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# Ideals generated by minors of a symmetric matrix

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## §0. Introduction

Let  $X$  be an  $n$  by  $n$  symmetric matrix with entries in a commutative Noetherian ring  $R$  with identity. R. Kutz investigated, in [11], ideals  $I_p(X)$  generated by all the  $p$  by  $p$  minors of  $X$ . His main results states:

$$\text{depth } I_p(X) \leq \nu(p, n) := \frac{(n-p+1)(n-p+2)}{2}$$

and in case of equality the ideal  $I_p(X)$  is perfect, i.e.  $\text{depth } I_p(X) = \text{pd } R/I_p(X)$ . Kutz used in his proof a technique applied for the first time by Hochster and Eagon in [10] to determinantal ideals associated with an arbitrary matrix.

In §2 of the present paper we extend some results of Kutz concerning the depth of  $I_p(X)$  and prove that the height of  $I_p(X)$  is also bounded by  $\nu(p, n)$ .

In §3 we construct a free complex,  $\mathbf{L}(X)$ , of length 3 which gives a free resolution of  $I_{n-1}(X)$  when  $\text{depth } I_{n-1}(X) = 3$ .

All the proofs in §§2, 3 depend heavily on a lemma stated in §1 which contains in particular the structure theorem for non-singular quadratic forms over a local ring (see [12], Lemme 2).

In §4 we utilize the complex  $\mathbf{L}(X)$  to describe the relation between the Poincaré series of local rings  $R$  and  $R/I_{n-1}(X)$  when  $\text{depth } I_{n-1}(X) = 3$ .

## §1. The fundamental lemma

(1.1) LEMMA (Micali-Villamayor). *Let  $R$  be a commutative ring with identity. Let  $X = (x_{ij})$  be an  $n$  by  $n$  symmetric matrix with entries in  $R$ . Let  $I_p(X)$  be the ideal of  $R$  generated by all the  $p$  by  $p$  minors of  $X$ ,  $1 \leq p \leq n$ .*

I) *If  $x_{11}$  is invertible in  $R$ , then there exists an invertible matrix  $C$  such that*

$$CXC = \left( \begin{array}{c|c} x_{11} & 0 \\ \hline 0 & X' \end{array} \right),$$

2) the  $n-1$  by  $n-1$  matrix  $X'$  is symmetric and

$$x'_{kj} = x_{kj} - \frac{x_{k1} x_{1j}}{x_{11}}, \quad k, j = 2, \dots, n,$$

3)  $I_p(X) = I_{p-1}(X')$  for  $p \geq 1$ .

II) If  $\det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} := a$  is invertible in  $R$ , then there exists an invertible matrix  $C$  such that

1)

$${}^t CXC = \left( \begin{array}{cc|c} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ \hline 0 & 0 & X'' \end{array} \right),$$

2) the  $n-2$  by  $n-2$  matrix  $X''$  is symmetric and  $x''_{kj} = x_{kj} - a_k x_{1j} - b_k x_{2j}$ , where  $a_k = \frac{x_{1k}x_{22} - x_{2k}x_{12}}{a}$ ,  $b_k = \frac{x_{2k}x_{11} - x_{1k}x_{21}}{a}$ ,  $k, j = 3, \dots, n$ ,

3)  $I_p(X) = I_{p-2}(X'')$  for  $p \geq 2$ .

(1.2) *Remark.* We adopt the convention  $I_0(X) = R$ .

*Proof.* I) We define

$$C = \begin{pmatrix} 1 & -\frac{x_{12}}{x_{11}} & \dots & -\frac{x_{1n}}{x_{11}} \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix}$$

II) We define

$$C = \left( \begin{array}{cc|ccc} 1 & 0 & -a_3 & \dots & -a_n \\ 0 & 1 & -b_3 & \dots & -b_n \\ \hline & & 1 & & \\ & 0 & & \ddots & 0 \\ & & & \ddots & \\ & & 0 & & 1 \end{array} \right)$$

(1.3) COROLLARY. *Let  $R$  be a local ring. If  $I_1(X) = R$ , then the hypothesis of either I) or II) of Lemma (1.1) holds (possibly after rearrangement of some rows and the same columns of  $X$ ).*

*Proof.* From  $I_1(X) = R$  it follows that some entry of  $X$  is invertible. If that entry lies on the main diagonal the hypothesis of I) holds. If this is not the case one may assume that  $x_{12}$  is invertible and all the entries on the main diagonal belong to the maximal ideal of  $R$ . Then  $a = x_{11}x_{22} - x_{12}^2$  is invertible.

