

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 53 (1978)

Artikel: On the decomposition of a class of plane quasiconformal mappings.
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DOI: <https://doi.org/10.5169/seals-40752>

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On the decomposition of a class of plane quasiconformal mappings

EDGAR REICH⁽¹⁾

Dedicated to Albert Pfluger on the occasion of his seventieth birthday

1. Introduction

Let Q_I denote the class of quasiconformal mappings $w = f(z)$ of the unit disk $U = \{|z| < 1\}$ onto $U = \{|w| < 1\}$ with the property that the boundary values of f are those of the identity:

$$f(e^{i\theta}) \equiv e^{i\theta}, \quad 0 \leq \theta < 2\pi.$$

If $f \in Q_I$, then the complex dilatation μ of f ,

$$\mu(z) = f_{\bar{z}}/f_z$$

will be said to belong to class \mathcal{F} . The maximal dilatation of f is

$$K[f] = \frac{1 + k[f]}{1 - k[f]}, \quad k[f] = \operatorname{ess\,sup}_{z \in U} |\mu(z)| = \|\mu\|_\infty.$$

To avoid triviality we assume $k[f] > 0$.

We will be concerned with certain questions related to the possibility of *decomposing* a given $f \in Q_I$ into factors

$$f = f_2 \circ f_1, \quad f_i \in Q_I, \quad K[f_i] < K[f], \quad i = 1, 2. \quad (1.1)$$

In particular, we will see that a decomposition satisfying (1.1) always exists. This should be contrasted with the known fact [4, pp. 215–216] that from the

¹ Work done with support from National Science Foundation grant MCS 75-06656A01.

assumptions that $\mu \in \mathcal{F}$ and $0 < t < 1$ it does *not* follow that $t\mu \in \mathcal{F}$. Thus, the attempt to construct f_i , $i = 1, 2$, as⁽²⁾

$$f_i = f^{t_i, \mu}, \quad 0 < t_i < 1, \quad i = 1, 2,$$

does *not* work. In fact, as shown by Gehring [2], a decomposition of f of type (1.1), where $K[f_i] = K^{1/2}$, $i = 1, 2$, does not necessarily exist.

In Theorem 1 the role played by f_1 will be a mapping

$$f_1(z) = h(z, t), \quad z \in U, \quad 0 \leq t \leq t_0(k), \quad k = k[f], \quad (1.2)$$

close to the identity, and depending on the positive parameter t . In this case f_2 assumes the role of a *variation* \tilde{f} of the mapping f ,

$$\tilde{f}(t, z) = f \circ h^{-1}, \quad 0 \leq t \leq t_0(k), \quad \tilde{f}(0, z) = f(z). \quad (1.3)$$

The construction of $h(z, t)$ for Theorem 1 requires the application of the Hahn-Banach theorem. On the other hand the process is sufficiently constructive so that the dilatations of all mappings involved are capable of being estimated explicitly in terms of k . An alternative construction of the f_i , avoiding the Hahn-Banach theorem, is found in Section 4. The resulting Theorem 3 has the advantage of leading to a very simple estimate for $K[f_i]$, but the disadvantage that a bound on $K[f]$ is assumed.

As a corollary of Theorems 2 and 3 a potentially quantitative version of a result of Earle and Eells [1] on the decomposition of $f \in Q_I$ into $(1 + \varepsilon)$ -quasiconformal mappings f_i ,

$$f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1, \quad f_i \in Q_I, \quad (1.4)$$

is obtained (Section 3).

A decomposition of type (1.4) give rise to an *interpolating chain* $\mathcal{C} = \{F_i\}$ for f within Q_I ,

$$F_0(z) = z, \quad F_n(z) = f(z), \quad F_i = f_i \circ f_{i-1} \circ \cdots \circ f_1, \quad i = 1, 2, \dots, n,$$

which connects f to the identity. If

$$K[F_i \circ F_{i-1}^{-1}] < 1 + \varepsilon, \quad i = 1, 2, \dots, n,$$

² f^κ denotes the quasiconformal mapping of U onto U with complex dilatation $\kappa(z)$, normalized so that $f^\kappa(1) = 1$, $f^\kappa(i) = i$, $f^\kappa(-1) = -1$.

we shall say that \mathcal{C} has *link size* ε . We define

$$R(\mathcal{C}) = \max_i K[F_i].$$

In Section 5 we shall see that $R(\mathcal{C})$ can be bounded in terms of $K[f]$ alone.

