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Reconstructing 1-coherent locally finite trees

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Abstract. We prove a theorem implying the conjecture of J. A. Bondy and R. L. Hemminger that an infinite, locally finite tree containing no two-way infinite path is uniquely determined, up to isomorphism, from its collection of vertex-deleted subgraphs.

Introduction and terminology

We say that two graphs G_1 and G_2 are *weakly hypomorphic* if there exist maps $\varphi: V(G_1) \to V(G_2)$ and $\psi: V(G_2) \to V(G_1)$ such that $G_1 - x \simeq G_2 - \varphi(x)$ and $G_2 - y \simeq G_1 - \psi(y)$ for each vertex x and y in $V(G_1)$ and $V(G_2)$, respectively. In other words, G_1 and G_2 have the same isomorphism classes of vertex-deleted subgraphs. If φ and ψ can be chosen to be bijections, then G_1 and G_2 are *hypomorphic*. The reconstruction conjecture asserts that any two hypomorphic graphs are isomorphic. The conjecture is open for finite graphs and false for infinite graphs in general (see e.g. [5]). However, no counter-examples are known to the Harary–Schwenk–Scott conjecture [6] that any two hypomorphic, locally finite trees are isomorphic. As a first step towards a proof of this conjecture, Bondy and Hemminger [1] demonstrate the validity of the conjecture for m-coherent, locally finite trees for $m \geq 2$ (a tree is m-coherent if it contains a set of m, but not $m+1$, pairwise disjoint one-way infinite paths) and conjecture in an early version of [4] (the problem first appear in [3], but is mistakenly listed as being for 2-coherent locally finite trees) that any two hypomorphic 1-coherent locally finite trees are isomorphic. In this note we prove that the same conclusion holds under the weaker assumption that the trees are weakly hypomorphic.

If F is a forest and x, y are distinct vertices of F, then $F[x, y]$ denotes the component of $F - x$ containing y rooted at y. A *branch* of F at x is a rooted tree of the form $F[x, y]$ where y is adjacent to x. When we speak of isomorphisms of branches we always mean root-preserving isomorphisms. The *height* $h(x, T)$ of a vertex x in a tree T is the total number of vertices belonging to finite branches at x. If x has finite degree in T, then obviously $h(x, T)$ is finite.

If T is a 1-coherent, locally finite tree, and P is a one-way infinite path of T, then the forest obtained from T by deleting all edges of P has only finite components, by König's Lemma. Furthermore, if P' is any one-way infinite path of T, then P and P' have an infinite path in common.

Isomorphism between weakly hypomorphic, 1-coherent, locally finite trees

THEOREM. *Let T_1 be a 1-coherent, locally finite tree and let T_2 be any graph which is weakly hypomorphic to T_1. Then $T_1 \simeq T_2$.*

Proof. Let $\varphi: V(T_1) \to V(T_2)$ and $\psi: V(T_2) \to V(T_1)$ be maps such that

$$T_1 - x \simeq T_2 - \varphi(x) \quad \text{and} \quad T_2 - y \simeq T_1 - \psi(y)$$

for each x and y in $V(T_1)$ and $V(T_2)$, respectively. It is easily verified that T_2 is a 1-coherent, locally finite tree. Let x be any endvertex of T_1 and $P_1: x_1x_2x_3\cdots$ be a one-way infinite path of $T_1 - x$. Let $P_2: y_1y_2y_3\cdots$ be the image of P_1 under some isomorphism of $T_1 - x$ onto $T_2 - \varphi(x)$. With no loss of generality we can assume that x (resp. $\varphi(x)$) is joined to a vertex in a branch of $T_1 - x$ (resp. $T_2 - \varphi(x)$) at x_1 (resp. y_1) for otherwise we consider paths of the form $x_kx_{k+1}\cdots$ and $y_ky_{k+1}\cdots$ instead of P_1 and P_2, respectively.

If T is a tree and x a vertex of T, then the degree of x in T equals the number of components of $T - x$. From this it follows that if one of T_1, T_2 has maximum degree 2, then each of T_1, T_2 is a one-way path, and hence $T_1 \simeq T_2$ in this case. So we can assume that each of T_1, T_2 contains a vertex of degree 3 or more.

If each of the vertices $x_2, x_3, \ldots$ has degree 2 in T_1 (and hence each of $y_2, y_3, \ldots$ has degree 2 in T_2), then we denote by k (resp. m) the maximum distance in T_1 (resp. T_2) between an endvertex and a vertex of degree 3 or more and we put $n = \max\{k, m\}$. It is obvious that for each $i \geq 1$, no vertex of T_2 other than y_i has height $h(y_i, T_2)$. Since

$$h(y_i, T_2) = h(x_i, T_1) = h(\varphi(x_i), T_2)$$

We conclude that $\varphi(x_i) = y_i$ for each $i \geq 1$. Now let

$$\sigma: T_1 - x_{n+3} \to T_2 - y_{n+3}$$

be an isomorphism. By definition of n, x_{n+2} (resp. y_{n+2}) is the only vertex of the finite component of $T_1 - x_{n+3}$ (resp. $T_2 - y_{n+3}$) which has distance greater than n to a vertex of degree 3 or more. Hence

$$\sigma(x_{n+2}) = y_{n+2}$$

and σ can be extended to an isomorphism of T_1 onto T_2.