(1.4) COROLLARY. *Let  $R$  be a local ring. If  $I_{n-1}(X) = R$ , then there exists an invertible matrix  $C$  over  $R$  such that*

$${}^tCXC = \begin{pmatrix} \begin{matrix} * & * \\ * & * \end{matrix} & & & O \\ & \ddots & & \\ & & \begin{matrix} * & * \\ * & * \end{matrix} & \\ O & & & * \dots * \\ & & & & u \end{pmatrix}$$

where the starred  $i$  by  $i$  minors on the main diagonal are invertible,  $i = 1$  and/or  $2$ ,  $u \in R$ .

*Proof.* If  $I_{n-1}(X) = R$ , then at least one entry of  $X$  is invertible. Using Lemma (1.1) one can transform  $X$  into a matrix of the kind I1) or II1) of the lemma with an  $r$  by  $r$  invertible matrix in the upper left corner,  $r = 1$  or  $2$ . If  $n - 1 = r$  we are done. If not, then the remaining  $n - r$  by  $n - r$  matrix in the lower right corner has also at least one invertible entry. We again apply Lemma (1.1). Proceeding in this way we get the required result.

## §2. Height and Depth of $I_p(X)$

The next theorem can be deduced from Kutz's results and the general theory of generically perfect ideals of Eagon and Northcott, [6]. To make the proof of Theorem (2.3) as self-contained as possible we indicate here a short proof of Theorem (2.1) along the lines presented in [5] for arbitrary determinantal ideals.

(2.1) THEOREM. *Let  $R$  be a commutative Noetherian ring with identity and  $X$  an  $n$  by  $n$  symmetric matrix with entries in  $R$ . Every minimal prime ideal of the ideal  $I_p(X)$  has height at most equal to  $\nu(p, n)$ .*

*Proof.* We only sketch the proof and refer to [5], pp. 202–203, where possible.

We use induction on  $n$ . If  $n \leq 2$  or  $p = 1$  the theorem follows from the generalized principal ideal theorem of Krull.



Suppose  $n > 2$ ,  $p > 1$  and let  $P$  be a minimal prime ideal of  $I_p(X)$ . One may assume that  $R$  is local with maximal ideal  $P$  and that  $I_p(X)$  is  $P$ -primary. If  $I_1(X) = R$ , then by Lemma (1.1) and Corollary (1.3)  $I_p(X) = I_{p-r}(\tilde{X})$  where  $r = 1$  or  $2$ , and  $\tilde{X}$  is an  $n-r$  by  $n-r$  symmetric matrix over  $R$ . By the induction hypothesis  $\text{ht } P \leq \nu(p-r, n-r) = \nu(p, n)$ .

If  $I_1(X) \subset P$  we consider a matrix

$$X + \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

where  $Z$  is an indeterminate over  $R$  and proceed as in [5].

For the proof of the next theorem we record the following easy lemma.

(2.2) LEMMA. *Let  $K$  be a subring of a commutative ring  $R$  with identity and let  $x_1, \dots, x_q$  be a sequence of elements in  $R$  which are algebraically independent over  $K$ . Assume that  $t$  is a non-zero divisor in  $R$  belonging to  $K[x_1, \dots, x_s]$ ,  $s < q$ , and write  $K' = K[x_1, \dots, x_s]_{(t^k)}$ ,  $R' = R_{(t^k)}$  for the localizations of the corresponding rings at the powers of  $t$ ; moreover let  $a'_{s+1}, \dots, a'_q$  be elements of  $K'$ .*

*Then  $K' \subset R'$  and the elements  $x_{s+1} - a'_{s+1}, \dots, x_q - a'_q$  are algebraically independent over  $K'$ .*

(2.3) THEOREM. *Let  $R$  be a commutative Noetherian ring with identity,  $K$  a Noetherian subring of  $R$  with the same identity. Let  $\{x_{ij}\}$ ,  $1 \leq i \leq j \leq n$ , be a sequence of elements of  $R$  which are algebraically independent over  $K$ . Assume that  $R$  is flat as an algebra over  $K[\{x_{ij}\}]$ . If we put  $x_{ji} = x_{ij}$  for  $j < i$  and define  $X = (x_{ij})$ , then  $\text{depth } I_p(X) = \nu(p, n)$ .*

*Proof.* We use certain arguments of Eagon from [4]. If  $p = 1$ , then  $I_p(X)$  is generated by  $\{x_{ij}\}$ ,  $i \leq j$ , and therefore  $\text{depth } I_1(X) = \nu(1, n)$ . In fact, the sequence  $\{x_{ij}\}$ ,  $i \leq j$ , is  $R$ -regular since  $R$  is flat over  $K[\{x_{ij}\}]$ .