Before proceeding we must list a number of known results and an immediate corollary of one of them for reference later. In what follows

$$\|\varphi\| = \iint_U |\varphi(z)| \, dx \, dy,$$

and \mathcal{B} denotes the Banach space of functions $\varphi(z)$ holomorphic in U , $\|\varphi\| < \infty$. \mathcal{N} will be the class of all complex valued measurable functions $\nu(z)$, $z \in U$, such that

$$\|\nu\|_\infty < \infty, \quad \iint_U \nu(z) \varphi(z) \, dx \, dy = 0 \quad \text{for all } \varphi \in \mathcal{B}.$$

For a given quasiconformal mapping g of U onto U , Q_g denotes the class of quasiconformal mappings of U onto U which agree with g on ∂U . Each class Q_g contains at least one *extremal* member, G , in the sense that $K[G]$ is minimal.

We recall the following ([4], [6]):

THEOREM A. *If $\mu \in \mathcal{F}$ then, for any function $\varphi \in \mathcal{B}$,*

$$\left| \iint_U \frac{\mu(z) \varphi(z)}{1 - |\mu(z)|^2} \, dx \, dy \right| \leq \iint_U \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} |\varphi(z)| \, dx \, dy. \quad (1.5)$$

THEOREM B. *Suppose g is a quasiconformal mapping of U onto U with complex dilatation $\kappa(z)$. If G is an extremal mapping in Q_g , $K[G] = (1 + k^*)/(1 - k^*)$, then*

$$\frac{k^*}{1 - k^*} \leq I[\kappa] + \Delta[\kappa], \quad (1.6)$$

where

$$I[\kappa] = \sup_{\left\{ \begin{array}{l} \varphi \in \mathfrak{B} \\ \|\varphi\| \leq 1 \end{array} \right\}} \left| \iint_U \frac{\kappa(z)\varphi(z)}{1-|\kappa(z)|^2} dx dy \right|, \quad (1.7)$$

and

$$\Delta[\kappa] = \sup_{\left\{ \begin{array}{l} \varphi \in \mathfrak{B} \\ \|\varphi\| \leq 1 \end{array} \right\}} \iint_U \frac{|\kappa(z)|^2}{1-|\kappa(z)|^2} |\varphi(z)| dx dy. \quad (1.8)$$

If $\kappa(z) = t\nu(z)$, $0 \leq t < 1/\|\nu\|_\infty$, then

$$\iint_U \frac{\kappa\varphi}{1-|\kappa|^2} dx dy = t \iint_U \nu\varphi dx dy + \iint_U \frac{\kappa|\kappa|^2}{1-|\kappa|^2} \varphi dx dy.$$

If $\nu \in \mathcal{N}$, then the first integral on the right hand side vanishes, and we obtain

$$I[\kappa] \leq \frac{t^3 \|\nu\|_\infty^3}{1-t^2 \|\nu\|_\infty^2}, \quad \Delta[\kappa] \leq \frac{t^2 \|\nu\|_\infty^2}{1-t^2 \|\nu\|_\infty^2}.$$

As a corollary of Theorem B, we therefore have the following.

THEOREM C. Suppose $\nu \in \mathcal{N}$, $0 \leq t < 1/\|\nu\|_\infty$, and suppose $g = f^\nu$. If G is an extremal mapping in Q_g , $K[G] = (1+k^*)/(1-k^*)$, then

$$\frac{k^*}{1-k^*} \leq \frac{t^2 \|\nu\|_\infty^2}{1-t \|\nu\|_\infty}. \quad (1.9)$$

2. Variation of f in the class Q_I .

Following the notation of Section 1, we will prove the following.

THEOREM 1. There exist functions $\delta(k) > 0$, $t_0(k) > 0$, and $C(k)$, defined for $0 < k < 1$, with the following properties. If $f \in Q_I$, $\|\mu\|_\infty = k = (K-1)/(K+1)$, $0 \leq t \leq t_0(k)$, then there exists a mapping $h(z, t) \in Q_I$ such that

$$K[h] \leq 1 + C(k)t \quad (2.1)$$

and

$$K[f \circ h^{-1}] \leq K - \delta(k)t. \quad (2.2)$$

Proof. The expression

$$L_\mu[\varphi] = \iint_U \frac{\mu}{1-|\mu|^2} \varphi \, dx \, dy$$

defines a bounded linear functional over \mathcal{B} . By the Hahn-Banach and Riesz representation theorems⁽³⁾ there exists a complex valued measurable function $\tau(z)$, $z \in U$, such that

$$\|\tau\|_\infty = \sup_{z \in U} |\tau(z)| = \|L_\mu\|,$$

and

$$\iint_U \frac{\mu}{1-|\mu|^2} \varphi \, dx \, dy = \iint_U \tau \varphi \, dx \, dy, \quad \text{for all } \varphi \in \mathcal{B}.$$

