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Remarks on the closest packing of convex discs

L. FEJES TÓTH

To Professor H. Hadwiger on his seventieth birthday

In the Euclidean plane let P be a packing of congruent replicas of a convex disc c . Let $r = r(P)$ be the supremum of the radii of those circles which have no common point with any disc of P . The smaller r is the “closer” is the packing. Thus $1/r$ can be considered as a measure of the closeness. If r for a certain packing $\bar{P} = \bar{P}(c)$ attains its infimum \bar{r} , then we speak of a closest packing or, in short, a close packing. A simple example for a close packing is given by a packing of unit circles in which each circle is touched by six others. Here we have

$$\bar{r} = \frac{2}{\sqrt{3}} - 1.$$

The above definitions can be extended to more general spaces. In Euclidean 3-space the closest packing of equal balls was determined by Böröczky [1]: the centres of the balls form a body-centred cubic lattice. The paper [2] deals with the same problem in spherical 2-space.

For the density of a packing of convex discs various results are known. In this paper we want to discuss similar problems for the closeness. Let us recall some theorems concerning the density [3, 4]. We shall denote a domain and its area by the same symbol.

THEOREM 1. *If d is the density of a packing of congruent replicas of a convex disc c and H is the hexagon of least area circumscribed about c then $d \leq c/H$.*

Theorem 1 implies

THEOREM 2. *The density of an arbitrary packing of congruent centro-symmetric convex discs cannot exceed the density of the densest lattice-packing of the discs.*

Theorem 2 implies

THEOREM 3. *The density of an arbitrary packing of translates of a convex disc cannot exceed the density of the densest lattice-packing of the discs.*

We start with the following

Remark 1. In a packing of congruent convex discs let r be the supremum of the radii of those circles which have no points in common with any of the discs. Let H be the hexagon of least area circumscribed about a disc. Let $h(x)$ be a hexagon of greatest area inscribed in the parallel domain of distance x of a disc. Then $h(r) \geq H$.

Since $h(x)$ is a strictly increasing function, the above inequality gives a lower bound for r , i.e. an upper bound for the closeness $1/r$.

The proof rests on Theorem 1 and an analogous theorem for the covering [3, 5]: If D is the density of a covering of the plane with non-crossing congruent replicas of a convex disc c and h is a hexagon of maximal area inscribed in c then $D \geq c/h$.

The term that two discs cross means that removing their intersection causes both discs to fall into disjoint pieces.

If d is the density of the packing considered in Remark 1 then we have $d \leq c/H$. On the other hand, let us observe that the parallel-domains of the discs at distance r cover the plane. The density of the parallel-domains is equal to $(c_r/c)d$, where c_r is the area of a parallel-domain. Since the parallel-domains of the same distance of two arbitrary non-overlapping convex domains do not cross, we can apply the above inequality for the covering density:

$$\frac{c_r}{c} d \geq c_r/h(r).$$

Thus we have $c/h(r) \leq d \leq c/H$ which implies the inequality to be proved.

If for a certain disc c H is a plane-filler and for a certain value r_0 the hexagon $h(r_0)$ is identical with H then $\bar{r} = r_0$. There is a great variety of discs with this property. The simplest example is the circle. The closest packing of such discs arises by tiling the plane with congruent replicas of H and inscribing in each hexagon a disc.

Remark 2. The statement arising from Theorem 2 by replacing the words “density” and “densest” by “closeness” and “closest” is false.

We shall show this by a special packing of directly and oppositely congruent discs. The question whether the statement under consideration becomes true by replacing the word “congruent” by “directly congruent” is still open.

Let $u = ABCDEF$ be a centro-symmetric hexagon such that $AB > BC = CD$

and $\sphericalangle ABC = \sphericalangle BCD = 135^\circ$. Let $v = A'A''BC'C''D'D''EF'F''$ be a centrosymmetric decagon arising from u by cutting off at the corners A , C , D and F small triangles such that $A'A = AA'' = CC''$ and

$$C'C = \left(1 - \frac{\sqrt{2}}{2}\right) AA'', = \rho,$$

where ρ is the radius of the incircle of the triangle $A'AA''$. We claim that in any lattice-packing of translates of v there is a gap into which a circle of radius greater than ρ can be inserted.