Now we argue by induction on  $n$ , assuming  $n > 1$ ,  $p > 1$ . Let  $u_1, \dots, u_l$  be a maximal  $R$ -regular sequence contained in  $I_p(X)$ . By Theorem (2.1) we know that  $l \leq \nu(p, n)$ , hence in view of  $p > 1$  we have  $l < (n^2 + n)/2 = \nu(1, n)$ . Write  $I = I_p(X)$ ,  $J = (u_1, \dots, u_l)$  for short. Since  $I$  consists of zero divisors on  $J$ , there exists a prime ideal  $P$  associated to  $J$  and containing  $I$ . Thus  $l = \text{depth } J = \text{depth } I = \text{depth } P$ . By  $l < (n^2 + n)/2$  we must have  $x_{ij} \notin P$  for some  $i, j$ . We consider two cases:

I)  $i = j$ ; one may assume without loss of generality that  $i = 1$ . Write  $t = x_{11}$ ,  $K' = K[x_{11}, x_{12}, \dots, x_{1n}]_{(t^k)}$ .

II)  $i \neq j$  and all elements on the main diagonal belong to  $P$ . As above one may

assume  $i = 1, j = 2$ . Write, in this case,  $t = x_{11}x_{22} - x_{12}^2$ ,  $K' = K[x_{11}, \dots, x_{1n}, x_{22}, \dots, x_{2n}]_{\{t^k\}}$ . Of course  $t \notin P$ .

In both cases write  $R' = R_{\{t^k\}}$ . For an ideal  $\mathfrak{A}$  of  $R$  let  $\mathfrak{A}'$  denote  $\mathfrak{A}R'$ . Thus we have  $J' \subset I' \subset P'$ ,  $P'$  is proper and  $\text{depth } J' = l$  since  $J'$  is generated by an  $R'$ -regular sequence  $u_1, \dots, u_l$ . On the other hand,  $P'$  is an associated prime of  $J'$  because  $t \notin P$ . Therefore  $l = \text{depth } J' = \text{depth } I' = \text{depth } P'$ . Observe that  $I' = I_p(X)R'$  is the ideal in  $R'$  generated by all the  $p$  by  $p$  minors of  $X$ . Since  $t$  is invertible in  $R'$  we may apply Lemma (1.1) to  $R'$  and  $X$ . We conclude that  $I' = I_{p-r}(\tilde{X})R'$ , where  $r = 1$  or  $2$  depending on case I) or II), and  $\tilde{X} = (x_{kj} - a'_{kj})$  is the  $n-r$  by  $n-r$  symmetric matrix with entries in  $R'$ ,  $a'_{kj} \in K'$ ,  $r < k, j \leq n$ . By Lemma (2.2) we infer that the elements  $\{x_{kj} - a'_{kj}\}$ ,  $r < k \leq j \leq n$ , are algebraically independent over  $K'$ . Moreover  $K'[\{x_{kj} - a'_{kj}\}]$ ,  $r < k \leq j \leq n$ , is equal to  $K[\{x_{ij}\}]_{\{t^k\}}$ ,  $1 \leq i \leq j \leq n$ , and  $R'$  is flat over  $K[\{x_{ij}\}]_{\{t^k\}}$ ,  $1 \leq i \leq j \leq n$ . Hence by the induction hypothesis we finally get  $l = \text{depth } I' = \text{depth } I_{p-r}(\tilde{X})R' = \nu(p-r, n-r) = \nu(p, n)$ .

(2.4) COROLLARY. Let  $K$  be a commutative Noetherian ring and  $R = K[\{x_{ij}\}]$ ,  $1 \leq i \leq j \leq n$ , a polynomial ring over  $K$  in  $(n^2 + n)/2$  indeterminates  $\{x_{ij}\}$ . Put  $X = (x_{ij})$ . Then  $\text{depth } I_p(X) = \nu(p, n)$ .

(2.5) Remark. Corollary (2.4) was proved by R. Kutz in [11, Proposition 6.2] under the additional assumption that  $K$  is an integral domain.

(2.6) COROLLARY. Let  $R$  be a local algebra over a field  $K$  and let  $\{x_{ij}\}$ ,  $1 \leq i \leq j \leq n$ , be a regular sequence in  $R$ . Then  $\text{depth } I_p(X) = \nu(p, n)$  for a symmetric matrix  $X = (x_{ij})$ .

Proof. Since  $\{x_{ij}\}$  are algebraically independent over  $K$  and  $R$  is flat over  $K[\{x_{ij}\}]$  ([8, Proposition 1]), the corollary follows immediately from Theorem (2.3).

Using the method of the proof of Theorem (2.3) one can also prove

(2.7) THEOREM. Let  $R$  be a commutative Noetherian ring with identity and  $K$  a Noetherian subring of  $R$  with the same identity. Let  $\{x_{ij}\}$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ , be a sequence of elements of  $R$  which are algebraically independent over  $K$  and let  $X$  denote an  $r$  by  $s$  matrix  $(x_{ij})$ . Assume that  $R$  is flat as an algebra over  $K[\{x_{ij}\}]$ . Then  $\text{depth } I_t(X) = (r-t+1)(s-t+1)$  where  $I_t(X)$  is an ideal of  $R$  generated by all the  $t$  by  $t$  minors of  $X$ .