Hence,

$$\nu(z) = \frac{\mu(z)}{1-|\mu(z)|^2} - \tau(z) \in \mathcal{N}, \quad (2.3)$$

while, by Theorem A,

$$\|\tau\|_\infty = \|L_\mu\| \leq \frac{k^2}{1-k^2}. \quad (2.4)$$

Let

$$g_1 = f^t, \quad g_2 = f \circ g_1^{-1}, \quad \left(0 \leq t < \frac{1}{\|\nu\|_\infty}\right). \quad (2.5)$$

³ Applications of the Hahn-Banach theorem in closely related situations can be found in [3] and [5].

The complex dilatation of g_2 is

$$\mu_{g_2}(\zeta) = \frac{\mu(z) - t\nu(z)}{1 - t\nu(z)\overline{\mu(z)}} \frac{g_{1z}}{\overline{g_{1z}}}, \quad (\zeta = g_1(z)). \quad (2.6)$$

We will show that there exist $\delta'(k) > 0$, and $t'_0(k) > 0$, such that

$$|\mu_{g_2}(\zeta)| \leq k - \delta'(k)t, \quad 0 \leq t \leq t'_0(k), \quad z \in U, \quad (\zeta = g_1(z)). \quad (2.7)$$

Let $\alpha = \alpha(k)$, $0 < \alpha(k) < 1$, be the solution of the equation

$$\frac{\alpha}{1 - \alpha^2} = \frac{1}{2} \left(\frac{k^2}{1 - k^2} + \frac{k}{1 - k^2} \right) = \frac{k}{2(1 - k)}. \quad (2.8)$$

Let

$$S_1 = \{z \in U : |\mu(z)| \leq \alpha\},$$

$$S_2 = \{z \in U : \alpha < |\mu(z)| \leq k\}.$$

Since $|\mu(z)| \leq \alpha < k$ for $z \in S_1$, it is obvious from (2.6) that there exist $\delta_1(k) > 0$, $t_1(k) > 0$, such that

$$|\mu_{g_2}(\zeta)| \leq k - \delta_1 t, \quad 0 \leq t \leq t_1, \quad z \in S_1. \quad (2.9)$$

By (2.6),

$$|\mu_{g_2}(\zeta)|^2 = \frac{|\mu|^2 - 2t \operatorname{Re}(\nu \bar{\mu}) + t^2 |\nu|^2}{1 - 2t \operatorname{Re}(\nu \bar{\mu}) + t^2 |\nu|^2 |\mu|^2}.$$

Therefore, for $z \in S_2$, we have the development

$$|\mu_{g_2}(\zeta)| = |\mu(z)| - t \frac{1 - |\mu(z)|^2}{|\mu(z)|} \operatorname{Re}[\nu(z) \overline{\mu(z)}] + 0(t^2), \quad (2.10)$$

where the $0(t^2)$ term is uniform both with respect to z and to k , providing k is bounded away from 1. By (2.3),

$$\operatorname{Re}(\nu \bar{\mu}) = \operatorname{Re} \left[\frac{|\mu|^2}{1 - |\mu|^2} - \tau \bar{\mu} \right] \geq \frac{|\mu|^2}{1 - |\mu|^2} - |\tau| |\mu| = |\mu| \left[\frac{|\mu|}{1 - |\mu|^2} - |\tau| \right].$$

Therefore, by (2.4), (2.8), the coefficient of $-t$ in (2.10) is bounded below by

$$|\mu| \left[1 - \frac{1 - |\mu|^2}{|\mu|} |\tau| \right] \geq \alpha \left[1 - \frac{1 - \alpha^2}{\alpha} \frac{k^2}{1 - k^2} \right] = \left(\frac{1 - k}{1 + k} \right) \alpha, \quad z \in S_2.$$

Hence, there exist $\delta_2(k) > 0$, $t_2(k) > 0$, such that

$$|\mu_{g_2}(\zeta)| \leq k - \delta_2 t, \quad 0 \leq t \leq t_2, \quad z \in S_2.$$

Taking $\delta'(k) = \min(\delta_1, \delta_2)$, $t'_0(k) = \min(t_1, t_2)$, therefore establishes (2.7).

