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## Remarks on the closest packing of convex discs

L. FEJES TÓTH

To Professor H. Hadwiger on his seventieth birthday

In the Euclidean plane let  $P$  be a packing of congruent replicas of a convex disc  $c$ . Let  $r = r(P)$  be the supremum of the radii of those circles which have no common point with any disc of  $P$ . The smaller  $r$  is the “closer” is the packing. Thus  $1/r$  can be considered as a measure of the closeness. If  $r$  for a certain packing  $\bar{P} = \bar{P}(c)$  attains its infimum  $\bar{r}$ , then we speak of a closest packing or, in short, a close packing. A simple example for a close packing is given by a packing of unit circles in which each circle is touched by six others. Here we have

$$\bar{r} = \frac{2}{\sqrt{3}} - 1.$$

The above definitions can be extended to more general spaces. In Euclidean 3-space the closest packing of equal balls was determined by Böröczky [1]: the centres of the balls form a body-centred cubic lattice. The paper [2] deals with the same problem in spherical 2-space.

For the density of a packing of convex discs various results are known. In this paper we want to discuss similar problems for the closeness. Let us recall some theorems concerning the density [3, 4]. We shall denote a domain and its area by the same symbol.

**THEOREM 1.** *If  $d$  is the density of a packing of congruent replicas of a convex disc  $c$  and  $H$  is the hexagon of least area circumscribed about  $c$  then  $d \leq c/H$ .*

Theorem 1 implies

**THEOREM 2.** *The density of an arbitrary packing of congruent centro-symmetric convex discs cannot exceed the density of the densest lattice-packing of the discs.*

Theorem 2 implies

**THEOREM 3.** *The density of an arbitrary packing of translates of a convex disc cannot exceed the density of the densest lattice-packing of the discs.*

We start with the following

**Remark 1.** In a packing of congruent convex discs let  $r$  be the supremum of the radii of those circles which have no points in common with any of the discs. Let  $H$  be the hexagon of least area circumscribed about a disc. Let  $h(x)$  be a hexagon of greatest area inscribed in the parallel domain of distance  $x$  of a disc. Then  $h(r) \geq H$ .

Since  $h(x)$  is a strictly increasing function, the above inequality gives a lower bound for  $r$ , i.e. an upper bound for the closeness  $1/r$ .

The proof rests on Theorem 1 and an analogous theorem for the covering [3, 5]: If  $D$  is the density of a covering of the plane with non-crossing congruent replicas of a convex disc  $c$  and  $h$  is a hexagon of maximal area inscribed in  $c$  then  $D \geq c/h$ .

The term that two discs cross means that removing their intersection causes both discs to fall into disjoint pieces.

If  $d$  is the density of the packing considered in Remark 1 then we have  $d \leq c/H$ . On the other hand, let us observe that the parallel-domains of the discs at distance  $r$  cover the plane. The density of the parallel-domains is equal to  $(c_r/c)d$ , where  $c_r$  is the area of a parallel-domain. Since the parallel-domains of the same distance of two arbitrary non-overlapping convex domains do not cross, we can apply the above inequality for the covering density:

$$\frac{c_r}{c} d \geq c_r/h(r).$$

Thus we have  $c/h(r) \leq d \leq c/H$  which implies the inequality to be proved.

If for a certain disc  $c$   $H$  is a plane-filler and for a certain value  $r_0$  the hexagon  $h(r_0)$  is identical with  $H$  then  $\bar{r} = r_0$ . There is a great variety of discs with this property. The simplest example is the circle. The closest packing of such discs arises by tiling the plane with congruent replicas of  $H$  and inscribing in each hexagon a disc.

**Remark 2.** The statement arising from Theorem 2 by replacing the words “density” and “densest” by “closeness” and “closest” is false.

We shall show this by a special packing of directly and oppositely congruent discs. The question whether the statement under consideration becomes true by replacing the word “congruent” by “directly congruent” is still open.

Let  $u = ABCDEF$  be a centro-symmetric hexagon such that  $AB > BC = CD$

and  $\angle ABC = \angle BCD = 135^\circ$ . Let  $v = A'A''BC'C''D'D''EF'F''$  be a centro-symmetric decagon arising from  $u$  by cutting off at the corners  $A, C, D$  and  $F$  small triangles such that  $A'A = AA'' = CC''$  and

$$C'C = \left(1 - \frac{\sqrt{2}}{2}\right) AA'', = \rho,$$

where  $\rho$  is the radius of the incircle of the triangle  $A'AA''$ . We claim that in any lattice-packing of translates of  $v$  there is a gap into which a circle of radius greater than  $\rho$  can be inserted.