(2.8) COROLLARY. Let  $R$  be a local algebra over a field  $K$  and let  $\{x_{ij}\}$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ , be a regular sequence in  $R$ . Then  $\text{depth } I_t(X) = (r-t+1)(s-t+1)$  where  $X = (x_{ij})$ .

### §3. A Free Resolution of $I_{n-1}(X)$

Let  $R$  be a commutative ring with identity and  $X = (x_{ij})$  a symmetric  $n$  by  $n$  matrix with entries in  $R$ . Write  $Y = (y_{ij})$  for the matrix of cofactors of  $X$ , i.e.

$y_{ij} = (-1)^{i+j} X_j^i$  where  $X_j^i$  stands for the minor of  $X$  obtained by deleting the  $i$ -th column and the  $j$ -th row of  $X$ . The matrix  $Y$  is also symmetric. We are fixing the matrix  $X$  (and hence  $Y$ ) throughout this section.

Let  $M_n(R)$  be the free  $R$ -module of all  $n$  by  $n$  matrices over  $R$  and  $A_n(R)$  the free submodule of  $M_n(R)$  consisting of all alternating matrices. Furthermore, let  $\text{tr}: M_n(R) \rightarrow R$  denote the trace map.

We have a free complex of length 3 associated with  $X$ :

$$\mathbf{L}(X): 0 \longrightarrow A_n(R) \xrightarrow{d_3} \text{Ker}(M_n(R) \xrightarrow{\text{tr}} R) \xrightarrow{d_2} M_n(R)/A_n(R) \xrightarrow{d_1} R,$$

where the corresponding differentials are defined as follows:

$$d_1(M \bmod A_n(R)) = \text{tr}(YM),$$

$$d_2(N) = XN \bmod A_n(R),$$

$$d_3(A) = AX.$$

$d_1$  and  $d_3$  are well defined because the trace of the product of a symmetric and an alternating matrices is 0. Observe that  $H_0(\mathbf{L}(X)) = R/I_{n-1}(X)$ .

Now we can state the main result of this section.

(3.1) THEOREM. *Let  $R$  be a commutative Noetherian ring with identity. Let  $X = (x_{ij})$  be an  $n$  by  $n$  symmetric matrix with entries in  $R$ . If  $\text{depth } I_{n-1}(X) = 3$  (the largest possible), then the complex  $\mathbf{L}(X)$  is acyclic and gives a free resolution of  $R/I_{n-1}(X)$ .*

The proof of (3.1) requires several preliminary lemmata.

(3.2) LEMMA. *Let  $\varphi: R \rightarrow R'$  be a ring homomorphism,  $X = (x_{ij})$  a symmetric matrix over  $R$ , and  $X' = (\varphi(x_{ij}))$ . Then the complexes  $\mathbf{L}(X) \otimes_R R'$  and  $\mathbf{L}(X')$  are isomorphic over  $R'$ .*

(3.3) LEMMA. *The complexes  $\mathbf{L}(CXC)$  and  $\mathbf{L}(X)$  are isomorphic for an arbitrary invertible  $n$  by  $n$  matrix  $C$ .*

*Proof.* Let  $F$  be a free  $R$ -module of rank  $n$  and let  $F^*$  be the dual module of  $F$ . A map  $f: F^* \rightarrow F$  is said to be *symmetric* if, with respect to some (and therefore every) basis and dual basis of  $F$  and  $F^*$ , the matrix of  $f$  is symmetric.

We are going to prove the lemma by assigning to a symmetric map  $f: F^* \rightarrow F$

a free complex  $\mathbf{L}(f)$  of length 3 and showing that  $\mathbf{L}(X)$  and  $\mathbf{L}({}^tCXC)$  are both isomorphic with  $\mathbf{L}(f)$ . The passage from  $\mathbf{L}(f)$  to  $\mathbf{L}(X)$  corresponds to fixing a basis of  $F$  and taking the dual basis of  $F^*$ , and further passage to  $\mathbf{L}({}^tCXC)$  corresponds to a change of bases.

An invariant basis-free description of our complex can be given as follows:

$$\mathbf{L}(f): 0 \longrightarrow \bigwedge^2(F^*) \xrightarrow{\partial_3} \text{Ker}(F^* \otimes F \xrightarrow{ev} R) \xrightarrow{\partial_2} S_2(F) \xrightarrow{\partial_1} R,$$

where  $ev$  stands for the evaluation map,  $S_2(F)$  is the second symmetric power of  $F$ , and  $\bigwedge^2(F^*)$  the second exterior power of  $F^*$ .

