let V denote the maximal complex subspace of U. Because $f = h \circ a$ is an immersion, the complex curve $a(X)$ has no tangency with the foliation on A determined by U. The same is true of the foliation determined by JU, where J denotes the complex structure on A; this is because $a(X)$ is complex.

LEMMA 1. *For genus $p > 1$, $\dim_C V = p - 3$.*

Proof. Since $p \geqq 2$, $f(X)$ lies in no affine subspace of T and so $\dim U = 2p - 3$. Writing $\dim_C V = p - r$, we have

$$\begin{aligned} 2p \geqq \dim (U + JU) &= \dim U + \dim JU - \dim V \\ &= 2(2p-3) - 2(p-r) \\ &= 2(p+r-3). \end{aligned}$$

So $3 \geqq r$, and since $V \subset U$ we also have $2r \geqq 3$. This leaves $r = 2$ or 3 as the only possibilities. Suppose $r = 2$. Choose a basis of V of the form $\{\xi_1, J\xi_1, \ldots, \xi_{p-2}, J\xi_{p-2}\}$. V has codimension one in U so we choose a vector ξ_0 complementary to V in U. Then the subspace W generated by ξ_0, $J\xi_0$ and V has complex dimension $p-1$ and every parallel translate of it in A is transversal to $a(X)$. The holomorphic normal bundle of $a(X)$ is therefore trivial. But the Whitney sum of this bundle with the tangent bundle of X is itself trivial, being the bundle induced by a from the tangent bundle of A. It follows that X has trivial tangent bundle, that is, X is a torus. The contradiction means $r = 3$, ending the proof of Lemma 1.

Remark. We do not know if, in general, V determines a complex subtorus of A.

2. Associates

There is a simple way in which we can *sometimes* construct a conformal minimal immersion of a compact Riemann surface X in a flat torus from another such immersion.

Given a conformal minimal immersion $f:X \to T$, normalised by $f(x_0)=e$, we have $f=h\circ a$, where $h:A\to T$ is a certain homomorphism. Assume for the moment that f is primitive in the sense of Section 1, that is, Ker h is a subtorus of A. Denoting by U the subspace tangent to this subtorus at the identity, we set

$$U_\theta=\{\cos\theta\cdot u+\sin\theta\cdot Ju \mid u\in U\}.$$

The foliation parallel to U_θ will have no tangency with the curve $a(X)$. *Suppose U_θ determines a subtorus* (also denoted U_θ) *of* A. Then the composition

$$f_\theta:X\xrightarrow{a}A\xrightarrow{h_\theta}A/U_\theta=T_\theta$$

defines an immersion of X into the 3-torus T_θ.

LEMMA 2. *For a suitable flat metric on T_θ, the immersion $f_\theta:X\to T_\theta$ is a conformal minimal immersion.*

Proof. f_θ is clearly harmonic in the sense of Section 1, so we only have to show that there is a natural flat metric on T_θ making f_θ conformal. Then f_θ is a conformal minimal immersion of X in the flat torus T_θ.

We now set about defining the natural metric on T_θ. f induces a Riemannian metric on X and the corresponding global inner product on forms determines a flat Hermitian structure g_0 on A. With respect to this structure the orthogonal complement of $V=U\cap JU$ in U is denoted K. By Lemma 1, K is a real 3-dimensional subspace of U and so $h_*:JK\to T_e(T)$ is an isomorphism. So we can pull back the flat metric $\langle\ ,\ \rangle$ on T to get an inner product on JK. Extend this to an inner product on $K\oplus JK$ such that J is orthogonal and JK is orthogonal to K. The direct sum of this inner product with the original Hermitian structure g_0 restricted to V gives a flat Hermitian structure on A, which we denote by g_f because the construction is unique and depends only on f.

Define an inner product $\langle\ ,\ \rangle_\theta$ on $T_e(T_\theta)$ by

$$\langle t_1,t_2\rangle_\theta=g_f(l_1,l_2),$$

where l_α is the unique element of

$$JK_\theta=\{\cos\theta\, Jk-\sin\theta k \mid k\in K\},$$

such that $h_\theta(l_\alpha)=t_\alpha$; here h_θ denotes the projection from A to A/U_θ. With respect to the flat metric $\langle\ ,\ \rangle_\theta$ on T_θ the map $f_\theta:X\to T_\theta$ is a conformal minimal immersion. In fact, if you check–and it is easy–you will find that the metric induced by f_θ on X is the same as that induced by f.

If $f:X \to T$ is not primitive, then taking a primitive f' of f we can apply the above construction to f', when θ is such that U_θ determines a subtorus of A, to obtain a conformal minimal immersion f'_θ of X in another flat 3-torus.

DEFINITION. Any conformal minimal immersion of X in a flat 3-torus which is equivalent to f'_θ (defined above) for some θ is called an *associate* of f; if $\theta = 0 \bmod \pi$, it is equivalent to f' and therefore to f and we call it a *trivial associate.*

Equivalently, we might say that two conformal minimal immersions $f_\alpha : X \to T_\alpha\,(\alpha = 1, 2)$ in flat 3-tori are associates if the corresponding homomorphisms $h_\alpha : A \to T_\alpha$ satisfy $\operatorname{Ker}(h_2)_* = e^{i\theta}\operatorname{Ker}(h_1)_*$ for some angle θ.

For what conformal minimal immersions $f:X \to T$ does a nontrivial associate exist? A criterion is given in the next section but the next lemma gives a simpler necessary condition. Under additional symmetry assumptions on f the variety A', appearing below, will be determined completely in Theorem 3.

LEMMA 3. *Let $f:X \to T$ be a conformal minimal immersion of a compact Riemann surface into a flat torus. If f has a nontrivial associate, then V determines a complex subtorus of A and f will factor, by a homomorphism, through the holomorphic immersion*

$$a' : X \xrightarrow{a} A \xrightarrow{\text{proj}} A/V = A'$$

of X into the 3-dimensional abelian variety A'.

Proof. Letting $h:A \to T$ denote the homomorphism arising from f and $U = \operatorname{Ker} h_{*e}$, the existence of a non-trivial associate implies that the subspace U_θ of $T_e(A)$ determines a subtorus of A for some angle θ which is not a multiple of π. This last condition tells us that the intersection of the subtori determined by U and U_θ has tangent space V at e and that V determines a subtorus of A (also denoted V). Since $V \subset \operatorname{Ker} h$, the homomorphism $h:A \to T$ projects to a homomorphism $h' : A' \to T$, where $A' = A/V$, and $f = h' \circ a'$, where a' is the projection of X into A'. Because A is an abelian variety, it follows from Poincaré's Complete Reducibility Theorem (cf. Swinnerton-Dyer [13]) that both V and A' are also.

We end this section by relating the notion of associates defined above to the classical notion of the associates of a minimal immersion of a simply-connected surface into Euclidean space. Recall that if $\phi: M \to R^3$ is an isometric minimal immersion of a Riemannian 2-manifold (normalised by $\phi(p_0) = (0, 0, 0)$, for some $p_0 \in M$), then ϕ can be considered as a triple of real harmonic functions on M. Assuming M is simply-connected, the harmonic conjugates of these functions exist and so define another triple ψ, which is unique when normalised by $\psi(p_0) = (0, 0, 0)$. Then $\phi_\theta = \cos\theta\phi + \sin\theta\psi$ gives a minimal isometric immersion of M into R^3 for each value of θ which, for all θ with $0 \leqq \theta < \pi$, are non-congruent (assuming $\phi(M)$ is not a portion of a plane in R^3). The ϕ_θ are called the *associates* of ϕ and $\phi_{\pi/2}$ is called the *conjugate* of ϕ. In addition to the invariance of the induced Riemannian metric under this deformation through minimal immersions, we point out that

i. The Gauss map of the immersion ϕ_θ is the same as for ϕ.
ii. $(\phi_{\pi/2})_* = -\phi_* \circ J$, where J is the complex structure on M.