So we assume that for some $t \geq 2$, the degree of x_t and y_t is at least 3. The union of finite branches of T_1 at x_t (except $T_1[x_t, x_{t-1}]$) is isomorphic to the union of finite branches of T_2 at y_t (except $T_2[y_t, y_{t-1}]$). We want to prove that

$$T_1[x_t, x_{t-1}] \simeq T_2[y_t, y_{t-1}].$$

This will imply $T_1 \simeq T_2$. If all finite branches at x_t are pairwise isomorphic and all finite branches at y_t are pairwise isomorphic, then obviously

$$T_1[x_t, x_{t-1}] \simeq T_2[y_t, y_{t-1}],$$

so we can assume, with no loss of generality, that there is a branch $B = T_1[x_t, z_1]$ which is not isomorphic to $T_1[x_t, x_{t-1}]$.

Consider a vertex z_2 of T_2 such that $T_2 - z_2 \simeq T_1 - z_1$. Let s be the integer such that z_2 and y_s belong to the same component of the forest obtained by deleting all edges of P_2. Suppose z_2 is chosen such that s is minimal. Let

$$\pi : T_1 - z_1 \to T_2 - z_2$$

be an isomorphism, and let

$$P_2' : y_1' y_2' y_3' \cdots$$

be the image of P_1 under π. Since T_2 is 1-coherent, the intersection of P_2 and P_2' is an infinite path $y_r' y_{r+1}' \cdots = y_q y_{q+1} \cdots$. For i sufficiently large, x_{r+i} and y_{r+i}' have the same height in T_1 and T_2, respectively, so

$$h(y_{q+i}, T_2) = h(y_{r+i}', T_2) = h(x_{r+i}, T_1) = h(y_{r+i}, T_2).$$

However,

$$h(y_1, T_2) < h(y_2, T_2) < \cdots,$$

so, for i sufficiently large, $y_{q+i} = y_{r+i}$ and hence $r = q$. Suppose π is chosen such that r is minimal.

We now prove by contradiction that

$$r \leq \max\{t, s\}.$$

For suppose $r > t$ and $r > s$. The union of branches of T_1 at x_r (except $T_1[x_r, x_{r-1}]$)

is isomorphic to the union of branches of T_2 at y_r (except $T_2[y_r, y_{r-1}]$) and since $\pi(x_r) = y_r$, the union of branches of $T_1 - z_1$ at x_r is isomorphic to the union of branches of $T_2 - z_2$ at y_r. Combining these observations, we conclude that

$$(T_1 - z_1)[x_r, x_{r-1}] \simeq (T_2 - z_2)[y_r, y_{r-1}]$$

and hence we can modify π so as to obtain an isomorphism $T_1 - z_1 \to T_2 - z_2$ which takes x_{r-1} to y_{r-1}. This contradicts the minimality of r. Hence $r \le \max\{t, s\}$.

We next prove that $s \le t$. For suppose $s > t$. (Hence $\pi(x_s) = y_s$). Since

$$h(y_s, T_2) = h(x_s, T_1) > h(z_1, T_1) = h(z_2, T_2),$$

we have $z_2 \neq y_s$, so z_2 belongs to a finite branch B' of T_2 at y_s (distinct from $T_2[y_s, y_{s-1}]$). Since $T_1 - x$ and $T_2 - \varphi(x)$ have the same branches at x_s and y_s, respectively, and since deleting z_2 from T_2 results in a forest with fewer branches at y_s isomorphic to B', it follows that π^{-1} maps $T_2[y_s, y_{s-1}]$ onto a branch of T_1 isomorphic to B'. But then $T_2[y_s, y_{s-1}]$ contains a vertex z_2' s.t. $T_2 - z_2 \simeq T_2 - z_2'$, a contradiction to the minimality of s.

Summarizing, $s \le t$ and π maps the path $x_t x_{t+1} \cdots$ onto $y_t y_{t+1} \cdots$. In particular, x_t and y_t have the same degree in $T_1 - z_1$ and $T_2 - z_2$, respectively. So z_2 is adjacent to y_t. Since the number of branches at x_t isomorphic to B is one less in $T_1 - z_1$ than in T_1 and since $T_1[x_t, x_{t-1}]$ is not isomorphic to B, it follows that $T_2[y_t, y_{t-1}]$ is not isomorphic to B (again using the isomorphism between $T_1 - x$ and $T_2 - \varphi(x)$) and that z_2 is the root of a branch at y_t isomorphic to B. In particular, $z_2 \neq y_{t-1}$ and π maps $T_1[x_t, x_{t-1}]$ onto a branch isomorphic to $T_2[y_t, y_{t-1}]$. Since $T_1[x_t, x_{t-1}] \simeq T_2[y_t, y_{t-1}]$ implies $T_1 \simeq T_2$, the proof is complete.

Further results and conjectures

We can prove that any two weakly hypomorphic, locally finite, m-coherent trees are isomorphic for $m \ge 3$ and believe that with a little more effort the same can be proved for $m = 2$. Furthermore, we can prove that any two hypomorphic m-coherent trees are isomorphic for $m \ge 3$ and Nešetřil [7, 8] proves it for $m = 0$. We conjecture that it also holds for $m = 1, 2$.

Halin (see [2]) conjectures that if G_1 and G_2 are infinite hypomorphic graphs, then G_i contains a subgraph isomorphic to G_{3-i} for $i = 1, 2$. We conjecture that the same conclusion holds under the weaker assumption that G_1 and G_2 are weakly hypomorphic.

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