Obviously, we can restrict ourselves to gaps bounded by three mutually touching translates of  $v$ . Again, we can restrict ourselves to such positions of the decagons in which the whole side  $A'A''$  (or, which is the same, the whole side  $D'D''$ ) belongs to the boundary of the gap, because otherwise the gap is "bigger" than the triangle  $A'AA''$  (Fig. 1). Now we have only to check that in such a position the whole triangle  $A'AA''$  belongs to the gap and that from among the two points at which the incircle of  $A'AA''$  touches the sides  $A'A$  and  $AA''$  one is always in the interior of the gap (Fig. 2).

We continue to construct a packing of congruent replicas of  $v$  with a closeness equal to  $1/\rho$ .

Besides the tiling with translates of the hexagon  $u$  there is another regular tiling consisting of alternate rows of translates of  $u$  and of translates of oppositely congruent replicas of  $u$ . This tiling generates a packing of congruent replicas of  $v$  in which there are equal gaps consisting of two triangles congruent with  $A'AA''$ .

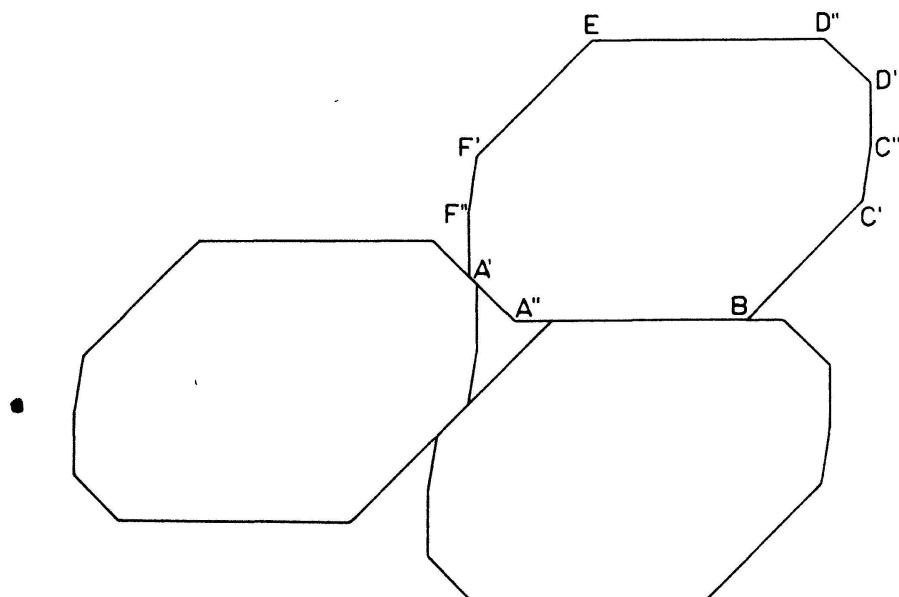


Figure 1

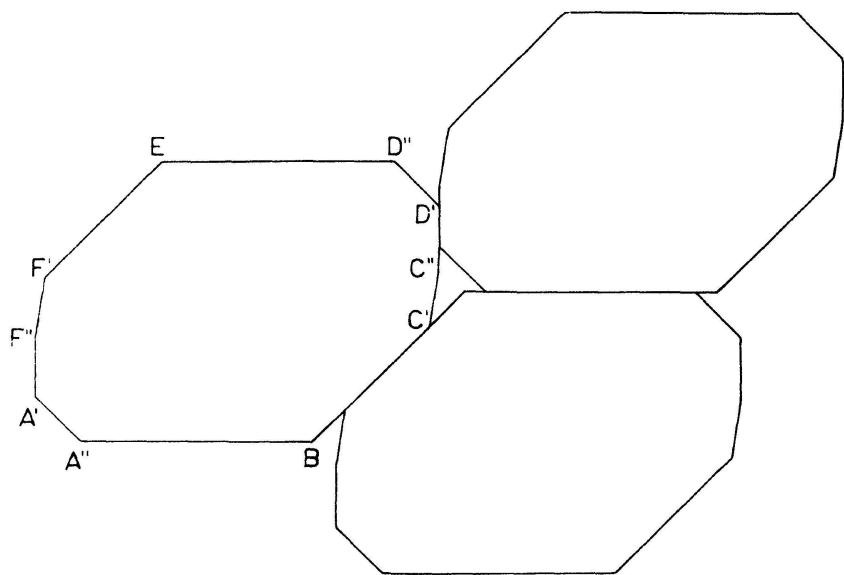


Figure 2

and  $C'CC''$  put side by side so that  $A$  and  $C$  coincide and  $C'$  lies on a side of  $A'AA''$  (Fig. 3). Consequently  $C'$  lies on the incircle of  $A'AA''$ . Thus the biggest circle contained in a gap is identical with the incircle of  $A'AA''$ .