Obviously, we can restrict ourselves to gaps bounded by three mutually touching translates of v . Again, we can restrict ourselves to such positions of the decagons in which the whole side $A'A''$ (or, which is the same, the whole side $D'D''$) belongs to the boundary of the gap, because otherwise the gap is "bigger" than the triangle $A'AA''$ (Fig. 1). Now we have only to check that in such a position the whole triangle $A'AA''$ belongs to the gap and that from among the two points at which the incircle of $A'AA''$ touches the sides $A'A$ and AA'' one is always in the interior of the gap (Fig. 2).

We continue to construct a packing of congruent replicas of v with a closeness equal to $1/\rho$.

Besides the tiling with translates of the hexagon u there is another regular tiling consisting of alternate rows of translates of u and of translates of oppositely congruent replicas of u . This tiling generates a packing of congruent replicas of v in which there are equal gaps consisting of two triangles congruent with $A'AA''$

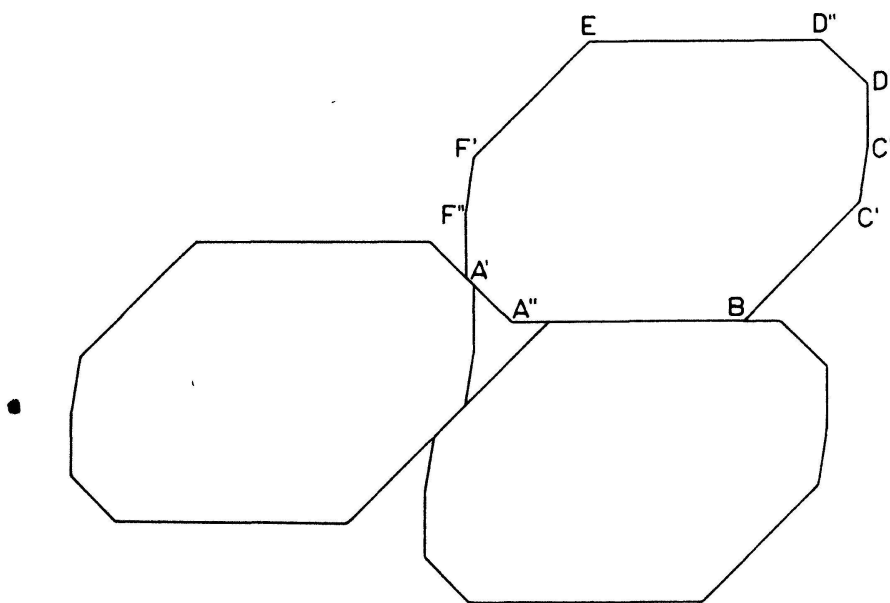


Figure 1

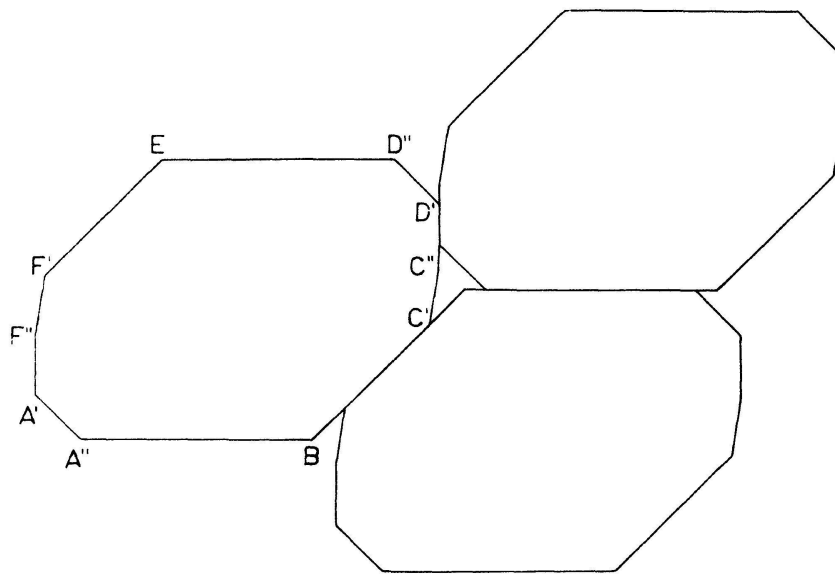


Figure 2

and $C'CC''$ put side by side so that A and C coincide and C' lies on a side of $A'AA''$ (Fig. 3). Consequently C' lies on the incircle of $A'AA''$. Thus the biggest circle contained in a gap is identical with the incircle of $A'AA''$.