To determine the differentials of  $\mathbf{L}(f)$  we define a map  $g: F \rightarrow F^*$  by requiring commutativity of the following diagram:

$$\begin{array}{ccc} \bigwedge^{n-1}(F^*) & \xrightarrow{f} & \bigwedge^{n-1}(F) \\ \parallel & & \parallel \\ F & \xrightarrow{g} & F^* \end{array}$$

where the vertical maps are the canonical isomorphisms. Then the composition  $F \otimes F \xrightarrow{1 \otimes g} F \otimes F^* \xrightarrow{ev} R$  induces  $\partial_1$  on  $S_2(F)$  and the map  $F^* \otimes F \xrightarrow{f \otimes 1} F \otimes F \xrightarrow{\eta} S_2(F)$  induces  $\partial_2$ , where  $\eta$  is the canonical epimorphism. Finally,  $\partial_3$  is induced by  $\bigwedge^2(F^*) \xrightarrow{\gamma} F^* \otimes F^* \xrightarrow{1 \otimes f} F^* \otimes F$ , where  $\gamma(u \wedge w) = u \otimes w - w \otimes u$ .

The next lemma needs some information about the  $n-2$  by  $n-2$  minors of  $X$ . To fix the notation let  $X_{ij}^{kl}$  be the minor of  $X$  obtained by leaving out the  $i$ -th and  $j$ -th rows, and the  $k$ -th and  $l$ -th columns of  $X$ ,  $i \neq j$ ,  $k \neq l$ . Observe that  $X_{ij}^{kl} = X_{kl}^{ij}$  because  $X$  is symmetric.

We define two functions:

$$\sigma(i, j) = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{if } i = j, \\ -1 & \text{if } i > j \end{cases} \quad i, j \in N;$$

$$T(i, j, k, l) = (-1)^{i+j+k+l} \sigma(i, j) \sigma(k, l) X_{ij}^{kl}, \quad i, j, k, l \in N$$

By the Laplace expansion we get the following formulas:

$$(\#) \sum_{l=1}^n x_{sl} T(i, j, k, l) = \begin{cases} 0 & \text{if } s \neq i, s \neq j, \\ -y_{ik} & \text{if } s = i, i \neq j, \\ y_{ik} & \text{if } s = j, i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

We write  $\{E_{ij}\}$  for the standard basis of  $M_n(R)$ ; if  $F_{ij} = E_{ij} - E_{ji}$ , then  $\{F_{ij}\}$ ,  $i < j$ , is a basis of  $A_n(R)$ .

(3.4) LEMMA. For  $i < j$

$$YF_{ij} = \left( \sum_{p < q} (-1)^{i+j+p+q} X_{ij}^{pq} F_{pq} \right) X.$$

*Proof.* Write  $\alpha = i + j + p + q$  for short. Using (#) we have

$$\begin{aligned} \sum_{p < q} (-1)^\alpha X_{ij}^{pq} F_{pq} X &= \\ \sum_{p < q} (-1)^\alpha X_{ij}^{pq} \left( \sum_s x_{qs} E_{ps} - \sum_s x_{ps} E_{qs} \right) &= \\ \sum_{p,s} \left( \sum_{q > p} (-1)^\alpha x_{qs} X_{ij}^{pq} + \sum_{q < p} (-1)^{\alpha-1} x_{qs} X_{ij}^{pq} \right) E_{ps} &= \\ \sum_{p,s} \left( \sum_q x_{sq} T(i, j, p, q) \right) E_{ps} &= \\ \sum_p y_{ip} E_{pj} - \sum_p y_{jp} E_{pi} = Y(E_{ij} - E_{ji}) = YF_{ij}. \end{aligned}$$

(3.5) COROLLARY. For an arbitrary alternating matrix  $B$  there exists an alternating matrix  $A$  such that  $YB = AX$ .

(3.6) LEMMA. If  $X$  is a symmetric invertible matrix, then  $L(X)$  is exact.

*Proof.*  $\text{Ker } d_1 = \text{Im } d_2$ . Let  $M \bmod A_n(R) \in \text{Ker } d_1$ ; this means that  $\text{tr}(YM) = 0$  and therefore  $YM \in \text{Ker}(M_n(R) \xrightarrow{\text{tr}} R)$ . Hence

$$M \bmod A_n(R) = d_2[(\det X)^{-1} YM] \in \text{Im } d_2.$$

$\text{Ker } d_2 = \text{Im } d_3$ . Let  $N \in \text{Ker } d_2$ , i.e.  $B := XN$  is alternating. Multiplying by  $Y$  and using Corollary (3.5) we get  $N = [(\det X)^{-1}A]X \in \text{Im } d_3$ , where  $A$  is an alternating matrix.