Next, we correct for the fact that g_1 does not necessarily belong to Q_I . By Theorem C, only a relatively small correction is required. Namely, let G_1 be extremal for the class Q_{g_1} . In view of (2.3), (2.4),

$$\|\nu\|_\infty \leq \frac{k}{1 - k}. \quad (2.11)$$

Thus, Theorem C provides an estimate of the type

$$K[G_1] \leq 1 + C'(k)t^2, \quad 0 \leq t \leq \frac{1 - k}{2k}. \quad (2.12)$$

The mapping

$$h = G_1^{-1} \circ g_1 \quad (2.13)$$

evidently belongs to Q_I , and in view of (2.5), (2.11), and (2.12), it has the desired property (2.1) when $\delta(k)$, $t_0(k)$, $C(k)$ are chosen appropriately. On the other hand,

$$\tilde{f} = f \circ h^{-1} = g_2 \circ G_1.$$

Thus, by employing (2.7) and (2.12), after possibly modifying $\delta(k)$, $t_0(k)$, $C(k)$, we obtain (2.2).

3. Decomposition of f

In Theorem 1 we may choose t as some specific value, say $t = \min[t_0(k), (K - 1)/(2C(k))]$. As is easily seen by following the computations of

Section 2, the functions $t_0(k)$, $\delta(k)$, and $C(k)$ occurring in assertions (2.1) and (2.2) can be chosen as *continuous* functions of k , $0 \leq k < 1$. Writing $h = f_1$, $f \circ h^{-1} = f_2$, we can therefore assert the following.

THEOREM 2. *There exists a function $\Phi(K)$, defined for $1 \leq K < \infty$, with the following properties:*

$$(i) \quad \Phi(K) \text{ is continuous, } 1 \leq K < \infty, \quad \Phi(1) = 1, \quad (3.1)$$

$$(ii) \quad \Phi(K) < K, \quad 1 < K < \infty, \quad (3.2)$$

$$(iii) \quad \text{If } f \in Q_I, \quad K[f] = K, \text{ then there exist } f_1 \in Q_I, \quad f_2 \in Q_I, \text{ such that } f = f_2 \circ f_1, \\ K[f_i] \leq \Phi(K), \quad i = 1, 2. \quad (3.3)$$

Suppose now the decomposition process referred to in (3.3) is iterated. Stage j results in a decomposition of f into 2^j factors each having a maximal dilatation not more than $\Phi_j(K[f])$, where

$$\Phi_1(x) = \Phi(x), \quad \Phi_{j+1}(x) = \Phi(\Phi_j(x)), \quad j = 1, 2, \dots,$$

As a consequence of (3.2)

$$\Phi_{j+1}(x) \leq \Phi_j(x), \quad j = 1, 2, \dots,$$

and, with the help of (3.1), it therefore follows that

$$\lim_{j \rightarrow \infty} \Phi_j(x) = 1, \quad 1 \leq x < \infty.$$

Thus we see that *an interpolating chain $\mathcal{C} = \{F_i\}$ connecting f to the identity within Q_I , with link size ε , exists⁴ [1].*

It would be straightforward to convert the above to an estimate of the value of $n = N(K, \varepsilon)$ required to achieve the decomposition (1.4).

4. An alternative decomposition algorithm

We will now indicate a more symmetric and somewhat more elementary procedure for arriving at the factors f_1, f_2 of Theorem 2. However, the success of

⁴ The desirability of approaching the Earle–Eells results in this fashion occurred to the author as a sequel to an oral communication from Professor A. Marden whom he would also like to thank for helpful remarks.

the procedure will be guaranteed only if $K[f]$ is sufficiently small. From Section 5 it appears, on the other hand, that under certain circumstances it may be useful to apply the decomposition process of Theorem 2 *in conjunction* with that of Theorem 3.