Let $f: X \to T$ be a conformal minimal immersion with $f(x_0) = e$ and corresponding homomorphism $h: A \to T$. Let $\langle\ ,\ \rangle$ denote the given flat metric on T and g_f the canonical Hermitian structure on A constructed out of f. Choose $\tilde{x}_0$ in the universal cover $\tilde{X}$ of X over the point x_0; the superscript tilda will always be used to denote universal covers. The map a lifts to a holomorphic immersion $\tilde{a}: \tilde{X} \to \tilde{A}$ with $\tilde{a}(\tilde{x}_0) = \mathcal{O}$, the origin of $\tilde{A}$. Let $\tilde{h}: \tilde{A} \to \tilde{T}$ denote the linear map of these vector spaces which covers the homomorphism h. Denote the corresponding inner products on $\tilde{A}$ and $\tilde{T}$ by g_f and $\langle\ ,\ \rangle$. Then $e^{i\theta}w = \cos\theta w + \sin\theta Jw$ defines a complex linear isometry of $(\tilde{A}, g_f)$. Let $\tilde{U} = \operatorname{Ker}\tilde{h}$ and $\tilde{U}_\theta = e^{i\theta}\tilde{U}$. The vector space $\tilde{A}/\tilde{U}_\theta$ inherits a natural inner product $\langle\ ,\ \rangle_\theta$ from g_f under the projection $\tilde{h}_\theta: \tilde{A} \to \tilde{A}/\tilde{U}_\theta$. From the definition of g_f we know $\langle\ ,\ \rangle_0 = \langle\ ,\ \rangle$ on $\tilde{A}/\tilde{U} = \tilde{T}$. The Euclidean spaces $(\tilde{A}/\tilde{U}_\theta, \langle\ ,\ \rangle_\theta)$ are all isometric, the isometry being induced from the transformation $e^{-i\theta}$ of $\tilde{A}$. Denote it also $e^{-i\theta}: \tilde{A}/\tilde{U}_\theta \to \tilde{A}/\tilde{U}$. Setting $\tilde{f}_\theta = e^{-i\theta} \circ \tilde{h}_\theta \circ \tilde{a}$ we have a one-parameter family of mappings

$$\tilde{f}_\theta: \tilde{X} \to \tilde{A}/\tilde{U} = \tilde{T}.$$

Because $\tilde{U}$ and its parallel translates have no tangency with the complex curve $\tilde{a}(\tilde{X})$, the same will be true of $\tilde{U}_\theta$, so each $\tilde{f}_\theta$ is an immersion. In fact, a straightforward computation from the definition of $\tilde{f}_\theta$ gives

$$\tilde{f}_\theta = \cos\theta\tilde{f}_0 + \sin\theta\tilde{f}_{\pi/2}$$

and $(\tilde{f}_{\pi/2})_* = -(\tilde{f}_0)_* \circ J$.

Therefore $\tilde{f}_{\pi/2}$ is the conjugate of the minimal immersion $\tilde{f}_0 = \tilde{f}$ and the $\tilde{f}_\theta$ are the minimal immersions associate to $\tilde{f}$ in the classical sense.

Finally note that $\tilde{f}_\theta$ covers a minimal immersion of X in a 3-torus if and only if the same is true of $\tilde{h}_\theta \circ \tilde{a}$. And the latter holds precisely when $\tilde{U}_\theta$ projects to a subtorus of A – or more explicitly when $\tilde{U}_\theta \cap \Delta$ is a lattice in $\tilde{U}_\theta$, where $A = \tilde{A}/\Delta$. Then since $e^{-i\theta}$ will not in general map the lattice $\tilde{h}_\theta(\Delta)$ into the lattice $\tilde{h}_0(\Delta)$, the flat torus $T_\theta = (\tilde{A}/\tilde{U}_\theta)/\tilde{h}_\theta(\Delta)$ is in general different from $T_0 = T$.

We have shown

LEMMA 4. *Let $f: X \to T$ be a minimal immersion of a compact oriented surface in a flat 3-torus. Let $\tilde{f}$ denote a lift of f to universal covers. A non-trivial associate of f exists if and only if for some $\theta \neq 0 \bmod \pi$ the classical associate $\tilde{f}_\theta$ of $\tilde{f}$ covers a minimal immersion of X into a flat 3-torus.*

3. A criterion for the existence of associates

THEOREM 1. *Let $f: X \to T$ be a minimal immersion of a compact oriented surface X of genus >1 in a flat 3-torus T. Let $\tilde{f}$ denote a lift of f to the level of universal covers and $\tilde{f}_\theta$ an associate of $\tilde{f}$ in the classical sense. Then f has a non-trivial associate if and only if, for some $\theta \neq 0 \bmod \pi$, the image $\tilde{f}_\theta(\tilde{X})$ is not dense in the Euclidean space $\tilde{T}$ covering T.*

Proof. The necessity should already be clear from Section 2. Assume that f has no non-trivial associates.

Consider X with the induced metric and complex structure. Then we know that $f = h \circ a$ where $a: X \to A$ is the Jacobi map based at $x_0 \in X$ and $h: A \to T$ is some homomorphism; we are assuming f is normalised by $f(x_0) = e$. In the notation of Section 2, our asumption implies that for each $\theta \neq 0 \bmod \pi$, $\tilde{U}_\theta = e^{i\theta}\tilde{U}$ does not project to a subtorus of A. A point $\beta \in \tilde{T}$ is the image, under the map $e^{-i\theta} \circ \tilde{h}_\theta$, of some affine subspace $\tilde{B}$ of $\tilde{A}$ parallel to $\tilde{U}_\theta$. If $\beta \notin \tilde{f}_\theta(\tilde{X})$ then the subspace $\tilde{B}$ never meets the complex curve $\tilde{a}(\tilde{X})$ and its projection B in A never meets $a(X)$. In A the submanifold B has real codimension 3 and is not closed, so its closure $\bar{B}$ is parallel to a subtorus of A of codimension $s = 0$, 1, or 2. The next step in the proof is to show that $\bar{B}$ must meet $a(X)$; we postpone the argument on this point preferring to show first how it is used to complete the proof of the theorem. It will then follow that there is a sequence $\{\tilde{b}_n\}$ in $\tilde{B}$ projecting to a sequence $\{b_n\}$ in B with $\lim b_n = a(x)$ for some $x \in X$. Fix $\tilde{x} \in \tilde{X}$ over the point $x \in X$. Let D be a fundamental region for the lattice Δ in $\tilde{A}$ containing the origin. For each positive integer n there exists $\delta_n \in \Delta$ such that $\tilde{b}_n - \tilde{a}(\tilde{x}) - \delta_n = \varepsilon(n) \in D$ and the convergence of b_n to $a(x)$ implies $\varepsilon(n) \to 0$. But the very definition of the Jacobi map tells us that $\tilde{a}(\tilde{x}) + \delta_n = \tilde{a}(\gamma_n x)$ for some deck transformation γ_n of X; write $\tilde{x}_n = \gamma_n \tilde{x}$. So $\lim \tilde{b}_n - \tilde{a}(\tilde{x}_n) = \lim \varepsilon(n) = 0$. Applying the map $e^{-i\theta}\tilde{h}_\theta$ we have at once $\lim \tilde{f}_\theta(\tilde{x}_n) = \beta$. Hence $\tilde{f}_\theta(\tilde{X})$ is dense in $\tilde{T}$.

It remains to verify the step: $\bar{B}$ must meet $a(X)$. This is trivially so when $s=0$. Suppose that $s=1$ but that $\bar{B}$ does not meet $a(X)$. Consider the parallel one-form w on A orthogonal to $\bar{B}$; as before we use the flat Hermitian metric g_f on A, defined in Section 2. The differential a^*w is harmonic on X and, assuming $a(X)$ and $\bar{B}$ do not meet, must also be exact. So $a^*w \equiv 0$ and this means $a(X)$ lies in a subtorus parallel to $\bar{B}$. But this cannot be, since $a(X)$ always generates the Jacobi variety of X. Therefore $\bar{B}$ must meet $a(X)$ when $s=1$. The final hurdle is to show s cannot be 2.