$\text{Ker } d_3 = 0$  is obvious.

In the course of the proof of Theorem (3.1) we will need the following corollary from the “*Lemme d’acyclicité*” of Peskine-Szpiro (see [3], Corollary 4.2).

(3.7) LEMMA. *Let  $R$  be a Noetherian ring, and let*

$$\mathbf{L}: 0 \rightarrow L_3 \rightarrow L_2 \rightarrow L_1 \rightarrow L_0$$

*be a complex of finitely generated free  $R$ -modules. If for every prime ideal  $P \subset R$  with  $\text{depth } P < 3$  the localized complex  $\mathbf{L}_P$  is exact, then  $\mathbf{L}$  is exact.*

*Proof of Theorem (3.1).* By Lemma (3.7) it is enough to prove that  $\mathbf{L}(X)_P$  is exact for every prime  $P$  with  $\text{depth } P < 3$ . Since  $\text{depth } I_{n-1}(X) = 3$  we infer that  $I_{n-1}(X) \not\subset P$  for such a  $P$ , and hence  $I_{n-1}(X_P) = R_P$  where  $X_P$  is a matrix  $X$  considered over  $R_P$ . Since  $\mathbf{L}(X)_P \simeq \mathbf{L}(X_P)$  by Lemma (3.2) it suffices to prove the theorem for  $R$  local and  $X$  with  $I_{n-1}(X) = R$ .

Under these assumptions and by Corollary (1.4) there exists an invertible matrix  $C$  such that  $CXC = X'X''$  where  $x'_{ij} = 0$  for  $i < n, j = n, i = n, j < n, x'_{nn} = 1, x''_{ii} = 1$  for  $i < n, x''_{nn} = u, x''_{ij} = 0$  for  $i \neq j$ , and the matrix  $X'$  is invertible. Observe that  $X'$  and  $X''$  commute with each other. By Lemma (3.3) it is enough to prove that  $\mathbf{L}(X'X'')$  is exact. By direct computation one proves that  $\mathbf{L}(X'')$  is exact.

Write  $d_p, d'_p, d''_p$  for the differentials of  $\mathbf{L}(X'X''), \mathbf{L}(X'), \mathbf{L}(X'')$ , respectively, and  $Y', Y''$  for the matrices of cofactors of  $X', X''$ , respectively. Note that  $Y = Y'Y''$ .  $\text{Ker } d_1 = \text{Im } d_2$

For any matrix  $Q = (q_{ij})$  we have an equality

$$Q = X''\tilde{Q} + (1-u) \sum_{j < n} q_{nj} E_{nj}, \quad (\#\#)$$

where  $\tilde{q}_{ij} = q_{ij}$  for  $(i, j) \neq (n, n)$ , and  $\tilde{q}_{nn} = -\sum_{i=1}^{n-1} q_{ii}$ ; hence  $\text{tr } \tilde{Q} = 0$ .

Suppose that  $M \bmod A_n(R) \in \text{Ker } d_1$ ; one can assume that  $M$  is triangular with zeros under the main diagonal. Since  $y'_{nj} = 0$  for  $j < n$  we get, by applying  $(\#\#)$  to  $Y'M$ , an equality  $Y'M = X''W$  with  $\text{tr } W = 0$ . Multiplying both sides by  $X'$  and using the invertibility of  $\det X'$  we finally get:

$$M \bmod A_n(R) = X'X''[(\det X')^{-1}W] \bmod A_n(R) \in \text{Im } d_2.$$

$\text{Ker } d_2 = \text{Im } d_3$

We can assume that the entry  $u$  in the lower right corner of  $X''$  belongs to the maximal ideal of  $R$  because otherwise  $X'X''$  is invertible and we are done by Lemma (3.6).

Let  $Q \in \text{Ker } d_2$ , i.e.  $\text{tr } Q = 0$  and  $B := X'X''Q$  is alternating. Multiplying by  $Y'$  gives  $X''Q = (\det X')^{-1} Y'B$  and  $\text{tr } (X''Q) = 0$ . This together with  $\text{tr } Q = 0$  implies that  $q_{nn} = 0$ , because  $1 - u$  is invertible. Note that  $X''QY' = (\det X')^{-1} Y'BY'$  is alternating and a simple calculation shows that  $\text{tr } (QY') = 0$ . This means that  $QY' \in \text{Ker } (M_n(R) \xrightarrow{\text{tr}} R)$ . Since  $X''(QY')$  is alternating we get  $QY' = DX''$ , for some alternating  $D$ , from the exactness of  $\mathbf{L}(X'')$ . Therefore  $Q = [(\det X')^{-1} D]X'X'' \in \text{Im } d_3$ .