This completes the proof of Remark 2.

In contrast with Theorem 2, there is an analogue of Theorem 3 for the closeness which we phrase as

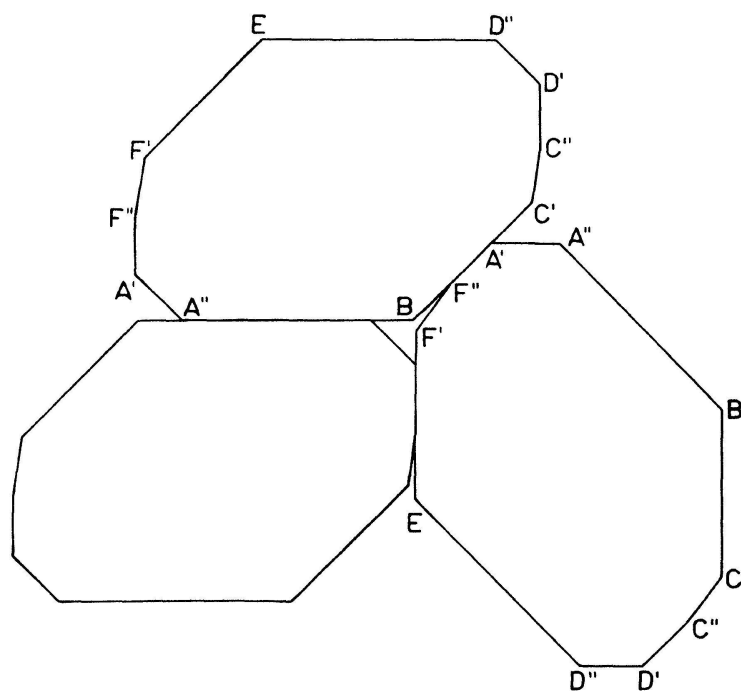


Figure 3

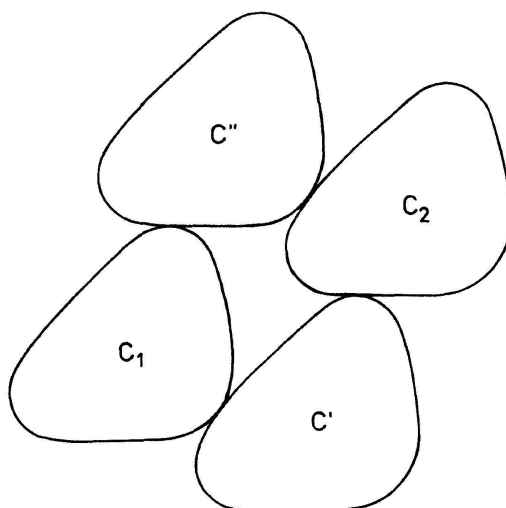


Figure 4

Remark 3. The closeness of an arbitrary packing of translates of a convex disc cannot exceed the closeness of the closest lattice-packing of the discs.

Let c_1, c_2, \dots be translates of a convex disc c forming a packing P . We may assume that in P there are two discs, say, c_1 and c_2 sufficiently near to one another in the following sense. There are two non-overlapping translates c' and c'' of c both touching simultaneously c_1 and c_2 (Fig. 4). Otherwise we could dilate the discs in the same ratio until the desired situation ensues. By a subsequent contraction we obtain a closer packing of translates of c than the original one.

If c is not strictly convex it may occur that the positions of c' and c'' are not uniquely determined. In this case let c'' be the image of c_1 under the translation $c' \rightarrow c_2$.

Obviously, none of the discs c_3, c_4, \dots can reach into the domain q enclosed by c_1, c', c_2 and c'' . (In general, q is a curvilinear quadrangle which can degenerate into two curvilinear triangles.) Thus $r = r(P)$ is at least as great as the radius r_0 of the biggest circle contained in q . On the other hand, we have for the lattice-packing L generated by any three of c_1, c', c_2 and c'' $r(L) = r_0$. Thus we have, in accordance with Remark 3, $r(P) \geq r(L)$.

The above considerations show that Remark 3 remains valid if we measure the closeness of a packing instead of circles by means of an arbitrary figure, say by the supremum of the area of the ellipses contained in the gaps of the packing.

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