We will show that if s were equal to 2, the tangent plane $\not{p}(x)$ to the curve $a(X)$ at $a(x)$ can neither be (i) parallel to $\bar{B}$, nor (ii) orthogonal to $\bar{B}$. Accepting this for the moment, it follows that there is a unique real line in $\not{p}(x)$ making least angle with the subspace parallel to $\bar{B}$. This then defines a continuous non-singular line-field on the surface X. By the Poincaré theorem on line fields, X must have genus one and this is a contradiction. To prove that (i) and (ii) cannot occur, denote by C the subspace of $T_e(A)$ parallel to $\bar{B}$. With the notation of Section 2, we note that C contains the subspace $V \oplus e^{i\theta}K$ as a subspace of codimension 1. Therefore

$$C = V \oplus e^{i\theta}K \oplus \{Je^{i\theta}k_1\}$$

for some unit vector $k_1 \in K$. Extend k_1 to an orthonormal basis $\{k_1, k_2, k_3\}$ of K. The maximal complex subspace of C is $V \oplus \{k_1, Jk_1\}$. When (i) holds, $\not{p}(x)$ is parallel to C and since $\not{p}(x)$ is complex, it must in fact be parallel to $V \oplus \{k_1, Jk_1\}$. It is then a simple matter to check that $\not{p}(x)$ contains a vector parallel to $V \oplus \{k_1\} = U$, and this contradicts the fact that f is an immersion. So (i) cannot occur. As to (ii), orthogonality of $\not{p}(x)$ to C, with respect to the metric g_f on A, implies that $Je^{i\theta}k_2$, $Je^{i\theta}k_3 \in \not{p}(x)$; this is impossible since $\not{p}(x)$ is complex. The proof of Theorem 1 is now complete.

Remark. You will now see from Theorem 1 and the remarks of Section 2 that for all but a countable number of θ with $0 \leqq \theta < \pi$, the minimal surfaces $\tilde{f}_\theta(\tilde{X})$ are *dense* in Euclidean 3-space. The existence of complete dense minimal surfaces in Euclidean space was new, to us at least.

4. A special kind of conformal minimal immersion and their classification

In studying a conformal minimal immersion $f: X \to T$ of a compact Riemann surface X into a flat torus, we need only consider the induced homomorphism

$h:A \to T$, where A is the Jacobi variety of X. Looking a little closer at the kernel of h, in fact at the maximal complex subspace V tangent to it at the identity, we realise that the case where V generates a subtorus (also denoted V) of A is of special interest. This case will be our sole interest in this section. V will then be referred to as *the complex codimension-3 subtorus of A annihilated by f*; recall that the codimension was determined in Lemma 1. We see from Lemma 3 that this is equivalent to saying that there exists a complex 3-torus A' a holomorphic immersion $a':X \to A'$ and a homomorphism $h':A' \to T$ such that $f = h' \circ a'$. So the immersions of this section might be thought of as "complexifiable."

As examples of this kind we have

(a) any conformal minimal immersion $f:X \to T$ which has a non-trivial associate (this is by Lemma 3).

(b) any conformal minimal immersion $f:X \to T$ where X has genus 3 (this follows from Lemma 1, since V is trivial when $p = 3$).

To classify those conformal minimal immersions of X in flat 3-tori which are complexifiable, it is first necessary to classify the complex codimension-3 subtori of the Jacobi variety of X. The next theorem shows that there is nothing more to be done, for two conformal minimal immersions annihilating the same complex codimension-3 subtorus of A must be associate.

THEOREM 2. *Let $f_\alpha:X \to T_\alpha$ ($\alpha = 1, 2$) be two conformal minimal immersions of a compact Riemann surface of genus >1 in flat 3-tori. If f_1 and f_2 annihilate the same complex codimension-3 subtorus of the Jacobi variety of X then f_1 and f_2 are associates.*

Remarks. (i) Theorem 2 is the converse of Lemma 3.

(ii) The next corollary is immediate from b) above.

COROLLARY. *If X is a compact Riemann surface of genus 3, then any two conformal minimal immersions of X in flat 3-tori must be associates.*

Proof of Theorem 2. This consists of a string of lemmas and will take up the rest of this section. It is enough to prove the theorem for primitives, so we will assume f_1 and f_2 primitive and also $f_\alpha(x_0) = e$ for each α.

For a fixed orientation of T there is a unique unit vector field ξ normal to X along f such that $\{v, Jv, \xi\}$ has the given orientation for any non-zero vector v tangent to X. For each $x \in X$, we let $\Gamma_f(x)$ denote the parallel translate of ξ_x to the identity in T giving thereby a map $\Gamma_f:X \to S^2$ where S^2 denotes the unit sphere in $T_e(T)$. the Gauss map Γ_f is well known to be antiholomorphic with respect to the unique complex structure J^0 on S^2 for which for every tangent vector u the vector product $u \times Ju$ always points away from the centre of S^2 (cf. [9]).

Denoting the metric connexion on T by D, the equation $D_v\xi = -f_*(Av)$ for any vector v tangent to X can be taken as the definition of the second fundamental form A of the immersion f. In a local complex coordinate z on X the induced metric ds^2 and the second fundamental form are of the form

$$ds^2 = 2F\,|dz|^2 \qquad (F>0)$$

$$A\frac{\partial}{\partial z} = (\alpha - i\beta)\frac{\partial}{\partial \bar{z}}$$

where α, β and F are local functions. The Codazzi equations for A reduce to the statement that $(\alpha - i\beta)F = w(z)$ is a holomorphic function of z. It is readily checked that $\Omega_f = w(z)\,dz^2$ is independent of the choice of local coordinate and so defines a holomorphic quadratic differential on the Riemann surface X. For the details the reader is referred to the lecture notes of H. Hopf [3]. A simple computation shows that $|\partial\xi/\partial z| = |w(z)|$; thus *the branch points of the Gauss map Γ_f coincide in order and location with the zeroes of the differential Ω_f.*

LEMMA 5. *If $f_\alpha : X \to T_\alpha$ $(\alpha = 1, 2)$ are two conformal minimal immersions of X in flat tori which are associates, then $\Omega_{f_2} = \lambda\Omega_{f_1}$ for some complex number $\lambda \neq 0$.*

Proof. From the remarks of Section 1, it will be seen that equivalent immersions induce the same metric and second fundamental forms to within constant factors. Consider a lift $\tilde{f}_2 : \tilde{X} \to \tilde{T}_2$ of f_2 with $\tilde{f}_2(\tilde{x}_0) = \mathcal{O} \in \tilde{T}_2$, where $\tilde{x}_0$ is any point of $\tilde{X}$ over $x_0 \in X$, and consider the classical associates $(\tilde{f}_2)_\theta$ of $\tilde{f}_2$. Since f_1 is an associate of f_2, we know from Section 2 that for some value of θ, $(\tilde{f}_2)_\theta$ covers a minimal immersion of X in a flat torus which is equivalent to f_1. But the sencond fundamental form of the projected immersion is $\cos\theta\, A_2 + \sin\theta JA_2$, where A_2 is the second fundamental form of f_2. It follows then from the definition of Ω that $\Omega_{f_2} = ke^{i\theta}\Omega_{f_1}$ for some real constant k.

The next is the most important step in the proof of Theorem 2.