$\text{Ker } d_3 = 0$  is obvious.

(3.8) *Remark.* The proof simplifies considerably when 2 is invertible in  $R$ . In this case one can transform  $X$  as in Corollary (1.4) to a diagonal matrix and the proof of the exactness of  $\mathbf{L}(X)$  for a diagonal matrix is straightforward.

(3.9) COROLLARY (Kutz). *If  $\text{depth } I_{n-1}(X) = 3$ , then  $I_{n-1}(X)$  is perfect.*  
From Corollary (2.4) we infer

(3.10) COROLLARY.  *$I_{n-1}(X)$  is a generically perfect ideal (see [6], §8, for the definition).*

Corollaries (2.7) and (3.9) give together

(3.11) COROLLARY. *Let  $R$  be a local algebra over a field  $K$  and  $X = (x_{ij})$  a symmetric matrix over  $R$  such that  $\{x_{ij}\}$ ,  $1 \leq i \leq j \leq n$ , form an  $R$ -sequence. Then*

- a)  *$I_{n-1}(X)$  is perfect and  $\mathbf{L}(X)$  is the minimal free resolution of  $R/I_{n-1}(X)$ .*
- b) *if  $R$  is regular,  $R/I_{n-1}(X)$  is a Cohen-Macaulay ring of type  $(n^2 - n)/2$ .*

The results of Eagon, Northcott and Hochster (see in particular [9, Theorem 3]) lead to the following corollary.

(3.12) COROLLARY. *Let  $X$  be a symmetric matrix over a Noetherian ring  $R$ . The complex  $\mathbf{L}(X)$  is depth-sensitive, i.e. for any finitely generated  $R$ -module  $E$  such that  $I_{n-1}(X)E \neq E$  we have*

$$\text{depth } (I_{n-1}(X), E) + q = 3,$$

where  $q$  is the index of the largest non-vanishing homology module of the complex  $\mathbf{L}(X) \otimes_R E$ .

(3.13) *Remark.* When the first version of this paper had been written I received from J. Herzog a preprint of S. Goto and S. Tachibana, [7]. The authors

constructed a complex of length 3 identical with  $\mathbf{L}(X)$  when 2 is invertible in  $R$ , and proved in this case (by different methods) Theorem (3.1).

#### §4. An Application to the Poincaré Series

We recall that if  $R$  is a local ring with residue field  $K$ , the Poincaré series  $\mathcal{P}_R$  of  $R$  is the power series

$$\sum_{p=0}^{\infty} (\dim_K \operatorname{Tor}_p^R(K, K)) t^p.$$

(4.1) THEOREM\*. *Let  $R$  be a local ring,  $Z$  an  $n$  by  $n$  symmetric matrix with entries in the maximal ideal  $\mathfrak{m}$  of  $R$ ,  $n > 1$ , and  $S = R/I_{n-1}(Z)$ . Assume that  $\operatorname{depth} I_{n-1}(Z) = 3$ . If  $n > 2$ , then  $I_{n-1}(Z)$  is a Golod ideal (see [1], Definition 3.6), and*

$$\mathcal{P}_R/\mathcal{P}_S = (1+t)/(1-t^2)^{3-r} \text{ if } n=2, \quad \text{where } r = \dim_K(I_1(Z) + \mathfrak{m}^2)/\mathfrak{m}^2,$$

$$\mathcal{P}_R/\mathcal{P}_S = 1 - \left(\frac{n^2+n}{2}\right)t^2 - (n^2-1)t^3 - \left(\frac{n^2-n}{2}\right)t^4 \quad \text{if } n > 2.$$

*Proof.* If  $n=2$ , then the ideal  $I_{n-1}(Z)$  is a complete intersection and the corresponding formula is well known.

Let  $n > 2$ ; since  $I_{n-1}(Z)$  generically perfect (Corollary (3.10)) we can use Theorem 6.2 of [1] which states that  $I_{n-1}(Z)$  is a Golod ideal in  $R$  if and only if  $I_{n-1}(X)$  is a Golod ideal in the power series ring  $K[[x_{ij}]]$ ,  $1 \leq i \leq j \leq n$ , where  $X = (x_{ij})$ . Observe that  $\operatorname{depth} I_{n-1}(X) = 3$  by Corollary (2.6). (It is Theorem 6.2 of [1] which needs the hypothesis that the entries of  $Z$  belong to the maximal ideal of  $R$ .) Write  $R' = K[[x_{ij}]]$ ,  $1 \leq i \leq j \leq n$ ,  $S' = R'/I_{n-1}(X)$  for short. To prove that  $I_{n-1}(X)$  is Golod it suffices (by Theorems 3.5 and 6.2 of [1]) to show that the algebra  $\operatorname{Tor}^{R'}(S', K)$  has trivial Massey products. Since  $\mathbf{L}(X)$  is a free resolution of  $S'$  over  $R'$  we know that  $\operatorname{Tor}^{R'}(S', K) = \mathbf{L}(X) \otimes_{R'} K$ .