This completes the proof of Remark 2.

In contrast with Theorem 2, there is an analogue of Theorem 3 for the closeness which we phrase as

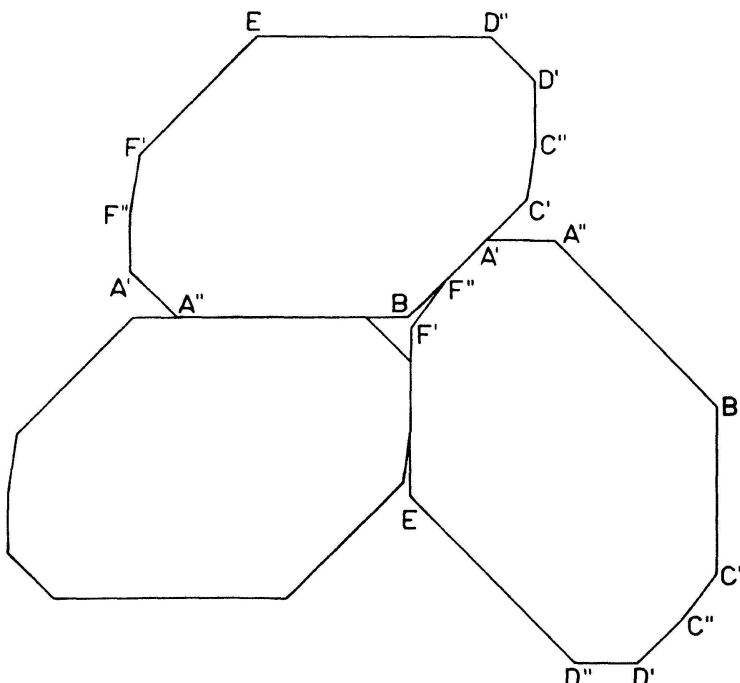


Figure 3

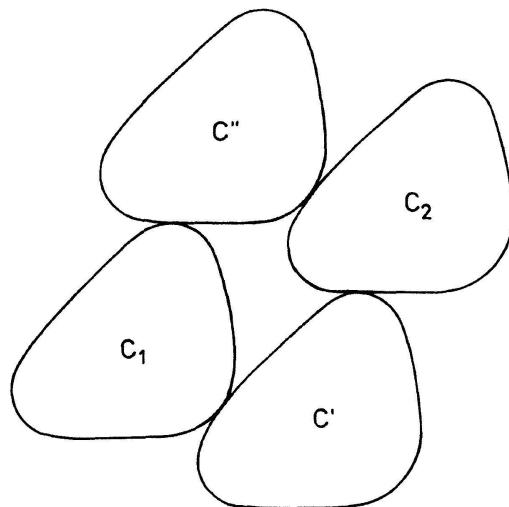


Figure 4

**Remark 3.** The closeness of an arbitrary packing of translates of a convex disc cannot exceed the closeness of the closest lattice-packing of the discs.

Let  $c_1, c_2, \dots$  be translates of a convex disc  $c$  forming a packing  $P$ . We may assume that in  $P$  there are two discs, say,  $c_1$  and  $c_2$  sufficiently near to one another in the following sense. There are two non-overlapping translates  $c'$  and  $c''$  of  $c$  both touching simultaneously  $c_1$  and  $c_2$  (Fig. 4). Otherwise we could dilate the discs in the same ratio until the desired situation ensues. By a subsequent contraction we obtain a closer packing of translates of  $c$  than the original one.

If  $c$  is not strictly convex it may occur that the positions of  $c'$  and  $c''$  are not uniquely determined. In this case let  $c''$  be the image of  $c_1$  under the translation  $c' \rightarrow c_2$ .

Obviously, none of the discs  $c_3, c_4, \dots$  can reach into the domain  $q$  enclosed by  $c_1, c', c_2$  and  $c''$ . (In general,  $q$  is a curvilinear quadrangle which can degenerate into two curvilinear triangles.) Thus  $r = r(P)$  is at least as great as the radius  $r_0$  of the biggest circle contained in  $q$ . On the other hand, we have for the lattice-packing  $L$  generated by any three of  $c_1, c', c_2$  and  $c''$   $r(L) = r_0$ . Thus we have, in accordance with Remark 3,  $r(P) \geq r(L)$ .

The above considerations show that Remark 3 remains valid if we measure the closeness of a packing instead of circles by means of an arbitrary figure, say by the supremum of the area of the ellipses contained in the gaps of the packing.

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