LEMMA 6. *If $f_\alpha : X \to T_\alpha$ $(\alpha = 1, 2)$ are conformal minimal immersions of X annihilating the same complex codimension-3 subtorus V of A, then $\Omega_{f_2} = \lambda\Omega_{f_1}$ for some non-zero complex number λ.*

Proof. Ω is the same for f_α as for its primitive. So we may assume the f_α are primitive. Let $a' : X \to A \to A/V = A'$ denote the obvious holomorphic immersion of X in the complex 3-torus $A' = A/V$. Because $V \subset \operatorname{Ker} h_\alpha$ we have homomorphisms $h'_\alpha : A' \to T_\alpha$ such that $f_\alpha = h'_\alpha \circ a'$ and because the f_α are primitive, the h'_α have connected kernels. U'_α denotes the tangent space to $\operatorname{Ker} h'_\alpha$ at the identity of A'. The complex structure on A' will be denoted also by J. Pull back the flat metric on T_α to JU'_α via the map h'_α and extend this to a Hermitian inner-product on $T_e(A')$ for which J is orthogonal and U'_α is orthogonal to JU'_α; denote the corresponding Hermitian metric on A' by g_α. For each $x \in X$ let $\Gamma(x)$ denote the element of the projective space $P^2(C)$ of all complex lines passing through the identity of A' which is parallel to the complex line $a'_*(T_x(X))$. The holomorphic map $\Gamma: X \to P^2(C)$ is called the Gauss map of the complex curve a'.

For any non-zero $v \in T_x(X)$, we have, in consequence of Lemma 1, the unique decomposition

$$a'_* v = z_\alpha + Jw_\alpha$$

with z_α, $w_\alpha \in U'_\alpha$; here, and frequently hereafter, we are identifying all vectors tangent to A' with their translation to the identity. Since $f_\alpha = h'_\alpha \circ a'$ is conformal, it follows that the pair $\{Jw_\alpha, Jz_\alpha\}$ are orthogonal and of equal length relative to g_α. The cross-product $Jw_\alpha \times Jz_\alpha$ in JU'_α determines a point of the unit sphere S_α in the subspace JU'_α at the identity. This point $\Gamma_\alpha(x)$ is independent of v. The inner-product g_α and the decomposition (U'_α, JU'_α) of $T_e(A')$ determine a complex bilinear form q_α on $T_e(A')$ for which

$$q_\alpha(a'_* v, a'_* v) = 0$$

for every vector v tangent to X. In other words, $q_\alpha(\Gamma(x), \Gamma(x)) = 0$. Thus $Q = \Gamma(X)$ is contained in a non-singular plane curve of degree 2 in $P^2(C)$. Q must then be the whole curve, that is, the Riemann sphere. But we have in fact constructed above an antiholomorphic diffeomorphism from Q to S_α; call it π_α. So $\Gamma_\alpha = \pi_\alpha \circ \Gamma$. Now h'_α identifies the sphere S_α with the unit sphere in $T_e(T_\alpha)$ and this identifies Γ_α with the classical Gauss map of the immersion $f_\alpha : X \to T_\alpha$. Now as we saw above, the zeroes of Ω_{f_α} coincide in order and location with the branch points of the Gauss map of f_α and therefore with those of Γ_α, and therefore with those of Γ (considered as a map from X onto the curve Q of genus zero in $P^2(C)$). Hence the zeroes of Ω_{f_1} and Ω_{f_2} are the same in order and location. Since X is compact, $\Omega_{f_2} = \lambda \Omega_{f_1}$ for some non-zero complex constant λ.

LEMMA 7. *With the same notation and assumptions as Lemma* 6, *there exist a pair of points x' and x'' on X such that $\{\Gamma_\alpha(x'), \Gamma_\alpha(x'')\}$ is a pair of antipodal points on the sphere S_α for both $\alpha = 1$ and $\alpha = 2$.*

Proof. Consider the mapping

$$\rho_\alpha : Q \xrightarrow{\pi_\alpha} S_\alpha \xrightarrow{-I} S_\alpha \xrightarrow{\pi_\alpha^{-1}} Q.$$

This is an antiholomorphic transformation of Q, the Riemann sphere. The holomorphic transformation $\rho_2^{-1} \circ \rho_1$ must therefore have a fixed point on Q, say $\Gamma(x')$. Then $\rho_1\Gamma(x') = \rho_2\Gamma(x') = \Gamma(x'')$ for some $x'' \in X$ different from x'. In fact

$$\begin{aligned}\Gamma_\alpha(x'') &= \pi_\alpha \circ \Gamma(x'') \\ &= \pi_\alpha \circ \rho_\alpha \circ \Gamma(x') \\ &= \pi_\alpha(\pi_\alpha^{-1} \circ -I \circ \pi_\alpha)\Gamma(x') \\ &= -\Gamma_\alpha(x'),\end{aligned}$$

completing the proof of the lemma.

In the proof of Lemma 6 the map h'_α identified the unit sphere S_α in the subspace JU'_α with the unit sphere in $T_e(T_\alpha)$. The maps Γ_α and π_α are then identified with maps – denoted by the same letters – into this latter sphere (which we will also denote S_α). With this understanding Γ_α now stands for the Gauss map of f_α. By Lemma 6, $\Omega_{f_2} = ke^{i\theta}\Omega_{f_1}$ for some positive number k. Changing the metric on T_2 by a suitable factor (which formally changes f_2 only to within the equivalence defined in Section 1) allows us to assume $\Omega_{f_2} = e^{i\theta}\Omega_{f_1}$. Fixing $\tilde{x}_0 \in \tilde{X}$ over $x_0 \in X$ we let $\tilde{f}_\alpha : \tilde{X} \to \tilde{T}_\alpha$ denote the lift of f_α for which $\tilde{f}_\alpha(x_0) = \mathcal{O}$, the origin of $\tilde{T}_\alpha$. The Gauss map and holomorphic differential arising from $\tilde{f}_\alpha$ are invariant by the deck transformations and the Gauss map is the same for all associates of $\tilde{f}_\alpha$. We see from the proof of Lemma 5 that the differentials corresponding to $\tilde{f}_1$ and the associate $(\tilde{f}_2)_\theta$ of $\tilde{f}_2$ are identical. Let $l : \tilde{T}_2 \to \tilde{T}_1$ be a proper linear isometry whose induced isometry $l : S_2 \to S_1$ carries $\Gamma_2(x')$ into $\Gamma_1(x')$ where x' is chosen as in Lemma 7. Then the minimal immersions $\tilde{f}_1$ and $\tilde{f}_3 = l \circ (\tilde{f}_2)_\theta$ of $\tilde{X}$ in $\tilde{T}_1$ induce the same differential on $\tilde{X}$. As the Gauss map of $\tilde{f}_3$ also projects to a map $\Gamma_3 : X \to S_1$ we can compare it easily with that of $\tilde{f}_1$. In fact

$$\begin{aligned}\Gamma_3 &= l \circ \Gamma_2 \\ &= l \circ \pi_2 \circ \pi_1^{-1} \circ \Gamma_1 \\ &= r \circ \Gamma_1\end{aligned}$$

where r is thought of as a holomorphic transformation of S_1. From Lemma 6 and the choice of l it follows that Γ_3 and Γ_1 coincide at x' and x'', so r fixes the antipodal pair $\{\Gamma_1(x'), \Gamma_1(x'')\}$ on S_1. Stereographic projection from $\Gamma_1(x')$ onto the corresponding equatorial plane of S_1 determines a complex coordinate s on $S_1 - \{\Gamma_1(x')\}$ in terms of which $r(s) = cs$ for some non-zero complex number c. The rotational freedom in the choice of l allows us to assume c real and positive.

Now choose an orthonormal basis $\{e_1, e_2, e_3\}$ for $\tilde{T}_1 = R^3$ such that $\Gamma_1(x')$ corresponds to e_3 and s corresponds to $x_1 + ix_2$ in the natural identification of $T_e(T_1)$ with $\tilde{T}_1$, where $\{x_1, x_2, x_3\}$ are the coordinates determined by this basis. To avoid confusion between the indices of the immersions and those of this basis, we relabel as follows: $\psi = \tilde{f}_1$, $\psi^c = \tilde{f}_3$, $\Gamma = \Gamma_1$ and $\Gamma^c = \Gamma_3$. We will show ψ and ψ^c coincide below. In doing so we make essential use of the fact that the immersions ψ and ψ^c induce the same quadratic differential Ω, have Gauss maps differing only by a real positive constant c (in the way described in the previous paragraph) and that the immersions project to the compact surface X. Here the Boundary Theorem is important; this says that the image of a compact orientable surface minimally immersed in a flat 3-torus is always a boundary. This was first proved by W. Meeks [5]; the right generalisation to arbitrary codimension is given in a forthcoming paper of the authors. Once we know that ψ and ψ^c coincide it follows then that $(\tilde{f}_2)_\theta$ projects to a conformal minimal immersion of X into a flat 3-torus which is equivalent to f_1. Thus, f_1 and f_2 are associates and this will complete the proof of Theorem 2.