THEOREM 3. *If $K[f] < M = (3 + \sqrt{5})/2 = 2.61803398\dots$, then the assertions of Theorem 2 hold with*

$$\Phi(x) = x^{3/2} - x + 1, \quad 1 \leq x < M. \quad (4.1)$$

Proof. Let $g_1(z)$ be a quasiconformal mapping of U onto U with complex dilatation

$$\mu_1(z) = t\mu(z), \quad t = \frac{K+1}{(K^{1/2}+1)^2}.$$

This makes

$$K[g_1] = \frac{1+tk}{1-tk} = K^{1/2}. \quad (4.2)$$

Let $g_2 = f \circ g_1^{-1}$. The complex dilatation μ_2 of g_2 is

$$\mu_2 \circ g_1(z) = \frac{\mu(z) - \mu_1(z)}{1 - \overline{\mu(z)}\mu_1(z)} \frac{g_{1z}}{g_{1\bar{z}}} = \frac{(1-t)\mu(z)}{1-t|\mu(z)|^2} \frac{g_{1z}}{g_{1\bar{z}}}.$$

Hence,

$$\|\mu_2\|_\infty = \frac{(1-t)k}{1-tk^2}, \quad \frac{1+\|\mu_2\|_\infty}{1-\|\mu_2\|_\infty} = K^{1/2}; \quad (4.3)$$

that is,

$$f = g_2 \circ g_1, \quad K[g_i] = K^{1/2}, \quad i = 1, 2, \dots \quad (4.4)$$

As in Section 2, we correct for the fact that g_1 need not belong to Q_I by introducing an extremal mapping G_1 from the class Q_{g_1} . Let

$$K[G_1] = K_1^* = \frac{1+k_1^*}{1-k_1^*}. \quad (4.5)$$

The mappings f_1, f_2 are then defined by means of

$$f_1 = G_1^{-1} \circ g_1, \quad f_2 = g_2 \circ G_1. \quad (4.6)$$

By (4.4) and (4.5),

$$K[f_i] \leq K^{1/2} K_1^*, \quad i = 1, 2. \quad (4.7)$$

We now proceed to apply Theorems A and B, and (4.2), to estimate K_1^* from above. Since $\mu_1(z) = t\mu(z)$,

$$\iint_U \frac{\mu_1 \varphi}{1 - |\mu_1|^2} dx dy = t \iint_U \frac{\mu \varphi}{1 - |\mu|^2} dx dy - t(1 - t^2) \iint_U \frac{\mu |\mu|^2 |\varphi|}{(1 - |\mu|^2)(1 - t^2 |\mu|^2)} dx dy.$$

Using Theorem A to estimate the first integral on the right side, we deduce that

$$\left| \iint_U \frac{\mu_1 \varphi}{1 - |\mu_1|^2} dx dy \right| \leq t \iint_U \frac{|\mu|^2 |\varphi|}{1 - |\mu|^2} dx dy + t(1 - t^2) \iint_U \frac{|\mu|^3 |\varphi|}{(1 - |\mu|^2)(1 - t^2 |\mu|^2)} dx dy.$$

Therefore,

$$I[\mu_1] \leq \frac{tk^2}{1 - k^2} + \frac{t(1 - t^2)k^3}{(1 - k^2)(1 - t^2 k^2)} = \frac{(1 - t^2 k)tk^2}{(1 - k)(1 - t^2 k^2)}.$$

Evidently,

$$\Delta[\mu_1] \leq \frac{t^2 k^2}{1 - t^2 k^2}.$$

Hence, by Theorem B,

$$\frac{k_1^*}{1 - k_1^*} \leq \frac{(1 + t)tk^2}{(1 - k)(1 + tk)},$$

and, therefore,

$$K_1^* \leq 1 + \frac{2(1 + t)tk^2}{(1 - k)(1 + tk)} \leq 1 + \frac{4tk^2}{(1 - k)(1 + tk)}. \quad (4.8)$$

Substituting $K = (1+k)/(1-k)$, and the value of t as specified in (4.2) into the outer inequality in (4.8), we obtain

$$K_1^* \leq 1 + K^{-1/2}(K-1)(K^{1/2}-1). \quad (4.9)$$

Therefore, by (4.7),

$$K[f_i] \leq K^{3/2} - K + 1, \quad i = 1, 2.$$

This completes the proof of Theorem 3. The upper bound M on $K[f]$ is the solution of the equation

$$M^{3/2} - M + 1 = M, \quad (M > 1).$$

5. Bounds for $R(\mathcal{C})$

The principal result of the present section will be to establish that for a given $f \in Q_I$ an interpolating chain \mathcal{C} may be constructed for which $R(\mathcal{C})$ is bounded in terms of $K[f]$.