We are going to prove that  $\mathbf{L}(X)$  can be endowed with the structure of a differential graded commutative algebra over  $R'$  in such a way that the induced multiplication on  $\mathbf{L}(X) \otimes_{R'} K$  is trivial. This result implies that  $\operatorname{Tor}^{R'}(S', K)$  has trivial Massey products, hence  $I_{n-1}(X)$  is a Golod ideal and consequently, applying once again Theorems 3.5 and 6.2 of [1], we get the required formula for  $\mathcal{P}_R/\mathcal{P}_S$ .

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\* I am grateful to L. Avramov who drew my attention to an erroneous formulation of Thm. (4.1) in an earlier version of the paper.



Write  $\mathbf{L} = \mathbf{L}(X)$  for short. Let  $S_2(\mathbf{L})$  denote the second symmetric power of the complex  $\mathbf{L}$ . A commutative multiplication on  $\mathbf{L}$  defines a differential graded homomorphism  $S_2(\mathbf{L}) \rightarrow \mathbf{L}$ , which is the identity on the canonical image of  $\mathbf{L}$  in  $S_2(\mathbf{L})$ , and vice versa (by Proposition (1.1) of [2]). Moreover, this multiplication is associative because  $\mathbf{L}$  is of length 3. So we only must define a map of complexes:

$$\begin{array}{ccccccc}
 S_2(\mathbf{L}): \cdots & \longrightarrow & L_1 \otimes L_2 + L_3 & \xrightarrow{\delta_3} & L_1 \otimes L_1 + L_2 & \xrightarrow{\delta_2} & L_1 \xrightarrow{\delta_1} L_0 \\
 \downarrow \varphi & & \varphi_2 \searrow \parallel & & \varphi_1 \searrow \parallel & & \parallel & \parallel \\
 \mathbf{L} & : & 0 \longrightarrow L_3 & \xrightarrow{d_3} & L_2 & \xrightarrow{d_2} & L_1 \xrightarrow{d_1} L_0
 \end{array}$$

To define  $\varphi_1$  we fix a basis  $U_{ij} = E_{ij} \bmod A_n(R)$ ,  $1 \leq i \leq j \leq n$ , of  $L_1$ , and a basis  $W_{pq} = E_{pq}$ ,  $p \neq q$ ,  $W_p = E_{pp} - E_{nn}$ ,  $p = 1, \dots, n-1$ , of  $L_2$ .

We put

$$\varphi_1(U_{ij} \otimes U_{kl}) = \sum_{\alpha \neq 1} T(k, j, i, \alpha) W_{\alpha 1} + \sum_{\alpha \neq j} T(l, i, k, \alpha) W_{\alpha j} +$$

$$\begin{cases} T(k, j, i, l)(W_l - W_j) & \text{if } j \neq n, l \neq n, \\ T(k, j, i, l) W_l & \text{if } j = n, l \neq n, \\ -T(k, j, i, l) W_j & \text{if } j \neq n, l = n, \\ 0 & \text{if } j = l = n. \end{cases}$$

Let  $\mathfrak{m}'$  denote the maximal ideal of  $R'$ . Note that  $\varphi_1(L_1 \otimes L_1) \subset \mathfrak{m}' L_2$  because  $n > 2$ . It follows from this definition of  $\varphi_1$  that  $\varphi_1 \delta_3(L_1 \otimes L_2) \subset \mathfrak{m}'^2 L_2$ . Since  $\mathbf{L}$  is exact there exists precisely one map  $\varphi_2$  making the above diagram commutative. We show that  $\varphi_2(L_1 \otimes L_2) \subset \mathfrak{m}' L_3$ . But this is equivalent to the implication  $d_3(b) \in \mathfrak{m}'^2 L_2 \Rightarrow b \in \mathfrak{m}' L_3$ ,  $b \in L_3$ . The last statement follows simply from the definition of  $d_3$  and linear independence of  $\{x_{ij} \bmod \mathfrak{m}'^2\}$ ,  $i \leq j$ , over  $K$ .

(4.2) *Remark.* If not all entries of  $Z$  belong to the maximal ideal of  $R$  and depth  $I_{n-1}(Z) = 3$ , then by Lemma (1.1)  $I_{n-1}(Z) = I_p(Z')$  for some symmetric matrix  $Z'$  with all the entries in the maximal ideal and for some  $p$ . Therefore, Theorem (4.1) applies also for such matrices.

(4.3) *Remark.* Theorem (4.1) has also been proved independently by J. Herzog and M. Steurich.

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