LEMMA 8. *ψ and ψ^c coincide.*

Proof. Let z denote a local complex coordinate on X. The choice of coordinates on $\mathbf{R}^3$ gives us a triple $(d\phi/dz) = (\phi_1, \phi_2, \phi_3)$ of local holomorphic functions satisfying $\sum_{\alpha=1}^{3} \phi_\alpha^2 \equiv 0$. (cf. [9]); this is the condition that ψ be conformal. It can be checked that $g = s \circ \Gamma = \phi_3/(\phi_1 - i\phi_2)(= -(\phi_1 + i\phi_2/\phi_3)$ where Γ is the Gauss map of ψ and s denotes stereographic projection from the north pole $(0, 0, 1)$ of the unit sphere in $\mathbf{R}^3$. Write $f = \phi_1 - i\phi_2$. A simple calculation shows that $\Omega = f(dg/dz)\,dz^2$ locally. Analogous entities for the immersion ψ^c are denoted similarly except for the superscript c. Our information on the quadratic differentials tells us that $f^c(dg^c/dz) = f(dg/dz)$ and our information on the Gauss maps tells us that $g^c = c \cdot g$. Thus $c \cdot f^c = f$ or

$$c(\phi_1^c - i\phi_2^c) = \phi_1 - i\phi_2 \tag{1}$$

It follows immediately from the equation $g^c = cg$ and (1) that

$$\phi_3^c = \phi_3 \tag{2}$$

Using $g^c = cg$ once more and the second expression for g above we have

$$\phi_1^c + i\phi_2^c = c(\phi_1 + i\phi_2). \tag{3}$$

Solving for ϕ^c we find

$$\begin{aligned} \phi_1^c &= \cosh\ a\phi_1 + i \sinh a\phi_2 \\ \phi_2^c &= -i \sinh\ a\phi_1 + \cosh a\phi_2 \\ \phi_3^c &= \phi_3, \end{aligned} \tag{4}$$

where $c=e^a$. We can assume ψ and its conjugate surface $\hat{\psi}$ coincide and have the same tangent plane at $\tilde{x}_0 \in \tilde{X}$ and $\psi(\tilde{x}_0)=\hat{\psi}(\tilde{x}_0)=0$, the origin. The identity

$$\phi(z)\,dz = d(\psi + i\hat{\psi})$$

between differential triplets is easily derived and implies

$$\int_{\tilde{x}_0}^{\tilde{x}} \phi(z)\,dz = \psi(\tilde{x}) + i\hat{\psi}(\tilde{x}),$$

where the integral on the left is along any path from $\tilde{x}_0$ to $\tilde{x}$ on the simply connected surface $\tilde{X}$. Thus

$$\psi(\tilde{x}) = \mathrm{Re}\int_{\tilde{x}_0}^{\tilde{x}} \phi(z)\,dz \tag{5}$$

$$\hat{\psi}(\tilde{x}) = \mathrm{Im}\int_{\tilde{x}_0}^{\tilde{x}} \phi(z)\,dz$$

Applied to the system (4) the equations (5) yield

$$\begin{aligned} \psi_1^c &= \cosh a\psi_1 - \sinh a\hat{\psi}_2 \\ \psi_2^c &= \sinh a\hat{\psi}_1 + \cosh a\psi_2 \\ \psi_3^c &= \psi_3 \end{aligned} \tag{6}$$

Next we use the Boundary Theorem to show that $a=0$. Each $d\psi_\alpha$ (or $d\psi_\alpha^c$) defines a harmonic differential η_α (resp. η_α^c) on X. Note that $d\hat{\psi}_\alpha$ then defines the differential $J\eta_\alpha$ where J denotes the complex structure of X. From (6) we quickly see that

$$\begin{aligned} \eta_1^c \wedge \eta_2^c &= 2(\cosh^2 a + \sinh^2 a)\eta_1 \wedge \eta_2 \\ &\quad + \sinh a \cosh a(\eta_1 \wedge J\eta_1 + \eta_2 \wedge J\eta_2). \end{aligned} \tag{7}$$

But the Boundary Theorem tells us that $\int_X \eta_1^c \wedge \eta_2^c = \int_X \eta_1 \wedge \eta_2 = 0$. Since $\int_X \eta_\alpha \wedge J\eta_\alpha < 0$, it follows from (7) that $a=0$. Now (6) says that $\psi^c = \psi$ and Lemma 8 and Theorem 2 are completed.

5. The number of associates

Given a conformal minimal immersion $f: X \to T$ of a compact Riemann surface in a flat 3-torus, we inquire how many associates it has. Clearly we are only interested in counting inequivalent associates. We saw in Section 1 that the number is at most countable. We need only consider for what angles θ, $0<\theta<\pi$, the subspace $e^{i\theta}U$ (in the notation of Section 2) determines a subtorus of A. We show in Theorem 3 that if a non-trivial associate of f exists and the immersion f has a high degree of symmetry, in a sense defined below, then associates f_θ exist for a countable dense set of angles θ. This part of Theorem 3 overlaps a result announced by William Meeks in a summary of his doctoral thesis [5]. The other part of Theorem 3 gives useful information on the complex structure of X under the same assumptions of f; in particular, X must cover an elliptic curve.

Let G denote the group of proper (i.e. orientation preserving isometries of X, relative to the metric induced by f, which extends to isometries of T under the immersion f. Let G_0 denote the subgroup of isometries having a proper extension to T. Denote this representation of G_0 on the isometry group of T by μ. The universality of the Jacobi map a of the Riemann surface X provides us with a faithful representation ρ of G on the group of complex affine transformations of A, which is equivariant with respect to a. With the normalisations $a(x_0)=e$ and $f(x_0)=e$, we have $f=h\circ a$ for some homomorphism $h:A\to T$. Now for $\tau\in G_0$,

$$\begin{aligned}\mu(\tau)\circ h\circ a&=\mu(\tau)\circ f\\&=f\circ\tau\\&=h\circ a\circ\tau\\&=h\circ\rho(\tau)\circ a.\end{aligned}\tag{1}$$

So the affine maps $\mu(\tau)\circ h$ and $h\circ\rho(\tau)$ agree along the curve $a(X)$; but it is a well-known property of the Jacobi map that $a(X)$ generates the Jacobi variety A, so

$$\mu(\tau)\circ h=h\circ\rho(\tau).\tag{2}$$

We could also argue that Equation (1) implies that these two affine maps agree on first cohomology and so the maps differ by a translation which must in turn be trivial since $a(x_0)=e$. At any rate Equation (2) will have a couple of simple consequences for us. First, each transformation $\rho(\tau)$ preserves the foliation of A determined by the subspace $U=\operatorname{Ker} h_*$, that is, $\rho(\tau)$ permutes the leaves of the U-foliation. Since the $\rho(\tau)$ are complex, the same applies to the foliation determined by $U_\theta=e^{i\theta}U$. Thus $\rho(\tau)$ preserves the V-foliation on A, where $V=U\cap JU$. Second, since the elements $\tau\in G_0$ are isometries of X, the transformations $\rho(\tau)$ preserve the flat Kähler metric g_0 on A arising from the global inner-product of one-forms on X (see Section 2). Thus $\rho(\tau)$ preserves g_0 and the foliations determined by U and V; therefore it must also preserve the foliation determined by K, the g_0-orthogonal complement of V in U. Since $\rho(\tau)$ is complex, it must also preserve the JK-foliation. Now recalling the definition of the flat Kähler metric g_f in Section 2, and applying Equation (2), it follows that $\rho(\tau)$ preserves g_f.