LEMMA 5.1. *Suppose $f \in Q_I$, $K[f] = K \leq \frac{5}{4}$. Then, for any $\varepsilon > 0$, there exists an interpolating chain \mathcal{C} for f with link size ε such that*

$$R(\mathcal{C}) \leq e^{2(K-1)}.$$

Proof. Let

$$x_0 = 1 + a_0, \quad a_0 = K - 1. \quad (5.1)$$

Let $\Phi(x)$ be defined by (4.1), and let

$$x_{j+1} = \Phi(x_j), \quad j = 1, 2, \dots, \quad (5.2)$$

In line with the remarks of Section 3 it will suffice to establish that the recursion formula (5.2) implies that

$$(x_j)^{(2j)} \leq e^{2(K-1)}, \quad j = 1, 2, \dots \quad (5.3)$$

We first note that

$$\Phi(x) = x^{3/2} - x + 1 \leq 1 + \frac{1}{2}(x-1) + \frac{1}{2}(x-1)^2, \quad x \geq 1. \quad (5.4)$$

Let $\{a_j\}$ be any sequence of real numbers such that

$$a_{j+1} - a_j \geq 2^{-j} a_j^2, \quad j = 0, 1, 2, \dots \quad (5.5)$$

Suppose

$$1 \leq x_m \leq 1 + 2^{-m} a_m \quad (5.6)$$

for some nonnegative integer m . It then follows, by (5.2), (5.4), (5.5), that

$$1 \leq x_{m+1} \leq 1 + 2^{-m-1} a_{m+1}.$$

Thus, by induction, (5.6) holds for $m = 0, 1, 2, \dots$. For any number $b > 0$, the sequence

$$a_j = \frac{b}{1 + 2^{-j+1} b}, \quad j = 0, 1, 2, \dots, \quad (5.7)$$

will satisfy (5.5). Therefore, if

$$\frac{b}{1 + 2b} = a_0 = K - 1, \quad \text{i.e.} \quad b = \frac{K-1}{3-2K}, \quad (5.8)$$

we conclude that

$$x_j \leq 1 + 2^{-j} a_j \leq 1 + 2^{-j} b, \quad j = 0, 1, 2, \dots,$$

and, therefore,

$$(x_j)^{(2^j)} \leq (1 + 2^{-j} b)^{(2^j)} \leq e^b \leq e^{2(K-1)}, \quad j = 1, 2, \dots,$$

as was to be shown.

THEOREM 4. *There exists a function $\Psi(K)$, defined for $1 \leq K < \infty$, with the following property: If $f \in Q_r$, then, for any $\varepsilon > 0$, there exists an interpolating chain \mathcal{C}*

for f with link size ε such that

$$R(\mathcal{C}) \leq \Psi(K).$$

Proof. Given $f \in Q_I$ we apply the process of Section 3 to find

$$u_l \in Q_I, \quad K[u_l] \leq \frac{5}{4}, \quad l = 1, 2, \dots, \quad N = N(K, 0.25),$$

such that

$$f = u_N \circ u_{N-1} \circ \dots \circ u_2 \circ u_1.$$

By applying Lemma 5.1 to each factor u_l we arrive at an interpolating chain \mathcal{C} for f with

$$R(\mathcal{C}) \leq e^{2(K-1)N(K, 0.25)}.$$

REFERENCES

- [1] CLIFFORD J. EARLE and JAMES EELLS, JR., *On the differential geometry of Teichmüller spaces*, J. d'Analyse Math., 19 (1967), 35–52.
- [2] F. W. GEHRING, *Quasiconformal mappings which hold the real axis pointwise fixed*, Mathematical Essays Dedicated To A. J. MacIntyre, Ohio University Press, 1970, 145–148.
- [3] S. L. KRUSHKAL', *On the theory of extremal quasiconformal mappings*, Sibirsk. Mat. Zh. 10 (1969), 573–583, (Russian). Translated under the title *Extremal quasiconformal mappings*, Sib. Math. J. 10 (1969), 411–418.
- [4] EDGAR REICH and KURT STREBEL, *On quasiconformal mappings which keep the boundary points fixed*, Trans. Am. Math. Soc., 138 (1969), 211–222.
- [5] — and —, *Extremal quasiconformal mappings with given boundary values*, Notes, Forschungsinstitut für Mathematik, Eidgenössische Technische Hochschule, Zürich, Sept. 1972.
- [6] — and —, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis, A Collection of Papers Dedicated to Lipman Bers, Academic Press, 1974, 375–391.

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Received May 16, 1977