First we will assume that

(i) *f has a non-trivial associate.* Then, by Lemma 2, V determines a complex subtorus of A. Since $\rho(G_0)$ acts isometrically on (A, g_f) preserving the V-foliation, ρ projects to a representation ρ' of G_0 on the group of complex isometries of the complex 3-torus $A' = A/V$ with the projected metric. Recall that f factors through the holomorphic immersion

$$a' : X \xrightarrow{a} A \xrightarrow{\text{proj}} A',$$

that is, $f = h' \circ a'$ where $h' : A' \to T$ is some homomorphism. Now a' is not universal (unless $p = 3$) and ρ' may not be faithful, but apart from this the situation for a' is the same as would obtain for a when X has genus 3. So to save ourselves the bother of setting up notation once more in A', *we proceed with the discussion assuming the genus is* 3, in which case $A' = A$; *for higher genus one just mimics the proof with $a' : X \to A'$ in place of the Jacobi map.*

Let μ_0 denote the representation of G_0 on the linear isometry group of $T_e(T)$ (or $\tilde{T}$) corresponding to the representation μ. We assume

(ii) *that f has irreducible symmetry,* by which we mean that μ_0 is an irreducible representation. But the finite rotation groups of 3-dimensional Euclidean space are well known to be either cyclic, dihedral, tetrahedral, octahedral or dodecahedral, and of these only the last three act irreducibly; however, by the "crystallographic restriction" a linear symmetry of a lattice in 3-space cannot be of order 5 and so the dodecahedral group is ruled out. Assumption (ii) then implies that $\mu_0(G_0)$ is the group of proper symmetries of a regular tetrahedron or a cube centred at the origin of $T_e(T)$. When necessary, we distinguish these two cases by saying that f has *tetrahedral symmetry* or *octahedral symmetry.*

By our assumption $\dim_C A = 3$ and so $\tilde{A}$ is the sum of the real 3-dimensional subspaces $\tilde{U}$ and $J\tilde{U}$ which are orthogonal, with respect to g_f. By Equation (2) and (ii) above, the linear representation ρ_0 corresponding to the affine representation ρ represents G_0 as the group of proper symmetries of a cube or regular tetrahedron in $J\tilde{U}$ with centre at the origin. Since each $\rho(\tau)$ is a complex isometry, the same remarks apply to $\rho_0(G_0)$ on the subspaces $\tilde{U}$ and $\tilde{U}_\theta$. As before, the lattice of A in $\tilde{A}$ is denoted Δ. The assumption (i) means that $\tilde{U}$ and $\tilde{U}_\theta$ project to subtori of A for some value of $\theta \neq 0 \bmod \pi$. Consequently, $\Delta_0 = \Delta \cap \tilde{U}$ and $\Delta_\theta = e^{-i\theta}(\Delta \cap \tilde{U}_\theta)$ are lattices in $\tilde{U}$ invariant by the group $\rho_0(G_0)$ acting on $\tilde{U}$;

here we are using also the fact that $\rho_0(G_0)$ preserves the lattice Δ in $\tilde{A}$. It follows from the next lemma that Δ_0 and Δ_θ each contain multiples of a fixed cubic lattice. We have not been able to find a reference for this very elementary fact, so we include a proof.

LEMMA 9. *Two lattices in Euclidean* 3*-space which are invariant by the octahedral* (*or tetrahedral*) *group must both contain multiples of a fixed cubic lattice.*

Proof. Whether the group N is octahedral or tetrahedral, it can be thought of as a subgroup of the proper symmetry group of the unit cube C centred at the origin $\mathcal{O}$. Let L be any lattice invariant by N and let Σ denote the smallest sphere centred at $\mathcal{O}$ which meets L. Note that $L \cap \Sigma$ cannot have 24 points (else there is a lattice point, other than $\mathcal{O}$, within Σ). When N is octahedral, it follows that N has non-trivial isotropy at each $l \in L \cap \Sigma$, and so the N-orbit of l is the set of vertices, the set of mid-edge points or the set of mid-face points of a cube parallel to C. Since L is N-invariant, it follows that L contains a cubic lattice with generators parallel to the edges of C. When N is tetrahedral, the same argument works, word for word, if we can find $l \in L \cap \Sigma$ whose orbit has less than 12 points. On the other hand, if every l has 12 points in its orbit, there can only be a single orbit (else $L \cap \Sigma$ has at least 24 points) in $L \cap \Sigma$. But $L \cap \Sigma$ is invariant by inversion, so for $l \in L \cap \Sigma$ there exists $\alpha \in N$ with $\alpha l = -l$, and consequently $\alpha^2 l = l$. Since the orbit of l has 12 points, it follows that $\alpha^2 = \mathrm{id}$. But considering the action of the elements of order two in the tetrahedral group, it follows that $L \cap \Sigma$ is the set of mid-face points of a cube parallel to C. In this case, L itself is a cubic lattice with generators parallel to the edges of C. The lemma now follows for any pair of N-invariant lattices.

So the lattice Δ_0 contains a cubic lattice Δ_0' and Δ_θ a cubic lattice Δ_θ' such that $\Delta_\theta' = r\,\Delta_0'$, for some positive real number r. Thus $e^{i\theta}\Delta_\theta' = re^{i\theta}\Delta_0'$ is a sublattice of $\Delta \cap \tilde{U}_\theta$ and, in particular, is contained in Δ. More explicitly, $r\cos\theta\delta + r\sin\theta J\delta \in \Delta$ for each $\delta \in \Delta_0'$. Using $\Delta_0' \subset \Delta$ it can be checked that, given any pair of integers (m, n) the angle θ' $(0 < \theta' < \pi)$ satisfying

$$\tan\theta' = \frac{mr\sin\theta}{mr\cos\theta + n}$$

has also the property that

$$r'\cos\theta'\delta + r'\sin\theta' J\delta \in \Delta$$

for each $\delta \in \Delta_0'$, where r' is some real number. Therefore $\tilde{U}_{\theta'}$ projects to a subtorus of A and so $\tilde{f}_{\theta'}$ projects to a conformal minimal immersion of X into a flat 3-torus. Clearly the set of all such θ' is a countable dense subset of the set of all angles. When $r \cos \theta$ is rational, it follows that the conjugate $\tilde{f}_{\pi/2}$ projects to X.

There is yet another conclusion to be drawn when (i) and (ii) hold, this time on the complex structure of X.

Let $\{\delta_1, \delta_2, \delta_3\}$ be an orthonormal basis of the cubic lattice Δ_0'. Let C_α stand for the complex line through the origin of $\tilde{A}$ spanned by $\{\delta_\alpha, J\delta_\alpha\}$. From the above $C_\alpha \cap \Delta$ is a lattice in C_α. We also saw above that $re^{i\theta}\delta_\alpha \in C_\alpha \cap \Delta$. Let Δ_α denote the lattice in C_α spanned by $\{\delta_\alpha, re^{i\theta}\delta_\alpha\}$. Now the natural map

$$C_1 \times C_2 \times C_3 \to \tilde{A}$$

carries the lattice $\Delta_1 \times \Delta_2 \times \Delta_3$ into Δ. Thus it induces a holomorphic homomorphism with finite kernel (or *isogeny* in the language of abelian varieties) of $T_1 \times T_2 \times T_3$ onto A, where T_α is the complex one-torus C_α/Δ_α.

The construction of Δ_0' is such that $\rho_*(G_0)$ contains a complex linear transformation mapping δ_1 to δ_2. This transformation identifies (C_1, Δ_1) with (C_2, Δ_2). Hence T_1 and T_2 are isomorphic. Thus, in the case of genus 3, the Jacobi variety A is isogenous to $T_1 \times T_1 \times T_1$ for some complex torus T_1. For arbitrary genus the same will be true of $A' = A/V$. We have shown

THEOREM 3. *Let $f: X \to T$ be a conformal minimal immersion of a compact Riemann surface X in a flat real 3-torus T. Suppose that*

(i) *f has a non-trivial associate,*

and (ii) *f has irreducible symmetry.*

Then (a) *the associates $\tilde{f}_\theta$ of $\tilde{f}$ (a lift of f) project to conformal minimal immersions of X in flat real* 3 *tori, for a countable dense subset of angles in $0 \leqq \theta < \pi$. So f has countably many inequivalent associates.*

(b) *f factors by a homomorphism through a holomorphic immersion of X in a* 3*-dimensional abelian variety A' which is isogenous to $T_1 \times T_1 \times T_1$, for some complex* 1*-torus T_1.*

In particular, from (b), *X covers an elliptic curve.*

Remarks. (i) Essentially the same result was obtained independently by W. Meeks [5].

(ii) Theorem 3 extends to minimal surfaces in n-tori; the proof will appear in a separate work.

6. The hyperelliptic case

A compact Riemann surface X of genus $p > 1$ is called *hyperelliptic* if it admits a meromorphic function $\pi: X \to S$ of degree 2; S denotes the Riemann sphere.

The transformation θ which exchanges the points of X where π takes equal values is holomorphic; θ is of order 2 (involutive) and has $(2p+2)$ fixed points by the Riemann-Hurwitz formula. The transformation θ is called the *hyperelliptic automorphism* of X and is the only involutive automorphism of X having $2p+2$ fixed points (cf. Gunning [1]).

We showed in [6] that if a hyperelliptic Riemann surface X admits a conformal minimal immersion in a flat 3-torus, then p is odd, and–more dramatic–if it admits such an imbedding then its genus must be 3; conversely, every Riemann surface of genus 3 admitting such an immersion is hyperelliptic. On the other hand, the method of constructing examples of compact minimal surfaces in tori by Weyl groups (see [8]) tends to give examples with a considerable amount of symmetry. In this section, we show that hyperelliptic minimal surfaces which have irreducible symmetry have a further restriction on their genera. This is

THEOREM 4. *Let X be a compact hyperelliptic Riemann surface of genus p and let $f:X \to T$ be a conformal minimal immersion of X in a flat 3-torus. If f has irreducible symmetry then $p = 3 \bmod 4$.*

Proof. Normalising f by $f(x_0)=e$, where x_0 is one of the fixed points of the hyperelliptic automorphism θ, we recall from [6] that θ extends to inversion in the identity element e of T. It follows easily that $\Gamma \circ \theta = \Gamma$ where Γ denotes the Gauss map of X onto the unit sphere S_0 in $T_e(T)$. Since Γ is holomorphic (with respect to one of the two complex structures on S_0) there exists a holomorphic function r completing the diagram

$$\begin{tikzcd} X \arrow[r, "\Gamma"] \arrow[d, "\pi"'] & S_0 \\ S \arrow[ur, "r"'] & \end{tikzcd}$$

The extension of θ to T has 8 fixed points, so it follows that when f is an imbedding θ itself has at most 8 fixed points. This then implies $2p+2 \leqq 8$, leaving $p=2$ and $p=3$ as the only possibilities. By Lemma 1, $p \neq 2$, so we have $p=3$ when f is an imbedding. We also showed in [6] that p must be odd. In fact, if B is the total number of branch points of r we have

$$\begin{aligned} p-1 &= \text{degree}\, \Gamma, \text{ by Gauss-Bonnet,} \\ &= 2 \text{ degree } r, \text{ by the diagram above,} \\ &= 2\left(1+\frac{B}{2}\right), \text{ by Riemann-Hurwitz,} \\ &= 2+B. \end{aligned}$$

So $p=3+B$. It remains only to show that the symmetry assumption of Theorem 4 implies that B is a multiple of 4.

Let G denote the group of all proper (i.e. orientation-preserving) isometries of X – relative to the induced metric – extending under f to isometries of T. Then G contains the hyperelliptic automorphism θ. Let G_0 denote the index 2 subgroup of those elements of G having proper extensions. Let μ_0 denote the associated special orthogonal representation of G_0 on the unit sphere S_0 in $T_e(T)$. Clearly

$$\Gamma \circ \tau = \mu_0(\tau) \circ \Gamma$$

for all $\tau \in G_0$, where Γ denotes the Gauss map of f. By uniqueness of the hyperelliptic automorphism, $\theta \circ \tau = \tau \circ \theta$ for all $\tau \in G_0$, and we then have another representation μ_1 of G_0 on the automorphism group of the Riemann sphere $S = X/\{\theta\}$ and this action will be effective (i.e. $\mu_1(\tau)$ is the identity automorphism of S only if τ is the identity in G_0). However the representation μ_0 might not be effective. The most we can say is that if $\mu_0(\tau)=\mathrm{id}$ (i.e. τ extends to a translation of T) then τ is the identity if it has a fixed point on X; but since $\mu_1(\tau)$ must have a fixed point on S it follows that τ or $\tau \circ \theta$ has a fixed point on X and therefore that τ or $\tau^2 = (\tau \circ \theta)^2$ has a fixed point on X; by the previous remark, it follows that $\mu_0(\tau)=\mathrm{id}$ always implies $\tau^2=\mathrm{id}$. On projecting the above equivariance law to S we obtain

$$r \circ \mu_1(\tau) = \mu_0(\tau) \circ r$$

for all $\tau \in G_0$. Suppose $z_1 \in S$ is a branch point of r of order l. Then the $\mu_1(G_0)$-orbit of z_1 consists of branch points of r of the same order. Letting H denote the isotropy group of the $\mu_1(G_0)$-action at z_1 and denoting its order by m, the total number of branch points of r in the orbit is $(l/m)\,|G_0|$, where $|G_0|$ denotes the order of G_0.

LEMMA 10. *H is cyclic of order $m=1$, 2, 3, 4 or 6.*

Proof. Choose any metric on S invariant by the μ_1-action of the finite group G_0. The isotropy and linear isotropy groups are isomorphic for all isometric action and so the isotropy group H at z_1 is a cyclic group of order m. Let τ be a generator of H. The assumption of irreducible symmetry tells us that $\mu_0(G_0)$ is the proper symmetry group of either a cube or a tetrahedron, in particular $\mu_0(\tau)$ has order $m_0=1$, 2, 3 or 4. Since $\mu_1(\tau)$ fixes z_1 it follows that either τ or $\tau \circ \theta$ must fix $x_1 \in X$, where $x_1 \in \pi^{-1}(z_1)$. If τ fixes x_1 so also does τ^{m_0}, and since τ^{m_0} extends to a translation of T we must have $\tau^{m_0}=\mathrm{id}$. Suppose instead that $\tau \circ \theta$ fixes x_1. Then if m_0 is even (i.e. $m_0=2$ or 4) it still follows that $\tau^{m_0}=(\tau^2)^{n_0}=((\tau \circ \theta)^2)^{n_0}$ fixes x_1 and consequently that $\tau^{m_0} = \mathrm{id}$. On the other hand if m_0 is odd (i.e. $m_0 = 1$ or 3), the remarks of the previous paragraph imply that $\tau^{m_0} \neq \mathrm{id}$ but $(\tau^{m_0})^2 = \mathrm{id}$. Thus $m = m_0$ except perhaps when $m_0 = 1$ or 3, and then $m = 2$ or 6. This ends the proof.

Let $\nu: S_0 \to S$ be any holomorphic transformation such that $\nu \circ r(z_1) = z_1$ and set $k = \nu \circ r$. The map ν transforms the representation μ_0 into another representation μ_2 of G_0 on the automorphism group of S. The old equivariance now appears as

$$k \circ \mu_1(\tau) = \mu_2(\tau) \circ k$$

for all $\tau \in G_0$. Now the generator τ of H chosen above may be assumed to be such that $\mu_1(\tau)$ acts on $T_{z_1}(S)$ as multiplication by $w_1 = e^{2\pi i/m}$ and then $\mu_2(\tau)$ must act as multiplication by $w_2 = e^{2\pi i s/m}$ where $0 \leqq s < m$. Choosing a complex coordinate on S with origin at z_1, then after differentiating the above equivariance law and evaluating at z_1, we obtain $w_1^{l+1} = w_2$ or

$$\frac{l}{m} = \frac{s-1}{m} + \alpha$$

for some non-negative integer α.

If s is relatively prime to m, then $\mu_0(\tau)$ has order m and so $m = 1$, 2, 3 or 4. We can very quickly see that l/m must then be of the form α or $\frac{1}{3} + \alpha$, when $m = 1$, 2 or 3 and of the form α or $\frac{1}{2} + \alpha$ when $m = 4$.

If s is not relatively prime to m, then we see from the proof of Lemma 10 that $m = 2$ or 6 and it again follows quickly that l/m must be of the form $\frac{1}{2} + \alpha$ or $\frac{1}{6} + \alpha$. Now we know that $|G_0|$ is a multiple of $|\mu_0(G_0)|$ in general. But when s is not relatively prime to m we also note from the proof of Lemma 10 that G_0 contains an element of order 2 extending to a translation of T; thus $|G_0|$ is a multiple of $2\,|\mu_0(G_0)|$ in this case.

We leave it to the reader to check that whether the immersion f has octahedral symmetry (i.e. $|\mu(G_0)| = 24$) or tetrahedral symmetry (i.e. $|\mu(G_0)| = 12$, $m \neq 4$), the total number of branch points in the orbit of z_1, namely $(l/m)\,|G_0|$, is divisible by 4. Hence the total number of branch points B is divisible by 4 and so $p = 3 \bmod 4$. This ends the proof of Theorem 4.

Remark. If f has octahedral symmetry, we can see from the above that $B = 0$ or else $B \geqq 8$. Thus $p = 3$ or else $p \geqq 11$. The classical periodic minimal surfaces of H. A. Schwarz determine five singularity-free compact orientable minimal surfaces in flat 3-tori with octahedral symmetry (see §7); their genera are all less than 11, so only in genus 3 have we hyperellipticity.

7. Remarks on examples

We add a few remarks on some examples of compact minimal surfaces in 3-tori to show the viability of the hypotheses in the results above and also some implications, especially of Theorem 2.

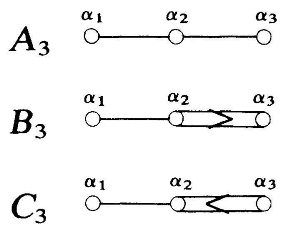
Perhaps it is well to note first of all that there are infinitely many geometrically distinct periodic minimal surfaces in R^n, for each n. This is implicit in the generation of such surfaces from Weyl groups [8]. Furthermore there is no difficulty in showing that there are infinitely many compact minimal surfaces in 3-tori satisfying the symmetry condition of Theorem 3.

The method of H. A. Schwarz for generating periodic minimal surfaces in R^3 goes as follows: select a skew (i.e. non-planar) r-gon P having affine projection onto some 2-plane which carries P bijectively onto the boundary of a convex domain D in this plane; the Plateau problem for P has then a unique solution Σ with no interior branch points (cf. [4]); let S denote the group of proper affine motions generated by the r reflexions in the edges of P; by the Schwarz reflexion principle (cf. [4]) $M = S \cdot \Sigma = \bigcup_{s \in S} s\Sigma$ is a minimal surface whose only possible branch points occur on the vertices of the 1-complex $S \cdot P$; when S is discrete and its maximal translational subgroup L is a lattice in R^3, the surface M is called triply-periodic. We will not be concerned with the case where M has branch points.

We next show how P then determines a canonical compact orientable surface X and a canonical minimal immersion of it in the flat 3-torus $T = R^3/L$. First consider the direct product $S \times \Sigma$ with the obvious equivalence relation (denoted $\sim$) which assembles $S \times \Sigma$ as M in R^3 (cf. [8]). Let $X' = S \times \Sigma/\sim$ denote the quotient connected surface and f' the obvious immersion of X' in R^3. The left action of S on itself defines an action of S on X' which is equivariant with respect to f'. We obtain then a minimal immersion f_0 of the compact surface $X_0 = X'/L$ in $T = R^3/L$. The action of the finite group $K = S/L$ on X_0 and T is equivariant with respect to f_0. Passing to the 2-fold cover if necessary, we obtain a compact orientable surface X and a minimal immersion $f: X \to T$. The action of K lifts also to X and is likewise equivariant for f and isometric with respect to the induced metric on X. With the induced conformal structure and a choice of orientation, X becomes a Riemann surface: apart from this trivial choice, the Riemann surface X, the immersion f and the torus T are cononically determined by P.

The genus p of X is easily computed. By the Gauss-Bonnet formula the total curvature of Σ is $\sigma = 2\pi - s$, where s is the sum of the exterior angles in the polygon P. The Gauss-Bonnet formula for the surface X states that $4\pi(1-p)$ equals $|K|\,\sigma$ or $2\,|K|\,\sigma$ according as X_0 is orientable or not.

When $r=4$, i.e. the polygon P is a quadrangle, the convexity condition on P is guaranteed. Schwarz considered the problem of determining all P which generate triply-periodic minimal surfaces. Acutally if Σ contains interior straight-line segments, we can replace P by another skew quadrangle generating the same periodic minimal surface and having no interior straight-line segments. With this latter normalisation on P there are to within congruence only six possible quadrangles P which generate triply-periodic minimal surfaces. The result is essentially due to Schwarz [11], but Schoenflies [10] later gave a very nice proof. These six quadrangles are pictured in Stessmann [12] and we follow his numbering, I–VI. These six quadrangles are the simplest polygons that can be made from the roots of a simple Lie group of rank 3. Indeed, if for each of the Dynkin diagrams of such groups we take the quadrangles whose first three edges are the



vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ (in that order) we obtain I, II and V; for the choice $\{\alpha_1, \alpha_3, \alpha_2\}$ we obtain III, VI, and IV. Actually the surface IV has branch points and we set it aside in the remarks to follow. In each case the group K is the proper octahedral group (order 24). Apart from Case III, (i) L is the maximal translation group leaving the set M invariant, (ii) X' is orientable and (iii) L contains elements reversing the orientation of X'; so X_0 is non-orientable and by the previous paragraph the genus p is 3, 9, 6 and 7 for the surface X arising from the polygons I, II, V and VI, respectively. In Case III, X' is orientable, but L contains only orientation-preserving translations so X_0 is orientable; so in this case the genus works out as 3.

The surfaces X arising in Cases I and III are hyperelliptic because the Gauss map gives a meromorphic function of degree 2 [1]. In fact in both cases this function is ramified over the 8 vertices $\{\alpha_j\}_{j=1}^{8}$ of a cube inscribed in the sphere. Thus X is the Riemann surface of

$$\sqrt{z^8+14z^4+1}=\left(\prod_{j=1}^{8}(z-\alpha_j)\right)^{1/2}$$

in both cases. By Theorem 2, the immersion of X arising from I and III are associate, because X has genus 3; it is not difficult to see they are not trivial associates. Both immersions have octahedral symmetry, so X satisfies the hypotheses of Theorem 3; in particular there are infinitely many associates.

Comparing second fundamental forms at a well-chosen point of X we quickly see that the immersions of X arising from I and III are even conjugate, as was very ingeniously proved by Neovius with other methods more than ninety years ago [14]. To him we owe also the construction of the conjugate of II; it is also triply-periodic and distinct from the other surfaces mentioned here (see Tafel IV [14]). It gives a further example, this time of genus 9, of a compact surface satisfying the hypotheses of Theorem 3.

As a further application of Theorem 2, we have the following result of W. Meeks.

COROLLARY. *Let $f: X \to T$ be a conformal minimal immerion of a compact Riemann surface of genus 3 into a flat 3-torus. If the isometry group of the induced metric on X contains the octahedral group as a subgroup, then X is the Schwarz surface of genus 3 and f is an associate of the Schwarz immersions of this surface* (Cases I and III above).

The condition on the isometry group can be used to show the Gauss map branches over the 8 vertices of a cube inscribed in the unit sphere. As we saw above this determines that X is the Schwarz surface of genus 3. Finally, Theorem 2 says that all such immersions of this surface are associate.

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